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A GENERALIZED SIMPLE FORMULA FOR EVALUATING RADON-NIKODYM DERIVATIVES OVER PATHS

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Abstract. Let $C[0,T]$ denote a generalized analogue of Wiener space, the space of real-valued continuous functions on the interval $[0,T]$. Define $Z_{\vec{e},n}: C[0,T] \to \mathbb{R}^{n+1}$ by

$$Z_{\vec{e},n}(x) = \left(x(0), \int_0^T e_1(t)dx(t), \ldots, \int_0^T e_n(t)dx(t) \right),$$

where $e_1, \ldots, e_n$ are of bounded variations on $[0,T]$. In this paper we derive a simple evaluation formula for Radon-Nikodym derivatives similar to the conditional expectations of functions on $C[0,T]$ with the conditioning function $Z_{\vec{e},n}$ which has an initial weight and a kind of drift. As applications of the formula, we evaluate the Radon-Nikodym derivatives of various functions on $C[0,T]$ which are of interested in Feynman integration theory and quantum mechanics. This work generalizes and simplifies the existing results, that is, the simple formulas with the conditioning functions related to the partitions of time interval $[0,T]$.

1. Introduction

Let $C_0[0,T]$ denote the classical Wiener space, the space of real-valued continuous functions x on the interval $[0,T]$ with $x(0) = 0$. When $\tau : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ is a partition of $[0,T]$ and $\xi_j \in \mathbb{R}$ for $j = 0, 1, \ldots, n$, the conditional expectation of time integral in which the paths of $C_0[0,T]$ pass through the point ξ_j at each time t_j is very useful in the Brownian motion theory. Park and Skoug [7] derived a simple formula for conditional Wiener integrals containing the time integral with the conditioning function $X_n : C_0[0,T] \to \mathbb{R}^n$ given by $X_n(x) = (x(t_1), \ldots, x(t_n))$. Furthermore, they [8] extended the formula in [7] with the conditioning function $Z_n : C_0[0,T] \to \mathbb{R}^n$ given by $Z_n(x) = (\int_0^T e_1(t)dx(t), \ldots, \int_0^T e_n(t)dx(t))$, where $e_1, \ldots, e_n$ are in

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$L^2[0,T]$. In their simple formulas, they expressed the conditional Wiener integrals directly in terms of ordinary Wiener integrals, which generalizes Yeh's inversion formula [12].

On the other hand, let $C[0,T]$ denote the space of continuous real-valued functions on the interval $[0,T]$. Ryu [10, 11] introduced a finite positive measure $w_{\alpha,\beta;\varphi}$ on $C[0,T]$, where $\alpha, \beta : [0,T] \to \mathbb{R}$ are appropriate functions and φ is a finite positive measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$. We note that $w_{\alpha,\beta;\varphi}$ is exactly the Wiener measure on $C_0[0,T]$ if $\alpha(t) = 0$, $\beta(t) = t$ for $t \in [0,T]$ and φ is the Dirac measure concentrated at 0. Let $X_\tau : C[0,T] \to \mathbb{R}^{n+1}$ and $Y_\tau : C[0,T] \to \mathbb{R}^n$ be the functions defined by $X_\tau(x) = (x(t_0), x(t_1), \ldots, x(t_n))$ and $Y_\tau(x) = (x(t_0), x(t_1), \ldots, x(t_{n-1}))$, respectively. In [4,6], the author investigated properties of the Fourier-transform of the function $W : C[0,T]\times[0,T] \to \mathbb{R}$ defined by $W(x,t) = x(t)$. In fact, using the Fourier-transform of W, he derived two simple evaluation formulas for Radon-Nikodym derivatives similar to the conditional expectations of functions on $C[0,T]$ for the conditioning functions X_τ and Y_τ which have a generalized drift α, a generalized variance function β and an initial weight φ. As applications of the formulas, he evaluated the Radon-Nikodym derivatives of the functions $F(x) \equiv \int_0^T [W(x,t)]^m d\lambda(t) (m \in \mathbb{N})$ and $G_3(x) \equiv [\int_0^T W(x,t) d\lambda(t)]^2$ on $C[0,T]$, where λ is a $\mathbb{C}$-valued Borel measure.

For $x \in C[0,T]$, let $Z_{\vec{e},n}(x) = (x(t_0), \int_0^T e_1(t)dx(t), \ldots, \int_0^T e_n(t)dx(t))$, where $e_1, \ldots, e_n$ are of bounded variations on $[0,T]$. In this paper we derive a simple evaluation formula for Radon-Nikodym derivatives similar to the conditional expectations of functions on $C[0,T]$ for the more generalized conditioning function $Z_{\vec{e},n}$ which also has a kind of drift α, the generalized variance function β and the initial weight φ. As applications of the formula, we evaluate the Radon-Nikodym derivatives of various functions on $C[0,T]$ which are of interested in Feynman integration theory and quantum mechanics. In fact, we calculate the derivatives of F, G_3, a cylinder type function and the functions in a Banach algebra which generalizes the Cameron-Storvick's one [1]. We note that W has a kind of drift α with the more generalized variance function β while it has no drifts on $C_0[0,T]$. Furthermore, our underlying space $C[0,T]$ may not be a probability space so that the results of this paper generalize and simplify those of [4, 6, 8, 12] and [7] in which the works are the first results among them.

2. A generalized analogue of Wiener space

In this section, we introduce a generalized analogue of Wiener space which is our underlying space of this work.

Let α be absolutely continuous on $[0,T]$ and let β be continuous, strictly increasing on $[0,T]$. Let φ be a positive finite measure on $\mathcal{B}(\mathbb{R})$. For $\vec{t}_k = (t_0, t_1, \ldots, t_k)$ with $0 = t_0 < t_1 < \cdots < t_k \le T$, let $J_{\vec{t}_k} : C[0,T] \to \mathbb{R}^{k+1}$ be the function given by $J_{\vec{t}_k}(x) = (x(t_0), x(t_1), \ldots, x(t_k))$. For $\prod_{j=0}^k B_j \in \mathcal{B}(\mathbb{R}^{k+1})$, the subset $J_{\vec{t}_k}^{-1}(\prod_{j=0}^k B_j)$ of $C[0,T]$ is called an interval I and let $\mathcal{I}$ be the set of all such intervals I. Define a premeasure $m_{\alpha,\beta;\varphi}$ on $\mathcal{I}$ by

$$m_{\alpha,\beta;\varphi}(I) = \int_{B_0}\int_{\prod_{j=1}^k B_j} \mathcal{W}(\vec{t}_k,\vec{u}_k,u_0)dm_L^k(\vec{u}_k)d\varphi(u_0),$$

where m_L is the Lebesgue measure on $\mathcal{B}(\mathbb{R})$, and for $u_0 \in \mathbb{R}$, $\vec{u}_k = (u_1,\ldots,u_k) \in \mathbb{R}^k$

$$\mathcal{W}(\vec{t}_k,\vec{u}_k,u_0) = \bigg[\frac{1}{\prod_{j=1}^k 2\pi[\beta(t_j)-\beta(t_{j-1})]}\bigg]^{\frac{1}{2}} \times \exp\bigg\{-\frac{1}{2}\sum_{j=1}^k \frac{[u_j-\alpha(t_j)-u_{j-1}+\alpha(t_{j-1})]^2}{\beta(t_j)-\beta(t_{j-1})}\bigg\}.$$

The Borel σ-algebra $\mathcal{B}(C[0,T])$ of $C[0,T]$ with the supremum norm, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique positive finite measure $w_{\alpha,\beta;\varphi}$ on $\mathcal{B}(C[0,T])$ with $w_{\alpha,\beta;\varphi}(I) = m_{\alpha,\beta;\varphi}(I)$ for all $I \in \mathcal{I}$. This measure $w_{\alpha,\beta;\varphi}$ is called a generalized analogue of Wiener measure on $(C[0,T],\mathcal{B}(C[0,T]))$ according to φ [10,11].

Now we introduce a useful lemma which is needed in the next section [4].

Lemma 2.1. *Let $0 \le s_1 \le s_2 \le s_3 \le T$. Then the Fourier-transform $\mathcal{F}(W(\cdot,s_1),\ W(\cdot,s_3)-W(\cdot,s_2))$ of $(W(\cdot,s_1),W(\cdot,s_3)-W(\cdot,s_2))$ can be expressed by*

$$\begin{aligned}&\mathcal{F}(W(\cdot,s_1),W(\cdot,s_3)-W(\cdot,s_2))(\xi_1,\xi_2)\\ &= \frac{1}{\varphi(\mathbb{R})}\mathcal{F}(W(\cdot,s_1))(\xi_1)\mathcal{F}(W(\cdot,s_3)-W(\cdot,s_2))(\xi_2)\end{aligned}$$

for $\xi_1,\xi_2 \in \mathbb{R}$ so that $W(\cdot,s_1)$ and $W(\cdot,s_3)-W(\cdot,s_2)$ are independent if φ is a probability measure.

Let $\nu_{\alpha,\beta}$ denote the Lebesgue-Stieltjes measure defined by $\nu_{\alpha,\beta}(E) = \int_E d(|\alpha| +\beta)(t)$ for each Lebesgue measurable subset E of $[0,T]$, where $|\alpha|$ denotes the total variation of α. Define $L^2_{\alpha,\beta}[0,T]$ to be the space of functions on $[0,T]$ that are square integrable with respect to $\nu_{\alpha,\beta}$ [9]; that is,

$$L^2_{\alpha,\beta}[0,T] = \bigg\{f:[0,T]\to\mathbb{R}\,\bigg|\,\int_0^T [f(t)]^2 d\nu_{\alpha,\beta}(t) < \infty\bigg\}.$$

The space $L^2_{\alpha,\beta}[0,T]$ is a Hilbert space and has the inner product

$$\langle f,g\rangle_{\alpha,\beta} = \int_0^T f(t)g(t)d\nu_{\alpha,\beta}(t).$$

We note that $L^2_{\alpha,\beta}[0,T] \subseteq L^2_{0,\beta}[0,T]$, where $L^2_{0,\beta}[0,T]$ denotes the space $L^2_{\alpha,\beta}[0,T]$ with $\alpha \equiv 0$. Since $\|\cdot\|_{0,\beta} \le \|\cdot\|_{\alpha,\beta}$, the two norms $\|\cdot\|_{0,\beta}$ and $\|\cdot\|_{\alpha,\beta}$ are equivalent on $L^2_{\alpha,\beta}[0,T]$ by the open mapping theorem. Let $S[0,T]$ be the collection of step functions on $[0,T]$ and let $\int_0^T \phi(t)dx(t)$ denote the Riemann-Stieltjes integral. For $f \in L^2_{\alpha,\beta}[0,T]$, let $\{\phi_n\}$ be a sequence of the step functions in $S[0,T]$ with $\lim_{n\to\infty} \|\phi_n - f\|_{\alpha,\beta} = 0$. Define $I_{\alpha,\beta}(f)$ by the $L^2(C[0,T])$-limit

$$I_{\alpha,\beta}(f)(x) = \lim_{n\to\infty} \int_0^T \phi_n(t)dx(t)$$

for all $x \in C[0,T]$ for which this limit exists or $I_{\alpha,\beta}(f)(x) = \lim_{n\to\infty} \int_0^T \phi_n(t)$ $dx(t)$ point-wisely if exists. We note that for $\in L^2_{\alpha,\beta}[0,T]$, $I_{\alpha,\beta}(f)(x)$ exists for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Moreover, we have the following theorems [3].

Theorem 2.2. *If f is of bounded variation on $[0,T]$, then $I_{\alpha,\beta}(f)(x) = \int_0^T f(t)$ $dx(t)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$.*

Throughout this paper, for $x \in C[0,T]$, we redefine

$$I_{\alpha,\beta}(f)(x) = \int_0^T f(t)dx(t)$$

if $\int_0^T f(t)dx(t)$ exists.

Theorem 2.3. *Let $f, g \in L^2_{\alpha,\beta}[0,T]$. Then we have the followings:*

(1) $\int_{C0,T]} I_{\alpha,\beta}(f)(x)dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})I_{\alpha,\beta}(f)(\alpha)$.
(2) $\int_{C[0,T]} [I_{\alpha,\beta}(f)(x)][I_{\alpha,\beta}(g)(x)]dw_{\alpha,\beta;\varphi}(x) = \varphi(\mathbb{R})[\langle f,g\rangle_{0,\beta} + [I_{\alpha,\beta}(f)(\alpha)]$ $\times[I_{\alpha,\beta}(g)(\alpha)]$.
(3) *$I_{\alpha,\beta}(f)$ is a normally distributed random variable with the mean $I_{\alpha,\beta}(f)$ (α) and the variance $\|f\|^2_{0,\beta}$ if $\varphi(\mathbb{R}) = 1$. In this case, the covariance of $I_{\alpha,\beta}(f)$ and $I_{\alpha,\beta}(g)$ is given by $\langle f,g\rangle_{0,\beta}$.*

Let k be a positive integer, let X be an $\mathbb{R}^k$-valued Borel measurable function defined for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and let $F : C[0,T] \to \mathbb{C}$ be integrable. Let m_X be the image measure on the Borel class $\mathcal{B}(\mathbb{R}^k)$ of $\mathbb{R}^k$ induced by X. By the Radon-Nikodym theorem, there exists an m_X-integrable function $\frac{d\mu_X}{dm_X}$ defined on $\mathbb{R}^k$ which is unique up to m_X a.e. such that for every $B \in \mathcal{B}(\mathbb{R}^k)$,

$$\int_{X^{-1}(B)} F(x)dw_{\alpha,\beta;\varphi}(x) = \int_B \frac{d\mu_X}{dm_X}(\vec{\eta})dm_X(\vec{\eta}).$$

Define the function $\frac{d\mu_X}{dm_X}$ as the generalized conditional expectation of F given X and it is denoted by $GE[F|X]$. We note that $GE[F|X]$ is a Radon-Nikodym derivative rather than a conditional expectation since m_X may not be a probability measure.

Lemma 2.4. *Let X and F be as given above. Let $\psi : \mathbb{R}^k \to \mathbb{R}^k$ be a bijective Borel measurable function. Then we have for $m_{\psi\circ X}$ a.e. $\vec{\xi} \in \mathbb{R}^k$*

$$GE[F|(\psi \circ X)](\vec{\xi}) = GE[F|X](\psi^{-1}(\vec{\xi})).$$

Proof. By the definition of generalized conditional expectation and the change of variable theorem, we have for $B \in \mathcal{B}(\mathbb{R}^k)$

$$\begin{aligned}\int_B GE[F|(\psi\circ X)](\vec{\xi})dm_{\psi\circ X}(\vec{\xi}) &= \int_{X^{-1}(\psi^{-1}(B))} F(x)dw_{\alpha,\beta;\varphi}(x) \\ &= \int_{\psi^{-1}(B)} GE[F|X](\vec{\xi})dm_X(\vec{\xi}) \\ &= \int_B GE[F|X](\psi^{-1}(\vec{\xi}))d(m_X\circ\psi^{-1})(\vec{\xi}) \\ &= \int_B GE[F|X](\psi^{-1}(\vec{\xi}))dm_{\psi\circ X}(\vec{\xi}).\end{aligned}$$

Now, by the uniqueness of Radon-Nikodym derivative, we have this lemma. □

3. A simple formula for the generalized conditional expectation

In this section, we derive a simple evaluation formula for the generalized conditional expectation.

Let $\{e_1,\dots,e_n\}$ be a set of functions in $L^2_{\alpha,\beta}[0,T]$ such that $\{e_1,\dots,e_n\}$ is orthonormal in $L^2_{0,\beta}[0,T]$. Such sets always exist:

Example 3.1. (1) Let $\{1,t,\dots,t^{n-1}\}$ be a set of polynomials on $[0,T]$ and let $\{f_1,\dots,f_n\}$ be the set obtained by the Gram-Schmidt orthonormalization process in $L^2_{0,\beta}[0,T]$. Then it is clear that the set $\{f_1,\dots,f_n\}$ satisfies the desired condition.

(2) Let $\tau : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0,T]$. For $j = 1,\dots,n$, let

$$g_j(s) = \frac{1}{\sqrt{\beta(t_j)-\beta(t_{j-1})}}\chi_{[t_{j-1},t_j]}(s) \text{ for } s\in[0,T]. \tag{1}$$

Then it is clear that the set $\{g_1,\dots,g_n\}$ satisfies the desired condition. Each g_j is also of bounded variation on $[0,T]$. Using this orthonormal set, we will simplify the results related to the simple formulas with the conditioning functions X_τ and Y_τ [4,6].

Let V_n be the subset of $L^2_{0,\beta}[0,T]$ generated by $\{e_1,\dots,e_n\}$ and let $V_n^\perp$ be the orthogonal complement of V_n. Let $\mathcal{P}_{\vec{e},n,\beta} : L^2_{0,\beta}[0,T]\to V_n$ and $\mathcal{P}^\perp_{\vec{e},n,\beta}$: $L^2_{0,\beta}[0,T]\to V_n^\perp$ be the orthogonal projections, where

$$\mathcal{P}_{\vec{e},n,\beta}v = \sum_{j=1}^n \langle v,e_j\rangle_{0,\beta}e_j \text{ for } v\in L^2_{0,\beta}[0,T].$$

It is clear that $\mathcal{P}_{\vec{e},n,\beta}v$ belongs to $L^2_{\alpha,\beta}[0,T]$ for $v \in L^2_{0,\beta}[0,T]$ and $\mathcal{P}^\perp_{\vec{e},n,\beta}v$ belongs to $L^2_{\alpha,\beta}[0,T]$ if $v\in L^2_{\alpha,\beta}[0,T]$. Let $z_0(x) = x(0)$ for $x \in C[0,T]$. For each $j = 1,\dots,n$, define z_j and Z_n by $z_j(x) = I_{\alpha,\beta}(e_j)(x)$ and

$$Z_{\vec{e},n}(x) = (z_0(x), z_1(x),\dots,z_n(x))$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. For $s \in [0,T]$, $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and $\vec{\xi} = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, let

$$x_{\vec{e},n,\beta}(s) = z_0(x) + I_{\alpha,\beta}(\mathcal{P}_{\vec{e},n,\beta}\chi_{[0,s]})(x)$$

and

$$\vec{\xi}_{\vec{e},n,\beta}(s) = \xi_0 + \sum_{j=1}^{n} \langle e_j, \chi_{[0,s]}\rangle_{0,\beta}\xi_j.$$

Note that for $0 \le s \le t \le T$, we have by the Schwarz's inequality

$$|\langle e_j, \chi_{[0,s]}\rangle_{0,\beta} - \langle e_j, \chi_{[0,t]}\rangle_{0,\beta}|^2 \le \|e_j\|_{0,\beta}^2[\beta(t) - \beta(s)]$$

so that $x_{\vec{e},n,\beta}$ and $\vec{\xi}_{\vec{e},n,\beta}$ are absolutely continuous on $[0,T]$ since β is increasing.

Throughout this paper, we assume that each e_j is of bounded variation on $[0,T]$. Note that $\mathcal{P}_{\vec{e},n,\beta}v(v \in L_{0,\beta}^2[0,T])$ is of bounded variation on $[0,T]$ and so is $\mathcal{P}_{\vec{e},n,\beta}^{\perp}v$ if v is of bounded variation on $[0,T]$. Moreover, we have the following properties:

- **(P1)** For $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and $s \in [0,T]$, we have by the linearity of $I_{\alpha,\beta}$, $x_{\vec{e},n,\beta}(s) = z_0(x) + \sum_{j=1}^{n}\langle e_j, \chi_{[0,s]}\rangle_{0,\beta} z_j(x)$.
- **(P2)** For $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$ and $s \in [0,T]$, we have by Theorem 2.2, $x(s) - x_{\vec{e},n,\beta}(s) = \int_0^T(\chi_{[0,s]} - \mathcal{P}_{\vec{e},n,\beta}\chi_{[0,s]})(u)dx(u) = \int_0^T(\mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,s]})(u)\,dx(u) = I_{\alpha,\beta}(\mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,s]})(x)$ so that $I_{\alpha,\beta}(\mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,\cdot]})(x)$ belongs to $C[0,T]$.
- **(P3)** For $0 \le s_1 \le s_2 \le T$, $\langle \mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,s_1]}, \mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,s_2]}\rangle_{0,\beta} = \beta(s_1) - \beta(0) - \sum_{l=1}^{n}\langle\chi_{[0,s_1]}, e_l\rangle_{0,\beta}\langle\chi_{[0,s_2]}, e_l\rangle_{0,\beta}$.

Theorem 3.2. *If $\varphi(\mathbb{R}) = 1$, then $\{I_{\alpha,\beta}(\mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,s]}) : 0 \le s \le T\}$ and z_j are stochastically independent for $j = 0, 1, 2, \dots, n$.*

Proof. For $s \in [0,T]$ and $j = 1, \dots, n$, we have by the orthonormality of e_js

$$\begin{aligned}\langle \mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,s]}, e_j\rangle_{0,\beta} &= \langle\chi_{[0,s]}, e_j\rangle_{0,\beta} - \sum_{l=1}^{n}\langle\chi_{[0,s]}, e_l\rangle_{0,\beta}\langle e_l, e_j\rangle_{0,\beta} \\ &= \langle\chi_{[0,s]}, e_j\rangle_{0,\beta} - \langle\chi_{[0,s]}, e_j\rangle_{0,\beta} = 0\end{aligned}$$

so that the independence of $\{I_{\alpha,\beta}(\mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,s]}) : 0 \le s \le T\}$ and z_j follows from Theorem 2.3. To complete the proof, we must prove that z_0 and $I_{\alpha,\beta}(\mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi[0,s])$ are independent. Let $\mathcal{F}$ denote the Fourier transform and for $l \in \mathbb{N}$, let $\tau_j = \frac{T}{l}j$ for $j = 0, 1, \dots, l$. Then for $\xi_1, \xi_2 \in \mathbb{R}$, we have by Lemma 2.1, **(P2)** and the dominated convergence theorem

$$\begin{aligned}&\mathcal{F}(z_0, I_{\alpha,\beta}(\mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,s]}))(\xi_1, \xi_2) \\ = &\int_{C[0,T]}\exp\{i[\xi_1 z_0(x) + \xi_2 I_{\alpha,\beta}(\mathcal{P}_{\vec{e},n,\beta}^{\perp}\chi_{[0,s]})(x)]\}dw_{\alpha,\beta;\varphi}(x)\end{aligned}$$

$$= \lim_{l\to\infty}\int_{C[0,T]}\exp\bigg\{i\bigg[\xi_1 W(x,0)+\xi_2\sum_{j=1}^{l}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s]})(\tau_j)[W(x,\tau_j)$$

$$- W(x,\tau_{j-1})]\bigg]\bigg\}dw_{\alpha,\beta;\varphi}(x)$$

$$= \mathcal{F}(W(\cdot,0))(\xi_1)\int_{C[0,T]}\exp\bigg\{i\xi_2\lim_{l\to\infty}\sum_{j=1}^{l}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s]})(\tau_j)[W(x,\tau_j)$$

$$- W(x,\tau_{j-1})]\bigg\}dw_{\alpha,\beta;\varphi}(x)$$

$$= \mathcal{F}(z_0)(\xi_1)\mathcal{F}(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi[0,s]))(\xi_2)$$

which completes the proof. □

By Theorem 3.2 and **(P1)**, we have the following corollary.

Corollary 3.3. *$\{I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s]}) : 0\le s\le T\}$ and $\{x_{\vec{e},n,\beta}(s) : 0\le s\le T\}$ are stochastically independent if $\varphi(\mathbb{R})=1$.*

Using the same process used in the proof of [8, Theorem 2] and [4, Theorem 4] with aid of **(P2)**, Theorem 3.2 and Corollary 3.3, we have the following theorem.

Theorem 3.4. *Let $\varphi_0=\frac{1}{\varphi(\mathbb{R})}\varphi$ and suppose that $F : C[0,T]\to\mathbb{C}$ is integrable. Then we have for $m_{Z_{\vec{e},n}}$ a.e. $\vec{\xi}\in\mathbb{R}^{n+1}$*

$$GE[F|Z_{\vec{e},n}](\vec{\xi}) = \int_{C[0,T]} F(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,\cdot]})(x)+\vec{\xi}_{\vec{e},n,\beta})dw_{\alpha,\beta;\varphi_0}(x).$$

For $s,t\in[t_{j-1},t_j]$, let $\gamma_j(t)=\frac{\beta(t)-\beta(t_{j-1})}{\beta(t_j)-\beta(t_{j-1})}$ and $\Phi_j(s,t)=[\beta(t_j)-\beta(s)]\gamma_j(t)$. For $s\in[0,T]$, $\vec{\eta}\in\mathbb{R}^n$ and $\vec{\xi}=(\xi_0,\xi_1,\dots,\xi_n)\in\mathbb{R}^{n+1}$, let

$$\Xi(n,\vec{\xi})(s)$$

$$=\xi_0+\sum_{j=1}^{n}\chi_{(t_{j-1},t_j]}(s)\bigg[\sum_{l=1}^{j-1}\xi_l\sqrt{\beta(t_l)-\beta(t_{l-1})}+\frac{\beta(s)-\beta(t_{j-1})}{\sqrt{\beta(t_j)-\beta(t_{j-1})}}\xi_j\bigg]$$

and

$$\Xi_{t_n}(\vec{\eta})(s)=\chi_{[0,t_{n-1})}(s)\Xi(n-1,\vec{\eta})(s)+\chi_{[t_{n-1},t_n]}(s)\Xi(n-1,\vec{\eta})(t_{n-1}).$$

For $x\in C[0,T]$, define the polygonal functions $P_\beta(x)$ and $P_{t_n,\beta}(x)$ of x by

$$P_\beta(x)(s) \tag{2}$$

$$=\chi_{\{0\}}(s)x(0)+\sum_{j=1}^{n}\chi_{(t_{j-1},t_j]}(s)[x(t_{j-1})+\gamma_j(s)[x(t_j)-x(t_{j-1})]]$$

and

$$P_{t_n,\beta}(x)(s) = \chi_{[0,t_{n-1})}(s)P_\beta(x)(s) + \chi_{[t_{n-1},t_n]}(s)P_\beta(x)(t_{n-1}) \tag{3}$$

for $s \in [0,T]$. Similarly, the polygonal functions $P_\beta(\vec{\xi})$ and $P_{t_n,\beta}(\vec{\xi})$ on $[0,T]$ are defined by (2) and (3), respectively, with replacing $x(t_j)$ by ξ_j for $j = 0,1,\ldots,n$. Throughout this paper, we will use the notation $\vec{g}$ in place of $\vec{e}$ when e_j is replaced by g_j which is given by (1).

Corollary 3.5. *Let $F : C[0,T] \to \mathbb{C}$ be integrable. Then the followings hold:*

(1) *For $m_{Z_{\vec{g},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$, we have*

$$GE[F|Z_{\vec{g},n}](\vec{\xi}) = \int_{C[0,T]} F(x - P_\beta(x) + \Xi(n,\vec{\xi}))dw_{\alpha,\beta;\varphi_0}(x), \tag{4}$$

where $m_{Z_{\vec{g},n}}$ is the measure on $\mathcal{B}(\mathbb{R}^{n+1})$ induced by $Z_{\vec{g},n}$.

(2) *For $m_{m_{Z_{\vec{g},n-1}}}$ a.e. $\vec{\eta} \in \mathbb{R}^n$, we have*

$$GE[F|Z_{\vec{g},n-1}](\vec{\eta}) = \int_{C[0,T]} F(x - P_{t_n,\beta}(x) + \Xi_{t_n}(\vec{\eta}))dw_{\alpha,\beta;\varphi_0}(x). \tag{5}$$

Proof. For $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, we have $x_{\vec{g},n,\beta}(0) = x(0) = P_\beta(x)(0)$ and for $s \in (t_{j-1},t_j]$

$$\begin{aligned}
&x_{\vec{g},n,\beta}(s)\\
&= z_0(x) + \sum_{l=1}^n \langle g_l, \chi_{[0,s]}\rangle_{0,\beta} I_{\alpha,\beta}(g_l)(x)\\
&= x(0) + \sum_{l=1}^n \frac{1}{\beta(t_l)-\beta(t_{l-1})}\int_0^s \chi_{[t_{l-1},t_l]}(u)d\beta(u)\int_0^T \chi_{[t_{l-1},t_l]}(u)dx(u)\\
&= x(0) + \sum_{l=1}^{j-1}[x(t_l)-x(t_{l-1})] + \gamma_j(s)[x(t_j)-x(t_{j-1})] = P_\beta(x)(s)
\end{aligned}$$

by Theorem 2.2 and **(P1)**. We also have $\vec{\xi}_{\vec{g},n,\beta}(0) = \xi_0 = \Xi(n,\xi)(0)$ and for $\vec{\xi} = (\xi_0,\xi_1,\ldots,\xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned}
\vec{\xi}_{\vec{g},n,\beta}(s) &= \xi_0 + \sum_{l=1}^n \xi_l \langle g_l, \chi_{[0,s]}\rangle_{0,\beta}\\
&= \xi_0 + \sum_{l=1}^n \frac{\xi_l}{\sqrt{\beta(t_l)-\beta(t_{l-1})}}\int_0^s \chi_{[t_{l-1},t_l]}(u)d\beta(u)\\
&= \xi_0 + \sum_{l=1}^{j-1}\xi_l\sqrt{\beta(t_l)-\beta(t_{l-1})} + \frac{\beta(s)-\beta(t_{j-1})}{\sqrt{\beta(t_j)-\beta(t_{j-1})}}\xi_j = \Xi(n,\vec{\xi})(s)
\end{aligned}$$

so that $x_{\vec{g},n,\beta} = P_\beta(x)$ and $\vec{\xi}_{\vec{g},n,\beta} = \Xi(n,\vec{\xi})$. By **(P2)** and Theorem 3.4, we have (4).

To prove (5), it suffices to prove by the above process that $\vec{\eta}_{\vec{g},n-1,\beta} = \Xi_{t_n}(\vec{\eta})$ and $x_{\vec{g},n-1,\beta} = P_{t_n,\beta}(x)$ for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$. Indeed, we clearly have $x_{\vec{g},n-1,\beta}(s) = P_{t_n,\beta}(x)(s)$ and $\vec{\eta}_{\vec{g},n-1,\beta}(s) = \Xi_{t_n}(\vec{\eta})(s)$ for $s \in [0, t_{n-1})$. Moreover, for $s \in [t_{n-1}, t_n]$, we have by Theorem 2.2 and **(P1)**

$$\begin{aligned}
&x_{\vec{g},n-1,\beta}(s)\\
&= x(0) + \sum_{l=1}^{n-1} \frac{1}{\beta(t_l) - \beta(t_{l-1})} \int_0^s \chi_{[t_{l-1},t_l]}(u)d\beta(u) \int_0^T \chi_{[t_{l-1},t_l]}(u)dx(u)\\
&= x(0) + \sum_{l=1}^{n-1}[x(t_l) - x(t_{l-1})] = x(t_{n-1}) = P_{t_n,\beta}(x)(s).
\end{aligned}$$

Similarly, we have $\vec{\eta}_{\vec{g},n-1,\beta}(s) = \eta_0 + \sum_{l=1}^{n-1} \eta_l \sqrt{\beta(t_l) - \beta(t_{l-1})} = \Xi(n-1, \vec{\eta})(t_{n-1}) = \Xi_{t_n}(\vec{\eta})(s)$, where $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n-1})$, so that $x_{\vec{g},n-1,\beta} = P_{t_n,\beta}(x)$ and $\vec{\eta}_{\vec{g},n-1,\beta} = \Xi_{t_n}(\vec{\eta})$ as desired. □

We now have Theorem 4 in [4] as a corollary of Theorem 3.4.

Corollary 3.6. *Let $F : C[0,T] \to \mathbb{C}$ be integrable. Then the followings hold:*

(1) *For m_{X_τ} a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$, $GE[F|X_\tau](\vec{\xi})$ is given by the right-hand side of (4) with replacing $\Xi(n, \vec{\xi})$ by $P_\beta(\vec{\xi})$.*
(2) *For m_{Y_τ} a.e. $\vec{\eta} \in \mathbb{R}^n$, $GE[F|Y_\tau](\vec{\eta})$ is given by the right-hand side of (5) with replacing $\Xi_{t_n}(\vec{\eta})$ by $P_{t_n\beta}(\vec{\eta})$.*

Proof. For $\vec{\xi} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, define $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$

$$\phi(\vec{\xi}) = \left(\xi_0, \frac{\xi_1 - \xi_0}{\sqrt{\beta(t_1) - \beta(t_0)}}, \frac{\xi_2 - \xi_1}{\sqrt{\beta(t_2) - \beta(t_1)}}, \ldots, \frac{\xi_n - \xi_{n-1}}{\sqrt{\beta(t_n) - \beta(t_{n-1})}}\right)$$

which is a bijective, bi-continuous function. Since $Z_{\vec{g},n} = \phi \circ X_\tau$, we have for m_{X_τ} a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$

$$\begin{aligned}
GE[F|X_\tau](\vec{\xi}) &= GE[F|Z_{\vec{g},n}](\phi(\vec{\xi}))\\
&= \int_{C[0,T]} F(x - P_\beta(x) + \Xi(n, \phi(\vec{\xi})))dw_{\alpha,\beta;\varphi_0}(x)
\end{aligned}$$

by Lemma 2.4 and (4). We also have $\Xi(n, \phi(\vec{\xi}))(0) = \xi_0 = P_\beta(\vec{\xi})(0)$ and

$$\Xi(n, \phi(\vec{\xi}))(s) = \xi_0 + \sum_{l=1}^{j-1}(\xi_l - \xi_{l-1}) + \gamma_j(s)(\xi_j - \xi_{j-1}) = P_\beta(\vec{\xi})(s)$$

for $s \in (t_{j-1}, t_j]$ so that $\Xi(n, \phi(\vec{\xi})) = P_\beta(\vec{\xi})$, which implies (1). For $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n-1}) \in \mathbb{R}^n$, let

$$\phi_1(\vec{\eta}) = \left(\eta_0, \frac{\eta_1 - \eta_0}{\sqrt{\beta(t_1) - \beta(t_0)}}, \frac{\eta_2 - \eta_1}{\sqrt{\beta(t_2) - \beta(t_1)}}, \ldots, \frac{\eta_{n-1} - \eta_{n-2}}{\sqrt{\beta(t_{n-1}) - \beta(t_{n-2})}}\right).$$

To prove (2), it suffices to show, by the above process, that $\Xi_{t_n}(\phi_1(\vec{\eta})) = P_{t_n,\beta}(\vec{\eta})$ by (5). Indeed, we have $\Xi_{t_n}(\phi_1(\vec{\eta}))(s) = P_{\vec{t}_n,\beta}(\vec{\eta})(s)$ for $s \in [0, t_{n-1})$. Moreover, for $s \in [t_{n-1}, t_n]$, we have

$$\Xi_{t_n}(\phi_1(\vec{\eta}))(s) = \eta_0 + \sum_{l=1}^{n-1}(\eta_l - \eta_{l-1}) = \eta_{n-1} = P_{\vec{t}_n,\beta}(\vec{\eta})(s)$$

so that we also have $\Xi_{t_n}(\phi_1(\vec{\eta})) = P_{t_n,\beta}(\vec{\eta})$ as desired. □

Letting $n = 1$ in (5), we have the following corollary.

Corollary 3.7. *We have for m_{z_0} a.e. $\eta \in \mathbb{R}$*

$$GE[F|z_0](\eta) = \int_{C[0,T]} F(x - x(0) + \eta)\,dw_{\alpha,\beta;\varphi_0}(x).$$

Remark 3.8. By Corollary 3.6 and Theorem 2.3 of [6], we have for m_{Y_τ} a.e. $\vec{\eta} = (\eta_0, \eta_1, \dots, \eta_{n-1}) \in \mathbb{R}^n$

$$\begin{aligned}
&\int_{C[0,T]} F(x - P_{t_n,\beta}(x) + P_{t_n,\beta}(\vec{\eta}))\,dw_{\alpha,\beta;\varphi_0}(x) \\
&= GE[F|Y_\tau](\vec{\eta}) = \int_{\mathbb{R}} \mathcal{W}(\eta_{n-1}, \eta_n) GE[F|X_\tau](\vec{\eta}_n)\,dm_L(\eta_n),
\end{aligned}$$

where $\vec{\eta}_n = (\eta_0, \eta_1, \dots, \eta_{n-1}, \eta_n)$ and

$$\begin{aligned}
&\mathcal{W}(\eta_{n-1}, \eta_n) \\
&= \left[\frac{1}{2\pi[\beta(t_n) - \beta(t_{n-1})]}\right]^{\frac{1}{2}} \exp\left\{-\frac{[\eta_n - \eta_{n-1} - \alpha(t_n) + \alpha(t_{n-1})]^2}{2[\beta(t_n) - \beta(t_{n-1})]}\right\}.
\end{aligned}$$

4. Applications to the time integrals

In this section we apply the simple formulas as given in the previous section, to various functions, in particular, the time integrals on $C[0,T]$.

Example 4.1. For $m \in \mathbb{N}$ and $t \in [0,T]$, let $F_t(x) = [x(t)]^m$ for $x \in C[0,T]$ and suppose that $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$. Then F_t is $w_{\alpha,\beta;\varphi}$-integrable by Theorem 7 of [4]. Now by Theorems 2.2, 2.3, 3.4 and Theorem 7 of [4], we have for $m_{Z_{\vec{e},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$,

$$\begin{aligned}
(6)\quad & GE[F_t|Z_{\vec{e},n}](\vec{\xi}) \\
&= \int_{C[0,T]} [I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},n,\beta}\chi_{[0,t]})(x) + \vec{\xi}_{\vec{e},n,\beta}(t)]^m\,dw_{\alpha,\beta;\varphi_0}(x) \\
&= \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{2^k k!(m-2k)!}[\vec{\xi}_{\vec{e},n,\beta}(t) + I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},n,\beta}\chi_{[0,t]})(\alpha)]^{m-2k}\|\mathcal{P}^\perp_{\vec{e},n,\beta}\chi_{[0,t]}\|^{2k}_{0,\beta},
\end{aligned}$$

where $[\cdot]$ denotes the greatest integer function. In addition, we have

$$I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{g},n,\beta}\chi_{[0,t]})(\alpha) = \alpha(t) - \alpha_{\vec{g},n,\beta}(t) = \alpha(t) - P_\beta(\alpha)(t)$$

by **(P2)** and Corollary 3.5. For $t \in [t_{j-1}, t_j]$, we also have by **(P3)**

$$\begin{aligned}\|\mathcal{P}^{\perp}_{\vec{g},n,\beta}\chi_{[0,t]}\|^2_{0,\beta} &= \beta(t) - \beta(0) - \sum_{k=1}^{j-1} \frac{[\beta(t_k) - \beta(t_{k-1})]^2}{\beta(t_k) - \beta(t_{k-1})} - \frac{[\beta(t) - \beta(t_{j-1})]^2}{\beta(t_j) - \beta(t_{j-1})} \\ &= \Phi_j(t,t).\end{aligned}$$

By Corollary 3.5, we now have for $m_{Z_{\vec{g},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$

$$\begin{aligned}GE[F_t|Z_{\vec{g},n}](\vec{\xi}) = \sum_{k=0}^{[\frac{m}{2}]} &\frac{m!}{2^k k!(m-2k)!}[\Xi(n,\vec{\xi})(t) + \alpha(t) \\ &- P_\beta(\alpha)(t)]^{m-2k}[\Phi_j(t,t)]^k \equiv G_1(t,\vec{\xi}).\end{aligned} \tag{7}$$

Note that we can obtain [6, Theorem 3.6] and [4, Theorem 7] by Corollary 3.6.

Example 4.2. Let the assumptions be as given in Example 4.1. Then for $m_{Z_{\vec{e},n-1}}$ a.e. $\vec{\eta} \in \mathbb{R}^n$, $GE[F_t|Z_{\vec{e},n-1}](\vec{\eta})$ is given by

$$\begin{aligned}&GE[F_t|Z_{\vec{e},n-1}](\vec{\eta}) \\ &= \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{2^k k!(m-2k)!}[\vec{\eta}_{\vec{e},n-1,\beta}(t) + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n-1,\beta}\chi_{[0,t]})(\alpha)]^{m-2k} \\ &\quad \times \|\mathcal{P}^{\perp}_{\vec{e},n-1,\beta}\chi_{[0,t]}\|^{2k}_{0,\beta}\end{aligned}$$

by Theorems 2.2, 2.3 and 3.4 if we use the same process in the proof of Theorem 7 in [4]. In addition, using the same process in Example 4.1 with aid of Corollary 3.5, we can prove that for $t \in [t_{j-1}, t_j]$ $(j = 1, \dots, n-1)$ and for $m_{Z_{\vec{g},n-1}}$ a.e. $\vec{\eta} \in \mathbb{R}^n$, $GE[F_t|Z_{\vec{g},n-1}](\vec{\eta})$ is given by the right-hand side of (7) with replacing $\Xi(n,\vec{\xi})(t)$ and $P_\beta(t)$ by $\Xi_{t_n}(\vec{\eta})(t)$ and $P_{t_n,\beta}(t)$, respectively. Moreover, we have for $t \in [t_{n-1}, t_n]$

$$I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,t]})(\alpha) = \alpha(t) - P_{t_n,\beta}(\alpha)(t) = \alpha(t) - \alpha(t_{n-1})$$

by **(P2)** and Corollary 3.5. We also have by **(P3)**

$$\|\mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,t]}\|^2_{0,\beta} = \beta(t) - \beta(0) - \sum_{k=1}^{n-1} \frac{[\beta(t_k) - \beta(t_{k-1})]^2}{\beta(t_k) - \beta(t_{k-1})} = \beta(t) - \beta(t_{n-1}).$$

Now, we have

$$\begin{aligned}GE[F_t|Z_{\vec{g},n-1}](\vec{\eta}) = \sum_{k=0}^{[\frac{m}{2}]} &\frac{m!}{2^k k!(m-2k)!}[\Xi_{t_n}(\vec{\eta})(t_{n-1}) + \alpha(t) \\ &- \alpha(t_{n-1})]^{m-2k}[\beta(t) - \beta(t_{n-1})]^k \equiv G_2(t,\vec{\eta})\end{aligned} \tag{8}$$

by Corollary 3.5. Using Corollary 3.6, we can also prove that for m_{Y_τ} a.e. $\vec{\eta} = (\eta_0, \eta_1, \dots, \eta_{n-1}) \in \mathbb{R}^n$, $GE[F_t|Y_\tau](\vec{\eta})$ is given by (8) with replacing $\Xi_{t_n}(\vec{\eta})(t_{n-1})$ by η_{n-1}. In particular, letting $n = 1$, we have for m_{z_0} a.e. $\eta \in \mathbb{R}$

$$GE[F_0|z_0](\eta) = \eta^m. \tag{9}$$

Note that, in Theorem 3.7 of [6], $GE[F_t|Y_\tau](\vec{\eta})$ is expressed by

$$GE[F_t|Y_\tau](\vec{\eta}) = \sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{[\frac{m}{2}-k]} \frac{m!}{2^{k+l}k!l!(m-2k-2l)!}[\eta_{n-1}+\alpha(t) - \alpha(t_{n-1})]^{m-2k-2l}[\Phi_n(t,t)]^k\left[\frac{[\beta(t)-\beta(t_{n-1})]^2}{\beta(t_n)-\beta(t_{n-1})}\right]^l \equiv K(\vec{\eta})$$

which coincides with our present result as above by the following calculation: We have by the binomial expansion theorem

$$\begin{aligned} K(\vec{\eta}) &= \sum_{k=0}^{[\frac{m}{2}]}\sum_{l=k}^{[\frac{m}{2}]} \frac{m!}{2^l k!(l-k)!(m-2l)!}[\eta_{n-1}+\alpha(t)-\alpha(t_{n-1})]^{m-2l}[\Phi_n(t,t)]^k \\ &\quad\times\left[\frac{[\beta(t)-\beta(t_{n-1})]^2}{\beta(t_n)-\beta(t_{n-1})}\right]^{l-k} \\ &= \sum_{l=0}^{[\frac{m}{2}]}\sum_{k=0}^{l} \frac{m!}{2^l l!(m-2l)!}\binom{l}{k}[\eta_{n-1}+\alpha(t)-\alpha(t_{n-1})]^{m-2l}[\Phi_n(t,t)]^k \\ &\quad\times\left[\frac{[\beta(t)-\beta(t_{n-1})]^2}{\beta(t_n)-\beta(t_{n-1})}\right]^{l-k} \\ &= \sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l l!(m-2l)!}[\eta_{n-1}+\alpha(t)-\alpha(t_{n-1})]^{m-2l}\bigg[\Phi_n(t,t) \\ &\quad+\frac{[\beta(t)-\beta(t_{n-1})]^2}{\beta(t_n)-\beta(t_{n-1})}\bigg]^l \\ &= \sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l l!(m-2l)!}[\eta_{n-1}+\alpha(t)-\alpha(t_{n-1})]^{m-2l}[\beta(t)-\beta(t_{n-1})]^l \end{aligned}$$

so that the formula in Theorem 3.4 can be used to simply the generalized conditional expectations which are evaluated by the formulas in [4, 6], that is, the formulas in this paper generalize and simplify those in [4, 6].

Now we can obtain the following example by Example 4.1.

Example 4.3. For $m \in \mathbb{N}$, let $F(x) = \int_0^T [x(t)]^m d\lambda(t)$ for $x \in C[0,T]$, where λ is a finite complex measure on the Borel class of $[0,T]$, and suppose that $\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty$. Then for $m_{Z_{\vec{e},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$, $GE[F|Z_{\vec{e},n}](\vec{\xi})$ is given by

$$GE[F|Z_{\vec{e},n}](\vec{\xi}) = \int_0^T GE[F_t|Z_{\vec{e},n}](\vec{\xi})d\lambda(t),$$

where $GE[F_t|Z_{\vec{e},n}](\vec{\xi})$ is expressed by (6). In addition, we have for $m_{Z_{\vec{g},n}}$ a.e. $\vec{\xi}=(\xi_0,\xi_1,\ldots,\xi_n)\in\mathbb{R}^{n+1}$

$$GE[F|Z_{\vec{g},n}](\vec{\xi})=\sum_{j=0}^{n}[\Xi(n,\vec{\xi})(t_j)]^m\lambda(\{t_j\})+\sum_{j=1}^{n}\int_{(t_{j-1},t_j)}G_1(t,\vec{\xi})d\lambda(t), \tag{10}$$

where $G_1(t,\vec{\xi})$ is given by the right-hand side of (7). We note that [4, Theorem 8] can be obtained from (10) by Corollary 3.6. In particular, if $\alpha(t)=P_\beta(\alpha)(t)$ and $\lambda(t)=\beta(t)$ for $t\in[0,T]$, then we have by Lemma 2.4, Corollary 3.6 and Corollary 3.9 of [6]

$$\begin{aligned}
&GE[F|Z_{\vec{g},n}](\vec{\xi})\\
&=GE[F|X_\tau](\phi^{-1}(\vec{\xi}))\\
&=\sum_{j=1}^{n}\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k}\frac{m!(l+k)![\beta(t_j)-\beta(t_{j-1})]^{\frac{l}{2}+k+1}[\Xi(n,\vec{\xi})(t_{j-1})]^{m-2k-l}\xi_j^l}{2^k l!(m-2k-l)!(l+2k+1)!}\\
&\equiv\Psi_n(\vec{\xi}).
\end{aligned}$$

Example 4.4. Let the assumptions be as given in Example 4.3. Then for $m_{Z_{\vec{e},n-1}}$ a.e. $\vec{\eta}\in\mathbb{R}^n$, $GE[F|Z_{\vec{e},n-1}](\vec{\eta})$ is given by

$$GE[F|Z_{\vec{e},n-1}](\vec{\eta})=\int_0^T GE[F_t|Z_{\vec{e},n-1}](\vec{\eta})d\lambda(t)$$

from Example 4.2. In addition, for $m_{Z_{\vec{g},n-1}}$ a.e. $\vec{\eta}=(\eta_0,\eta_1,\ldots,\eta_{n-1})\in\mathbb{R}^n$, we have by Example 4.2

$$\begin{aligned}
GE[F|Z_{\vec{g},n-1}](\vec{\eta})=&\sum_{j=0}^{n-1}[\Xi_{t_n}(\vec{\eta})(t_j)]^m\lambda(\{t_j\})+\sum_{j=1}^{n-1}\int_{(t_{j-1},t_j)}G_1(t,\vec{\eta})d\lambda(t)\\
&+\int_{(t_{n-1},T]}G_2(t,\vec{\eta})d\lambda(t),
\end{aligned}$$

where G_1 and G_2 are given by (7) and (8), respectively. We note that [6, Theorem 3.8] can be obtained from the above equality by Corollary 3.6. In particular, letting $n=1$, we have for m_{z_0} a.e. $\eta\in\mathbb{R}$

$$GE[F|z_0](\eta)=\sum_{k=0}^{[\frac{m}{2}]}\frac{m!}{2^k k!(m-2k)!}\int_0^T[\eta+\alpha(t)-\alpha(0)]^{m-2k}[\beta(t)-\beta(0)]^k d\lambda(t)$$

by (9). Moreover, if the support of λ is contained in $\{t_0,t_1,\ldots,t_n\}$, then $GE[F|Z_{\vec{g},n-1}](\vec{\eta})$ is reduced to

$$GE[F|Z_{\vec{g},n-1}](\vec{\eta})=\sum_{j=0}^{n-1}\lambda(\{t_j\})[\Xi_{t_n}(\vec{\eta})(t_j)]^m+\lambda(\{t_n\})G_2(t_n,\vec{\eta}).$$

Note that the final result of [6, Theorem 3.8] can be also obtained from the above equality by Corollary 3.6. Furthermore, if $\alpha(t) = P_{t_n,\beta}(\alpha)(t)$ and $\lambda(t) = \beta(t)$ for $t \in [0,T]$, then we have by (8) and Corollary 3.9 of [6]

$$GE[F|Z_{\vec{g},n-1}](\vec{\eta}) = \Psi_{n-1}(\vec{\eta}) + \sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l(l+1)!(m-2l)!} \times [\Xi_{t_n}(\vec{\eta})(t_{n-1})]^{m-2l}[\beta(t_n)-\beta(t_{n-1})]^{l+1}. \tag{11}$$

By Lemma 2.4, Corollary 3.6 and (11), we have for m_{Y_τ} a.e. $\vec{\eta} \in \mathbb{R}^n$

$$\begin{aligned} &GE[F|Y_\tau](\vec{\eta}) \\ &= GE[F|Z_{\vec{g},n-1}](\phi_1(\vec{\eta})) \\ &= \sum_{j=1}^{n-1}\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{m-2k} \frac{m!(l+k)![\beta(t_j)-\beta(t_{j-1})]^{k+1}\eta_{j-1}^{m-2k-l}(\eta_j-\eta_{j-1})^l}{2^k l!(m-2k-l)!(l+2k+1)!} \\ &\quad + \sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l(l+1)!(m-2l)!}\eta_{n-1}^{m-2l}[\beta(t_n)-\beta(t_{n-1})]^{l+1}. \end{aligned} \tag{12}$$

Note that in Corollary 3.9 of [6], the last term of (12) is expressed by

$$\sum_{k=0}^{[\frac{m}{2}]}\sum_{l=0}^{[\frac{m}{2}-k]} \frac{m!(2l+k)!\eta_{n-1}^{m-2k-2l}[\beta(t_n)-\beta(t_{n-1})]^{l+k+1}}{2^{l+k}l!(m-2k-2l)!(2l+2k+1)!} \equiv K_1(\eta_{n-1}).$$

Indeed, we have by Chu Shih-Chieh's identity [2]

$$\begin{aligned} K_1(\eta_{n-1}) &= \sum_{k=0}^{[\frac{m}{2}]}\sum_{l=k}^{[\frac{m}{2}]} \frac{m!(2l-k)!\eta_{n-1}^{m-2l}[\beta(t_n)-\beta(t_{n-1})]^{l+1}}{2^l(l-k)!(m-2l)!(2l+1)!} \\ &= \sum_{l=0}^{[\frac{m}{2}]}\sum_{k=0}^{l} \frac{m!(2l-k)!\eta_{n-1}^{m-2l}[\beta(t_n)-\beta(t_{n-1})]^{l+1}}{2^l(l-k)!(m-2l)!(2l+1)!} \\ &= \sum_{l=0}^{[\frac{m}{2}]} \frac{m!l!\eta_{n-1}^{m-2l}[\beta(t_n)-\beta(t_{n-1})]^{l+1}}{2^l(m-2l)!(2l+1)!}\sum_{k=0}^{l}\binom{2l-k}{l} \\ &= \sum_{l=0}^{[\frac{m}{2}]} \frac{m!l!\eta_{n-1}^{m-2l}[\beta(t_n)-\beta(t_{n-1})]^{l+1}}{2^l(m-2l)!(2l+1)!}\binom{2l+1}{l+1} \\ &= \sum_{l=0}^{[\frac{m}{2}]} \frac{m!}{2^l(l+1)!(m-2l)!}\eta_{n-1}^{m-2l}[\beta(t_n)-\beta(t_{n-1})]^{l+1} \end{aligned}$$

which coincides with the last term of (12).

Example 4.5. Let $s_1, s_2 \in [0,T]$ and let $G(s_1,s_2,x) = x(s_1)x(s_2)$ for $x \in C[0,T]$. Then $G(s_1,s_2,\cdot)$ is $w_{\alpha,\beta;\varphi}$-integrable by [4, Theorem 5] so that we have for $m_{Z_{\vec{e},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$

$$\begin{aligned}
&GE[G(s_1, s_2, \cdot)|Z_{\vec{e},n}](\vec{\xi})\\
&= \int_{C[0,T]} [I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s_1]})(x) + \vec{\xi}_{\vec{e},n,\beta}(s_1)]\\
&\quad \times [I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s_2]})(x) + \vec{\xi}_{\vec{e},n,\beta}(s_2)] dw_{\alpha,\beta;\varphi_0}(x)\\
&= \langle \mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s_2]}\rangle_{0,\beta}\\
&\quad + [\vec{\xi}_{\vec{e},n,\beta}(s_1) + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s_1]})(\alpha)][\vec{\xi}_{\vec{e},n,\beta}(s_2) + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s_2]})(\alpha)]
\end{aligned}$$

by Theorems 2.3 and 3.4.

Lemma 4.6. *Let $s_1 \in [t_{j-1}, t_j]$, $s_2 \in [t_{k-1}, t_k]$ for $1 \le j \le k \le n$.*

(1) *If $j \neq k$, then we have $\langle \mathcal{P}^{\perp}_{\vec{g},n,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},n,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} = 0$.*

(2) *If $j = k$ and $s_1 \le s_2$, then $\langle \mathcal{P}^{\perp}_{\vec{g},n,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},n,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} = \Phi_j(s_2, s_1)$.*

Proof. Suppose that $j \neq k$. Now we have $s_1 \le s_2$ and we have by **(P3)**

$$\begin{aligned}
\langle \mathcal{P}^{\perp}_{\vec{g},n,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},n,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} &= \beta(s_1) - \beta(0) - \sum_{l=1}^{j-1} \frac{[\beta(t_l) - \beta(t_{l-1})]^2}{\beta(t_l) - \beta(t_{l-1})}\\
&\quad - \frac{[\beta(s_1) - \beta(t_{j-1})][\beta(t_j) - \beta(t_{j-1})]}{\beta(t_j) - \beta(t_{j-1})} = 0
\end{aligned}$$

which proves (1). If $j = k$ and $s_1 \le s_2$, then we have by **(P3)**

$$\begin{aligned}
\langle \mathcal{P}^{\perp}_{\vec{g},n,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},n,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} &= \beta(s_1) - \beta(0) - \sum_{k=1}^{j-1} \frac{[\beta(t_k) - \beta(t_{k-1})]^2}{\beta(t_k) - \beta(t_{k-1})}\\
&\quad - \frac{[\beta(s_1) - \beta(t_{j-1})][\beta(s_2) - \beta(t_{j-1})]}{\beta(t_j) - \beta(t_{j-1})}\\
&= \Phi_j(s_2, s_1)
\end{aligned}$$

which proves (2). □

Lemma 4.7. *Under the assumptions as in Lemma 4.6, we have the followings*:

(1) *If $j \neq k$, then $\langle \mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} = 0$.*

(2) *If $1 \le j = k \le n-1$ and $s_1 \le s_2$, then*

$$\langle \mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} = \Phi_j(s_2, s_1).$$

(3) *If $j = k = n$ and $s_1 \le s_2$, then $\langle \mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} = \beta(s_1) - \beta(t_{n-1})$.*

Proof. The proofs of (1) and (2) are similar to the proofs of (1) and (2), respectively, in Lemma 4.6. To complete the proof, it suffices to prove (3).

Indeed we have

$$\begin{aligned}\langle \mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{g},n-1,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} &= \beta(s_1) - \beta(0) - \sum_{l=1}^{n-1} \frac{[\beta(t_l)-\beta(t_{l-1})]^2}{\beta(t_l)-\beta(t_{l-1})} \\ &= \beta(s_1) - \beta(t_{n-1}).\end{aligned}$$

which completes the proof. □

Example 4.8. Let $s_1 \in [t_{j-1}, t_j]$, $s_2 \in [t_{k-1}, t_k]$ for $1 \le j \le k \le n$ and let $G(s_1, s_2, x) = x(s_1)x(s_2)$ for $x \in C[0,T]$. Then, by Corollary 3.5, Example 4.5 and Lemma 4.6, we have the followings:

(1) If $j \ne k$, then for $m_{Z_{\vec{g},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$, we have

$$\begin{aligned}&GE[G(s_1,s_2,\cdot)|Z_{\vec{g},n}](\vec{\xi}). \\ &= [\Xi(n,\vec{\xi})(s_1) + \alpha(s_1) - P_\beta(\alpha)(s_1)][\Xi(n,\vec{\xi})(s_2) + \alpha(s_2) - P_\beta(\alpha)(s_2)].\end{aligned}$$

(2) If $j = k$ and $s_1 \le s_2$, then we have for $m_{Z_{\vec{g},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$

$$\begin{aligned}&GE[G(s_1,s_2,\cdot)|Z_{\vec{g},n}](\vec{\xi}) \\ &= [\Xi(n,\vec{\xi})(s_1) + \alpha(s_1) - P_\beta(\alpha)(s_1)][\Xi(n,\vec{\xi})(s_2) + \alpha(s_2) - P_\beta(\alpha)(s_2)] \\ &\quad + \Phi_j(s_2, s_1).\end{aligned}$$

Example 4.9. Let the assumptions be as in Example 4.8. Then, by Corollary 3.5, Examples 4.2, 4.5 and Lemma 4.7, we have the followings:

(1) If $1 \le j < k \le n-1$, then for $m_{Z_{\vec{g},n-1}}$ a.e. $\vec{\eta} \in \mathbb{R}^n$, we have

$$\begin{aligned}&GE[G(s_1,s_2,\cdot)|Z_{\vec{g},n-1}](\vec{\eta}) \\ &= [\Xi_{t_n}(\vec{\eta})(s_1) + \alpha(s_1) - P_{t_n,\beta}(\alpha)(s_1)][\Xi_{t_n}(\vec{\eta})(s_2) + \alpha(s_2) - P_{t_n,\beta}(\alpha)(s_2)].\end{aligned}$$

(2) If $1 \le j = k \le n-1$ and $s_1 \le s_2$, then for $m_{Z_{\vec{g},n-1}}$ a.e. $\vec{\eta} \in \mathbb{R}^n$,

$$\begin{aligned}&GE[G(s_1,s_2,\cdot)|Z_{\vec{g},n-1}](\vec{\eta}) \\ &= [\Xi_{t_n}(\vec{\eta})(s_1) + \alpha(s_1) - P_{t_n,\beta}(\alpha)(s_1)][\Xi_{t_n}(\vec{\eta})(s_2) + \alpha(s_2) - P_{t_n,\beta}(\alpha)(s_2)] \\ &\quad + \Phi_j(s_2, s_1).\end{aligned}$$

(3) If $1 \le j \le n-1$ and $k = n$, then for $m_{Z_{\vec{g},n-1}}$ a.e. $\vec{\eta} \in \mathbb{R}^n$, we have

$$\begin{aligned}&GE[G(s_1,s_2,\cdot)|Z_{\vec{g},n-1}](\vec{\eta}) \\ &= [\Xi_{t_n}(\vec{\eta})(s_1) + \alpha(s_1) - P_{t_n,\beta}(\alpha)(s_1)][\Xi_{t_n}(\vec{\eta})(t_{n-1}) + \alpha(s_2) - \alpha(t_{n-1})].\end{aligned}$$

(4) If $j = k = n$ and $s_1 \le s_2$, then for $m_{Z_{\vec{g},n-1}}$ a.e. $\vec{\eta} \in \mathbb{R}^n$, we have

$$\begin{aligned}&GE[G(s_1,s_2,\cdot)|Z_{\vec{g},n-1}](\vec{\eta}) \\ &= [\Xi_{t_n}(\vec{\eta})(t_{n-1}) + \alpha(s_1) - \alpha(t_{n-1})][\Xi_{t_n}(\vec{\eta})(t_{n-1}) + \alpha(s_2) - \alpha(t_{n-1})] \\ &\quad + \beta(s_1) - \beta(t_{n-1}).\end{aligned}$$

In particular, we have for m_{z_0} a.e. $\eta \in \mathbb{R}$

$$GE[G(s_1,s_2,\cdot)|z_0](\eta) = [\eta + \alpha(s_1) - \alpha(0)][\eta + \alpha(s_2) - \alpha(0)] + \beta(s_1) - \beta(0).$$

Remark 4.10. Note that [4, Theorem 5] and [6, Theorem 3.2] can be also obtained from Corollary 3.6.

We now have the following theorem from [6, Theorem 3.3], Theorem 3.4 and Example 4.5.

Theorem 4.11. *For $x \in C[0,T]$, let $G_3(x) = [\int_0^T x(t)d\lambda(t)]^2$, where λ is a finite complex measure on the Borel class of $[0,T]$. Suppose that $\int_0^T [\alpha(t)]^2 d|\lambda|(t) < \infty$ and $\int_{\mathbb{R}} u^2 d\varphi(u) < \infty$. Then for $m_{Z_{\vec{e},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$, we have*

$$GE[G_3|Z_{\vec{e},n}](\vec{\xi}) = \int_0^T \int_0^T \langle \mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s_1]}, \mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s_2]}\rangle_{0,\beta} d\lambda^2(s_1,s_2) + \left[\int_0^T [\vec{\xi}_{\vec{e},n,\beta}(s) + I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,s]})(\alpha)]d\lambda(s)\right]^2.$$

Using the same method as used in the proofs in Theorems 3.3 and 3.5 of [6] with aid of Lemmas 4.6 and 4.7, Examples 4.8 and 4.9, and Theorem 4.11, we can prove the following corollary.

Corollary 4.12. *Let the assumptions be as given in Theorem 4.11.*

(1) *For $m_{Z_{\vec{g},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$, we have*

$$GE[G_3|Z_{\vec{g},n}](\vec{\xi}) = \int_0^T \int_0^T \Lambda(s_1 \vee s_2, s_1 \wedge s_2) d\lambda^2(s_1,s_2) + \left[\int_0^T [\Xi(n,\vec{\xi})(s) + \alpha(s) - P_\beta(\alpha)(s)]d\lambda(s)\right]^2,$$

where $\Lambda(s,t) = \sum_{j=1}^n \chi_{[t_{j-1},t_j]^2}(s,t)\Phi_j(s,t)$ for $(s,t) \in [0,T]^2$, $s_1 \vee s_2 = \max\{s_1,s_2\}$ and $s_1 \wedge s_2 = \min\{s_1,s_2\}$.

(2) *For $m_{Z_{\vec{g},n-1}}$ a.e. $\vec{\eta} = (\eta_0, \eta_1, \ldots, \eta_{n-1}) \in \mathbb{R}^n$, we have*

$$\begin{aligned} &GE[G_3|Z_{\vec{g},n-1}](\vec{\eta}) \\ &= \int_0^{t_{n-1}} \int_0^{t_{n-1}} \Lambda(s_1 \vee s_2, s_1 \wedge s_2) d\lambda^2(s_1,s_2) + \int_{t_{n-1}}^T \int_{t_{n-1}}^T [\beta(s_1 \wedge s_2) \\ &\quad - \beta(t_{n-1})] d\lambda^2(s_1,s_2) + \left[\int_0^{t_{n-1}} [\Xi_{t_n}(\vec{\eta})(s) + \alpha(s) - P_\beta(\alpha)(s)]d\lambda(s) \right. \\ &\quad \left. + \int_{(t_{n-1},T]} [\alpha(t) - \alpha(t_{n-1}) + \Xi_{t_n}(\vec{\eta})(t_{n-1})]d\lambda(t)\right]^2. \end{aligned}$$

In particular, for m_{z_0} a.e. $\eta \in \mathbb{R}$ we have

$$\begin{aligned} &GE[G_3|z_0](\eta) \\ &= \int_0^T \int_0^T [\beta(s_1 \wedge s_2) - \beta(0)] d\lambda^2(s_1,s_2) + \left[\int_0^T [\alpha(t) - \alpha(0) + \eta] d\lambda(t)\right]^2. \end{aligned}$$

Remark 4.13. Theorems 3.3 and 3.5 of [6] can be also obtained from Corollaries 3.6 and 4.12 so that Theorem 4.11 extends the results of [6].

5. More applications of the simple formula

In this section we apply the simple formulas as given in the previous section, to the cylinder type functions and the functions in a Banach algebra [5] which are of significant in Feynman integration theory and quantum mechanics. For these purposes, we need the following lemma.

Lemma 5.1. *For $f \in L^2_{\alpha,\beta}[0,T]$, we have for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$*

$$I_{\alpha,\beta}(f)(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,\cdot]})(x)) = I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}f)(x).$$

Proof. Let $\{\phi_l\}_{l=1}^{\infty}$ be a sequence in $S[0,T]$ such that $\lim_{l\to\infty}\|\phi_l - f\|_{\alpha,\beta} = 0$. Since the two norms $\|\cdot\|_{0,\beta}$ and $\|\cdot\|_{\alpha,\beta}$ are equivalent on $L^2_{\alpha,\beta}[0,T]$ and $\mathcal{P}^{\perp}_{\vec{e},n,\beta}$ is bounded with the norm $\|\cdot\|_{0,\beta}$, we have $\lim_{l\to\infty}\|\mathcal{P}^{\perp}_{\vec{e},n,\beta}\phi_l - \mathcal{P}^{\perp}_{\vec{e},n,\beta}f\|_{\alpha,\beta} = 0$. By Corollary 3.11 of [3], we have $\lim_{l\to\infty} I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\phi_l) = I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}f)$ in $L^2(C[0,T])$ so that without loss of generality, we have for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$

$$I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}f)(x) = \lim_{l\to\infty} I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\phi_l)(x).$$

For each $l \in \mathbb{N}$, let ϕ_l have the form $\phi_l(t) = \sum_{k=1}^{r_l} c_{lk}\chi_{I_{lk}}(t)$ for $t \in [0,T]$, where $r_l \in \mathbb{N}$, $c_{lk} \in \mathbb{R}$ and the intervals $I_{lk} = (t_{lk-1}, t_{lk}] \subseteq [0,T]$ are mutually disjoint. Now, we have by **(P2)**

$$\begin{aligned}
&I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}f)(x) \\
&= \lim_{l\to\infty}\sum_{k=1}^{r_l} c_{lk} I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{(t_{lk-1},t_{lk}]})(x) \\
&= \lim_{l\to\infty}\sum_{k=1}^{r_l} c_{lk}[I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,t_{lk}]})(x) - I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,t_{lk-1}]})(x)] \\
&= \lim_{l\to\infty} I_{\alpha,\beta}(\phi_l)(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,\cdot]})(x)) \\
&= I_{\alpha,\beta}(f)(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,\cdot]})(x))
\end{aligned}$$

which is the desired result. $\square$

5.1. The cylinder type functions

Let $\{v_1, v_2, \ldots, v_r\}$ be an orthonormal subset of $L^2_{\alpha,\beta}[0,T]$. For convenience, let

$$I_{\alpha,\beta}(\vec{v})(x) = (I_{\alpha,\beta}(v_1)(x), \ldots, I_{\alpha,\beta}(v_r)(x)) \text{ for } x \in C[0,T].$$

Let F_r be the cylinder type function of the form given by

$$F_r(x) = f_r(I_{\alpha,\beta}(\vec{v})(x))$$

for $w_{\alpha,\beta;\varphi}$ a.e. $x \in C[0,T]$, where $f_r : \mathbb{R}^r \to \mathbb{C}$ is Borel measurable. Assume that F_r is integrable on $C[0,T]$. Then we have for $m_{Z_{\vec{e},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$

$$\begin{aligned} GE[F_r|Z_{\vec{e},n}](\vec{\xi}) &= \int_{C[0,T]} F_r(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,\cdot]})(x) + \vec{\xi}_{\vec{e},n,\beta}) dw_{\alpha,\beta;\varphi_0}(x) \\ &= \int_{C[0,T]} f_r(I_{\alpha,\beta}(v_1)(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,\cdot]})(x) + \vec{\xi}_{\vec{e},n,\beta}), \dots, \\ &\qquad I_{\alpha,\beta}(v_r)(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\chi_{[0,\cdot]})(x) + \vec{\xi}_{\vec{e},n,\beta})) dw_{\alpha,\beta;\varphi_0}(x) \\ &= \int_{C[0,T]} f_r(I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\vec{v})(x) + I_{\alpha,\beta}(\vec{v})(\vec{\xi}_{\vec{e},n,\beta})) dw_{\alpha,\beta;\varphi_0}(x) \end{aligned}$$

by Theorem 3.4 and Lemma 5.1.

(1) Suppose that $\mathcal{P}^{\perp}_{\vec{e},n,\beta}v_j = 0$ for $j = 1, \dots, r$, that is, $v_j \in V_n$ for $j = 1, \dots, r$. In this case, we have for $m_{Z_{\vec{e},n}}$ a.e. $\vec{\xi} \in \mathbb{R}^{n+1}$

$$GE[F_r|Z_{\vec{e},n}](\vec{\xi}) = \int_{C[0,T]} f_r(I_{\alpha,\beta}(\vec{v})(\vec{\xi}_{\vec{e},n,\beta})) dw_{\alpha,\beta;\varphi_0}(x) = F_r(\vec{\xi}_{\vec{e},n,\beta}).$$

(2) Suppose that $\mathcal{P}^{\perp}_{\vec{e},n,\beta}v_j \neq 0$ for some j, that is, $v_j \notin V_n$ for some j. Let $\{w_1, \dots, w_{r_1}\}$ be a maximal independent set obtained from $\{\mathcal{P}^{\perp}_{\vec{e},n,\beta}v_l : l = 1, \dots, r\}$. Now, for $j = 1, \dots, r$, let $\mathcal{P}^{\perp}_{\vec{e},n,\beta}v_j = \sum_{l=1}^{r_1} \alpha_{jl} w_l$ be the linear combination of the w_ls and let $A_{\vec{e}} = [\alpha_{lj}]_{r_1 \times r}$ be the transpose of the coefficient matrix of the combinations. Then we have by Theorem 3.6 of [3]

$$\begin{aligned} GE[F_r|Z_{\vec{e},n}](\vec{\xi}) &= \int_{C[0,T]} f_r\bigg(\bigg(\sum_{l=1}^{r_1} \alpha_{1l} I_{\alpha,\beta}(w_l)(x), \dots, \sum_{l=1}^{r_1} \alpha_{rl} I_{\alpha,\beta}(w_l)(x)\bigg) \\ &\qquad + I_{\alpha,\beta}(\vec{v})(\vec{\xi}_{\vec{e},n,\beta})\bigg) dw_{\alpha,\beta;\varphi_0}(x) \\ &= \left[\frac{1}{(2\pi)^{r_1}|M_{\vec{e}}|}\right]^{\frac{1}{2}} \int_{\mathbb{R}^{r_1}} f_r(\vec{u}A_{\vec{e}} + I_{\alpha,\beta}(\vec{v})(\vec{\xi}_{\vec{e},n,\beta})) \exp\bigg\{-\frac{1}{2}\langle M_{\vec{e}}^{-1}[\vec{u} \\ &\qquad - I_{\alpha,\beta}(\vec{w})(\alpha)], \vec{u} - I_{\alpha,\beta}(\vec{w})(\alpha)\rangle_{r_1}\bigg\} dm_L^{r_1}(\vec{u}), \end{aligned}$$

where $M_{\vec{e}} = [\langle w_i, w_j\rangle_{0,\beta}]_{r_1 \times r_1}$, $I_{\alpha,\beta}(\vec{w})(\alpha) = (I_{\alpha,\beta}(w_1)(\alpha), \dots, I_{\alpha,\beta}(w_{r_1})(\alpha))$ and $\langle\cdot,\cdot\rangle_{r_1}$ denotes the dot product on $\mathbb{R}^{r_1}$. In particular, if $\{\mathcal{P}^{\perp}_{\vec{e},n,\beta}v_j : j = 1, \dots, r\}$ itself is an orthonormal set in $L^2_{0,\beta}[0,T]$, then we have

$$\begin{aligned} GE[F_r|Z_{\vec{e},n}](\vec{\xi}) &= \left(\frac{1}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f_r(\vec{u}) \exp\bigg\{-\frac{1}{2}\|\vec{u} - I_{\alpha,\beta}(\vec{v})(\vec{\xi}_{\vec{e},n,\beta}) \\ &\qquad - I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}\vec{v})(\alpha)\|_r^2\bigg\} dm_L^r(\vec{u}). \end{aligned}$$

(3) Replacing e_j by g_j, we can obtain $GE[F_r|Z_{\vec{g},n}](\vec{\xi})$ and $GE[F_r|X_\tau](\vec{\xi})$ by Corollaries 3.5 and 3.6, where $\vec{\xi}_{\vec{e},n,\beta}$ is replaced by $\Xi(n,\vec{\xi})$ and $P_\beta(\vec{\xi})$, respectively. We can also obtain $GE[F_r|Z_{\vec{g},n-1}](\vec{\eta})$ and $GE[F_r|Y_\tau](\vec{\eta})$ for $\vec{\eta}\in\mathbb{R}^n$ by the same corollaries, where $\vec{\xi}_{\vec{e},n,\beta}$ is replaced by $\Xi_{t_n}(\vec{\eta})$ and $P_{t_n,\beta}(\vec{\eta})$, respectively. In each case, $A_{\vec{e}}$, $M_{\vec{e}}$ and $\vec{w}$ depend on the g_js. In aprticular, we have for m_{z_0} a.e. $\eta\in\mathbb{R}$

$$GE[F|z_0](\eta)=\left[\frac{1}{(2\pi)^r|M_1|}\right]^{\frac12}\int_{\mathbb{R}^r} f(\vec{u})\exp\bigg\{-\frac12\langle M_1^{-1}[\vec{u}\\ -I_{\alpha,\beta}(\vec{v})(\alpha)],\vec{u}-I_{\alpha,\beta}(\vec{v})(\alpha)\rangle_r\bigg\}dm_L^r(\vec{u})$$

by Corollary 3.7 and Theorem 3.6 of [3], where $M_1=[\langle v_i,v_j\rangle_{0,\beta}]_{r\times r}$.

5.2. The functions in a Banach algebra

In this subsection we give an additional condition that $|\alpha|'(t)+\beta'(t)>0$ for $t\in[0,T]$. Let $\mathcal{M}_{\alpha,\beta}$ be the class of complex measures of finite variations on the Borel class $\mathcal{B}(L^2_{\alpha,\beta}[0,T])$ of $L^2_{\alpha,\beta}[0,T]$. If $\mu\in\mathcal{M}_{\alpha,\beta}$, then we set $\|\mu\|=\mathrm{var}\mu$, the total variation of μ over $L^2_{\alpha,\beta}[0,T]$. Now let $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ be the space of functions of the form

$$F(x)=\int_{L^2_{\alpha,\beta}[0,T]}\exp\{iI_{\alpha,\beta}(f)(x)\}d\mu(f) \tag{13}$$

for all $x\in C[0,T]$ for which the integral exists, where $\mu\in\mathcal{M}_{\alpha,\beta}$. Here we take $\|F\|=\inf\{\|\mu\|\}$, where the infimum is taken for all μ's so that F and μ are related by (13). We note that $\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ is a Banach algebra [5].

For $F\in\bar{\mathcal{S}}_{\alpha,\beta;\varphi}$ given by (13), we have for $m_{Z_{\vec{e},n}}$ a.e. $\vec{\xi}\in\mathbb{R}^{n+1}$

$$\begin{aligned}
&GE[F|Z_{\vec{e},n}](\vec{\xi})\\
&=\int_{C[0,T]}F(I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},n,\beta}\chi_{[0,\cdot]})(x)+\vec{\xi}_{\vec{e},n,\beta})dw_{\alpha,\beta;\varphi_0}(x)\\
&=\int_{L^2_{\alpha,\beta}[0,T]}\int_{C[0,T]}\exp\{i[I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},n,\beta}f)(x)+I_{\alpha,\beta}(f)(\vec{\xi}_{\vec{e},n,\beta})]\}dw_{\alpha,\beta;\varphi_0}(x)\\
&\quad d\mu(f)\\
&=\int_{L^2_{\alpha,\beta}[0,T]}\exp\bigg\{-\frac12\|\mathcal{P}^\perp_{\vec{e},n,\beta}f\|^2_{0,\beta}+i[I_{\alpha,\beta}(\mathcal{P}^\perp_{\vec{e},n,\beta}f)(\alpha)+I_{\alpha,\beta}(f)(\vec{\xi}_{\vec{e},n,\beta})]\bigg\}\\
&\quad d\mu(f)
\end{aligned}$$

by Theorem 2.3 and Lemma 5.1. Replacing e_j by g_j, we can obtain $GE[F|Z_{\vec{g},n}]$ $(\vec{\xi})$ and $GE[F|X_\tau](\vec{\xi})$ by Corollaries 3.5 and 3.6, where $\vec{\xi}_{\vec{e},n,\beta}$ is replaced by $\Xi(n,\vec{\xi})$ and $P_\beta(\vec{\xi})$, respectively. We can also obtain $GE[F_r|Z_{\vec{g},n-1}](\vec{\eta})$ and $GE[F_r|Y_\tau](\vec{\eta})$ for $\vec{\eta}\in\mathbb{R}^n$ by the same corollaries, where $\vec{\xi}_{\vec{e},n,\beta}$ is replaced by

$\Xi_{t_n}(\vec{\eta})$ and $P_{t_n,\beta}(\vec{\eta})$, respectively. In each case, $\mathcal{P}^{\perp}_{\vec{e},n,\beta}$ depends on the g_js. In particular, we have for m_{z_0} a.e. $\eta \in \mathbb{R}$

$$GE[F|z_0](\eta) = \int_{L^2_{\alpha,\beta}[0,T]} \exp\bigg\{-\frac{1}{2}\|f\|^2_{0,\beta} + iI_{\alpha,\beta}(f)(\alpha)\bigg\}d\mu(f)$$

by Corollary 3.7.

Remark 5.2. (1) Note that for $f \in L^2_{\alpha,\beta}[0,T]$ and $\vec{\xi} = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, we have the following more detailed expressions:

(a) $I_{\alpha,\beta}(f)(\vec{\xi}_{\vec{e},n,\beta}) = \sum_{j=1}^n \xi_j \langle f, e_j\rangle_{0,\beta}$,
(b) $I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{e},n,\beta}f)(\alpha) = I_{\alpha,\beta}(f)(\alpha) - \sum_{j=1}^n \langle f, e_j\rangle_{0,\beta} I_{\alpha,\beta}(e_j)(\alpha)$ and
(c) $\|\mathcal{P}^{\perp}_{\vec{e},n,\beta}f\|^2_{0,\beta} = \|f\|^2_{0,\beta} - \sum_{j=1}^n \langle f, e_j\rangle^2_{0,\beta}$

by Theorem 2.2 and the mean value theorem for the Riemann-Stieltjes integral.

(2) For m_{Y_τ} a.e. $\vec{\eta} \in \mathbb{R}^n$, we have by (2) of Corollary 3.6

$$\begin{aligned} GE[F|Y_\tau](\vec{\eta}) = \int_{L^2_{\alpha,\beta}[0,T]} &\exp\bigg\{-\frac{1}{2}\|\mathcal{P}^{\perp}_{\vec{g},n-1,\beta}f\|^2_{0,\beta} \\ &+ i[I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{g},n-1,\beta}f)(\alpha) + I_{\alpha,\beta}(f)(P_{t_n,\beta}(\vec{\eta}))]\bigg\}d\mu(f). \end{aligned}$$

In view of Remark 3.8, the above equality can be also obtained by Theorem 2.3 of [6] using the following long calculations: For $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_{n-1}) \in \mathbb{R}^n$ and $\vec{\eta}_n = (\eta_0, \eta_1, \ldots, \eta_{n-1}, \eta_n) \in \mathbb{R}^{n+1}$, we have by formulas as above in (1)

$$\begin{aligned} &\exp\bigg\{-\frac{1}{2}\|\mathcal{P}^{\perp}_{\vec{g},n,\beta}f\|^2_{0,\beta} + i[I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{g},n,\beta}f)(\alpha) + I_{\alpha,\beta}(f)(P_\beta(\vec{\eta}_n))]\bigg\} \\ = \ &\exp\bigg\{-\frac{1}{2}\bigg[\|f\|^2_{0,\beta} - \sum_{j=1}^n \langle f, g_j\rangle^2_{0,\beta}\bigg] + i\bigg[I_{\alpha,\beta}(f)(\alpha) - \sum_{j=1}^n \langle f, g_j\rangle_{0,\beta} \\ &\times I_{\alpha,\beta}(g_j)(\alpha) + I_{\alpha,\beta}(P_{t_n,\beta}(\vec{\eta})) + \langle f, g_n\rangle_{0,\beta}\frac{\eta_n - \eta_{n-1}}{\sqrt{\beta(t_n) - \beta(t_{n-1})}}\bigg]\bigg\} \end{aligned}$$

and by the Fourier-transform of normal random variable, we also have

$$\begin{aligned} &\int_{\mathbb{R}} \mathcal{W}(\eta_{n-1}, \eta_n)\exp\bigg\{i\langle f, g_n\rangle_{0,\beta}\frac{\eta_n - \eta_{n-1}}{\sqrt{\beta(t_n) - \beta(t_{n-1})}}\bigg\}dm_L(\eta_n) \\ = \ &\exp\bigg\{-\frac{1}{2}\langle f, g_n\rangle^2_{0,\beta} + \langle f, g_n\rangle_{0,\beta} I_{\alpha,\beta}(g_n)(\alpha)\bigg\}. \end{aligned}$$

Hence we have by Theorem 2.3 of [6]

$$GE[F|Y_\tau](\vec{\eta}) = \int_{\mathbb{R}} \mathcal{W}(\eta_{n-1}, \eta_n) GE[F|X_\tau](\vec{\eta}_n) dm_L(\eta_n)$$

$$
\begin{aligned}
&= \int_{L^2_{\alpha,\beta}[0,T]} \exp\bigg\{-\frac{1}{2}\bigg[\|f\|^2_{0,\beta} - \sum_{j=1}^{n-1}\langle f, g_j\rangle^2_{0,\beta}\bigg] + i\bigg[I_{\alpha,\beta}(f)(\alpha) \\
&\quad - \sum_{j=1}^{n-1}\langle f, g_j\rangle_{0,\beta} I_{\alpha,\beta}(g_j)(\alpha) + I_{\alpha,\beta}(P_{t_n,\beta}(\vec{\eta}))\bigg]\bigg\} d\mu(f) \\
&= \int_{L^2_{\alpha,\beta}[0,T]} \exp\bigg\{-\frac{1}{2}\|\mathcal{P}^{\perp}_{\vec{g},n-1,\beta} f\|^2_{0,\beta} + i[I_{\alpha,\beta}(\mathcal{P}^{\perp}_{\vec{g},n-1,\beta} f)(\alpha) \\
&\quad + I_{\alpha,\beta}(f)(P_{t_n,\beta}(\vec{\eta}))]\bigg\} d\mu(f)
\end{aligned}
$$

which is the desired result. Once again we note that the formulas in this paper can be used to simply the generalized conditional expectations which are evaluated by the formulas in [4, 6–8, 12].

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