J. Korean Math. Soc. **58** (2021), No. 3, pp. 643–702
https://doi.org/10.4134/JKMS.j200219
pISSN: 0304-9914 / eISSN: 2234-3008

ON PETERSON'S OPEN PROBLEM AND REPRESENTATIONS OF THE GENERAL LINEAR GROUPS

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Dedicated to Professor Frank Williams of New Mexico State University

Abstract. Fix $\mathbb{Z}/2$ is the prime field of two elements and write $\mathcal{A}_2$ for the mod 2 Steenrod algebra. Denote by $GL_d := GL(d, \mathbb{Z}/2)$ the general linear group of rank d over $\mathbb{Z}/2$ and by $\mathscr{P}_d$ the polynomial algebra $\mathbb{Z}/2[x_1, x_2, \ldots, x_d]$ as a connected unstable $\mathcal{A}_2$-module on d generators of degree one. We study the *Peterson* "*hit problem*" of finding the minimal set of $\mathcal{A}_2$-generators for $\mathscr{P}_d$. Equivalently, we need to determine a basis for the $\mathbb{Z}/2$-vector space

$$Q\mathscr{P}_d := \mathbb{Z}/2 \otimes_{\mathcal{A}_2} \mathscr{P}_d \cong \mathscr{P}_d/\mathcal{A}_2^+\mathscr{P}_d$$

in each degree $n \geq 1$. Note that this space is a representation of GL_d over $\mathbb{Z}/2$. The problem for $d = 5$ is not yet completely solved, and unknown in general.

In this work, we give an explicit solution to the hit problem of five variables in the generic degree $n = r(2^t - 1) + 2^t s$ with $r = d = 5,\ s = 8$ and t an arbitrary non-negative integer. An application of this study to the cases $t = 0$ and $t = 1$ shows that the Singer algebraic transfer of rank 5 is an isomorphism in the bidegrees $(5, 5 + (13.2^0 - 5))$ and $(5, 5 + (13.2^1 - 5))$. Moreover, the result when $t \geq 2$ was also discussed. Here, the Singer transfer of rank d is a $\mathbb{Z}/2$-algebra homomorphism from GL_d-coinvariants of certain subspaces of $Q\mathscr{P}_d$ to the cohomology groups of the Steenrod algebra, $\mathrm{Ext}_{\mathcal{A}_2}^{d,d+*}(\mathbb{Z}/2, \mathbb{Z}/2)$. It is one of the useful tools for studying these mysterious Ext groups.

1. Introduction and statement of results

Throughout this article, we shall work only at the prime 2. Let Sq^k : $H^*(\mathbb{X}) \to H^{k+*}(\mathbb{X})$ be the stable cohomology operation of degree $k \geq 0$, which is introduced by Steenrod in 1947 (see [51]). Here $H^*(\mathbb{X})$ is the singular cohomology group of the topological space $\mathbb{X}$ with coefficients in $\mathbb{Z}/2$. The $\mathbb{Z}/2$-graded algebra $\mathcal{A}_2$ generated by the operations Sq^k is called *the mod* 2 *Steenrod algebra* and acts in a natural way on the cohomology of any space $\mathbb{X}$. For d

Received April 25, 2020; Revised October 6, 2020; Accepted December 9, 2020.
2010 *Mathematics Subject Classification.* Primary 55S10, 55S05, 55T15.
Key words and phrases. Steenrod algebra, Peterson hit problem, Singer algebraic transfer.

a natural number, we denote by $B(\mathbb{Z}/2)^{\times d}$ the classifying space of elementary abelian 2-group $(\mathbb{Z}/2)^{\times d}$ of rank d and by $\mathscr{P}_d = \mathbb{Z}/2[x_1, x_2, \ldots, x_d]$ the polynomial algebra on d variables of degree 1. Note that $B(\mathbb{Z}/2)^{\times d}$ is homotopy equivalent to $(\mathbb{R}P(\infty))^{\times d}$, where $\mathbb{R}P(\infty)$ denotes the infinite real projective space. Since $\mathscr{P}_d$ is isomorphic to the cohomology with $\mathbb{Z}/2$-coefficients of $B(\mathbb{Z}/2)^{\times d}$, it has a connected unstable left $\mathcal{A}_2$-module structure. The left action of $\mathcal{A}_2$ on $\mathscr{P}_d$ is determined by the unstable condition $Sq^1(x_i) = x_i^2,\ Sq^k(x_i) = 0$ for $k > 1$ and Cartan's formula (see [51]).

The investigation of the homotopy classification of topological spaces leads us to the study of the cohomology groups of the Steenrod algebra, $\mathrm{Ext}_{\mathcal{A}_2}^{d,*}(\mathbb{Z}/2, \mathbb{Z}/2)$. It has been thoroughly studied for homological degrees $d \leq 5$ (see Adams [2], Adem [3], Wall [68], Wang [69], Tangora [59], Lin [20], Chen [9]). However, for d higher, the calculations seem to be difficult. Moreover, it has a deep connection with the "hit problem" of our interest in determining the minimal set of $\mathcal{A}_2$-generators for $\mathscr{P}_d$. Equivalently, we need to find a basis for the $\mathbb{Z}/2$-graded vector space

$$Q\mathscr{P}_d := \mathbb{Z}/2 \otimes_{\mathcal{A}_2} \mathscr{P}_d \cong \mathscr{P}_d/\mathcal{A}_2^+\mathscr{P}_d,$$

where $\mathcal{A}_2^+$ denotes the *augmentation ideal* of $\mathcal{A}_2$ and $\mathbb{Z}/2$ is viewed as a right $\mathcal{A}_2$-module concentrated in grading 0. This problem was posed by Peterson [34] in 1987. However, it remains open for $d \geq 5$.

As well known, the general linear group $GL_d := GL(d, \mathbb{Z}/2)$ acts regularly on $\mathscr{P}_d$ by matrix substitution. Further, the two actions of $\mathcal{A}_2$ and GL_d upon $\mathscr{P}_d$ commute with each other; hence there is an inherited action of GL_d on $Q\mathscr{P}_d$. From this event, one of the applications of the hit problem of Peterson is to study the representations of the general linear groups over $\mathbb{Z}/2$. Therefrom the hit problem has attracted great interest of many algebraic topologists (see Crabb and Hubbuck [11], Crossley [12], Kameko [18], Mothebe and his collaborators [24–26], Nam [27], Pengelley and Williams [30,32], Priddy [45], Silverman and Singer [47], Singer [49], Peterson [35], the present author [36–39, 43, 44], Sum [52–58], Walker and Wood [65–67], Wood [70, 71] and others).

Several other aspects of the hit problems were then studied by many authors. For instance, the hit problem for the symmetric of polynomials $\mathscr{P}_d^{S_d}$ as an $\mathcal{A}_2$-submodule of $\mathscr{P}_d$, has been of interest in [17], where S_d is the symmetric group on d letters acting on the right of $\mathscr{P}_d$, and $\mathscr{P}_d^{S_d}$ is isomorphic to the cohomology of the classifying space $BO(d)$ of the orthogonal group $O(d)$. In [14,16], Hưng and his collaborators have studied the hit problem for the rank d Dickson algebra, $\mathscr{P}_d^{GL_d}$ (the algebra of GL_d-invariants, which is also an unstable $\mathcal{A}_2$-module). Note that the Dickson algebra is dual to the coalgebra of Dyer-Lashof operations of the length d (see Madsen [22]). The relationship between Kudo-Araki-May algebra and the hit problem has been investigated by Pengelley and Williams [29,31,33], and by Singer [50]. In [5], Ault and Singer have examined the dual problem of the Peterson hit problem, which is to determine the set of $\mathcal{A}_2^+$-annihilated elements in the homology of $B(\mathbb{Z}/2)^{\times d}$. Recently, Zare [72] has used geometric methods to study the hit problem for $H_*(B(\mathbb{Z}/2)^{\times d})$ (the dual of the hit problem of Peterson) as well as the hit problem for $H_*(BO(d))$ (the dual of the symmetric hit problem of Janfada and Wood). His main idea is based on the relation between the Dyer-Lashof algebra and these dual hit problems.

Let $P_{\mathcal{A}_2}H_*(B(\mathbb{Z}/2)^{\times d})$ be the subspace of $H_*(B(\mathbb{Z}/2)^{\times d})$ consisting of all elements that are $\mathcal{A}_2^+$-annihilated. With the idea of describing the cohomology groups of the Steenrod algebra by means of modular representations of the general linear groups, William Singer [48] constructed a transfer homomorphism of rank d from GL_d-coinvariants of the $\mathcal{A}_2^+$-annihilated elements of the dual of $\mathscr{P}_d$ to the cohomology of the Steenrod algebra:

$$Tr_d : \mathbb{Z}/2 \otimes_{GL_d} P_{\mathcal{A}_2}H_*(B(\mathbb{Z}/2)^{\times d}) \to \mathrm{Ext}_{\mathcal{A}_2}^{d,d+*}(\mathbb{Z}/2, \mathbb{Z}/2),$$

which is related to the geometrical transfer $tr_d : \pi_*^{\mathbb{S}}(B(\mathbb{Z}/2)_+^{\times d}) \to \pi_*^{\mathbb{S}}(\mathbb{S}^0)$ of the stable homotopy of spheres. More explicitly, tr_d induces Tr_d at the E^2-term of the Adams spectral sequence [1]. Singer [48] has pointed out the non-trivial value of Tr_d by proving that it is an isomorphism for $d \leq 2$. In 1993, by using a basis consisting of the all the classes represented by certain polynomials in $\mathscr{P}_3$, Boardman showed in [6] that Tr_3 is also an isomorphism. Through these events, the d-th transfer homomorphism can be considered as a useful tool in the study of the groups $\mathrm{Ext}_{\mathcal{A}_2}^{d,d+*}(\mathbb{Z}/2, \mathbb{Z}/2)$. Many mathematicians then investigated Singer's algebraic transfer (see Bruner et al. [8], Cho'n and Hà [10], Crossley [13], M. H. Lê [19], Hu'ng [15], Minami [23], Nam [28], the present author [36–39, 41, 42], Sum [53, 55, 56, 58] and others). In [48], by using the invariant theory, Singer also showed that Tr_d is an isomorphism for $d = 4$ in a range of internal degrees. However, he proved that Tr_5 is not an epimorphism when acting on $\mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2}H_9(B(\mathbb{Z}/2)^{\times 5})$. Later, he gave a conjecture that *Tr_d is a monomorphism for any positive integer d.* The conjecture is still open in general.

For a non-negative integer n, let $(\mathscr{P}_d)_n$ be the subspace of $\mathscr{P}_d$ consisting of all the homogeneous polynomials of degree n in $\mathscr{P}_d$. Denote by $(Q\mathscr{P}_d)_n$ the subspace of $Q\mathscr{P}_d$ consisting of all the classes represented by the homogeneous polynomials in $(\mathscr{P}_d)_n$. One of the extremely useful tools for computing the hit problem and studying Singer's transfer is the Kameko squaring operation [18] $(\widetilde{Sq^0_*})_{(d,2n+d)} : (Q\mathscr{P}_d)_{2n+d} \to (Q\mathscr{P}_d)_n$, which is an epimorphism of $\mathbb{Z}/2(GL_d)$-modules. We refer to Section 2 for its precise meaning. Let $\mu(n)$ denote the smallest number u such that $\alpha(n+u) \leq u$, where $\alpha(k)$ is the number of 1's in the dyadic expansion of the positive integer k. By Kameko [18], if $\mu(2n+d) = d$, then $(\widetilde{Sq^0_*})_{(d,2n+d)}$ is an $\mathbb{Z}/2(GL_d)$-module isomorphism.

Recall that to solve the hit problem of Peterson, we will determine $Q\mathscr{P}_d$ in each degree $n \geq 1$. However, as explicitly shown in [38], we need only to survey in "generic degrees" n of the following form:

$$n = r(2^t - 1) + 2^t s, \tag{1.1}$$

where r, t, s are non-negative integers such that $0 \leq \mu(s) < r \leq d$. The problem has been completely investigated in [18, 34, 52, 54] for $d \leq 4$. For $r = d-1$ and $s > 0$, the problem was solved by Crabb-Hubbuck [11], Nam [27], Repka-Selick [46], and Sum [54]. For $r = d-1$ and $s = 0$, it is partially studied by Mothebe [24] and by us [37, 43, 44]. The case $r = d - 2 = 3$ was investigated by the present author [38] for $s = 1$, and by Sum [57] for $s = 2^{m+u} + 2^m - 2$, $m \geq 0$, $u > 0$, $t \geq 6$. The recent results when $r = d = 5$ were explicitly determined in [56, 58, 61–63] for $s \in \{2, 3, 5, 7, 10\}$, and by the present author [36, 39] for $s = 6$. The case $r = d = 5$, $s = 26$ and $t = 0$ was studied by Walker-Wood [65]. The authors showed in [65] that in any minimal generating set for the $\mathcal{A}_2$-module $\mathscr{P}_d$, there are $2^{\binom{d}{2}}$ elements in degree $2^d - d - 1$. For $d = 5$, we see that $2^5 - 6 = 26 = 5(2^0 - 1) + 26.2^0$ and $\dim((Q\mathscr{P}_5)_{5(2^0-1)+26.2^0}) = 2^{\binom{5}{2}} = 1024$. More generally, in generic degree of form (1.1) for $d = r = 5$, $s = 26$ and $t \geq 0$, we have $\mu(5(2^t - 1) + 26.2^t) = 5$ for all $t > 0$. This leads to the iterated Kameko map

$$((\widetilde{Sq^0_*})_{(5,5(2^t-1)+26.2^t)})^t : (Q\mathscr{P}_5)_{5(2^t-1)+26.2^t} \to (Q\mathscr{P}_5)_{26}$$

is an isomorphism for all $t \geq 0$. So, $Q\mathscr{P}_5$ has dimension 1024 in degree $5(2^t - 1) + 26.2^t$ for any $t \geq 0$. This event and the result for the case $d = 6$, $t \geq 5$ have also been studied in [64]. For $d = r = 5$, $s = 42$, $t \geq 0$, and $d = 6$, $r = 5$, $s = 42$, $t \geq 5$, we have the following result: Since $\mu(5(2^t - 1) + 42.2^t) = 5$ for $t > 0$, the iterated Kameko homomorphism

$$((\widetilde{Sq^0_*})_{(5,5(2^t-1)+42.2^t)})^t : (Q\mathscr{P}_5)_{5(2^t-1)+42.2^t} \to (Q\mathscr{P}_5)_{42}$$

is an isomorphism for every $t \geq 0$. By using a computer program of Robert R. Bruner written in MAGMA, we get $\dim(Q\mathscr{P}_5)_{5(2^t-1)+42.2^t} = 2520$ for all $t \geq 0$. The direct proofs of this result will be published in detail elsewhere. On the other hand, since $\mu(42) = 4$ and $\alpha(42 + \mu(42)) = 4 = \mu(42)$, by Sum [54, Theorem 1.3], we claim

$$\dim((Q\mathscr{P}_6)_{5(2^t-1)+42.2^t}) = (2^6 - 1)\dim((Q\mathscr{P}_5)_{42}) = 158760 \text{ for all } t \geq 5.$$

It is currently difficult to solve the hit problem in the general case.

In this paper, based on our works in [36–39, 43, 44], we continue our study of the hit problem of five variables in generic degree of (1.1). At the same time, by using these computational techniques, we examine Singer's algebraic transfer of rank 5 in some internal degrees. More precisely, we explicitly determine a basis of $Q\mathscr{P}_d$ for $d = 5$ and generic degree of (1.1) with $r = d = 5$, $s = 8$ and t an arbitrary non-negative integer. (A basis of this space is a set consisting of all the classes represent by the *admissible monomials* of degree n in $\mathscr{P}_d$. We refer to Sect. 2 for the concept of the admissible monomial.) Using this result combining with the computations of $\mathrm{Ext}^{5,13.2^t}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$ (see Tangora [59], Chen [9], Lin [20]), and a direct computation using a result in [10] on the representation in the $\mathbb{Z}/2$-lambda algebra Λ of the Singer transfer homomorphism of rank 5, we show that Tr_5 is an isomorphism when acting on $\mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2} H_{13.2^t-5}(B(\mathbb{Z}/2)^{\times 5})$

for $t \in \{0, 1\}$. (The information on the algebra Λ can be found below in this section.) This gives a quite efficient method to study the isomorphism of the fifth transfer in some internal degrees of (1.1). Furthermore, our approach is different from that of Boardman [6], and of Singer [48]. The following is our first main result.

Theorem 1.1. *Let us consider the generic degree* $13.2^t - 5$ *with* t *a non-negative integer. Then, we have*

$$\dim(Q\mathscr{P}_5)_{13.2^t-5} = \begin{cases} 174 & \text{if } t = 0, \\ 840 & \text{if } t = 1, \\ 1894 & \text{if } t \geq 2. \end{cases}$$

Note that $13.2^t - 5 = 5(2^t - 1) + 8.2^t$. The theorem will be proved by determining explicitly an admissible monomial basis for $Q\mathscr{P}_5$ in generic degree $13.2^t - 5$. The first idea for the proof of the theorem is from Kameko's squaring operation. More clearly, since

$$5(2^t - 1) + 8.2^t = 2^{t+3} + 2^{t+2} + 2^{t-1} + 2^{t-2} + 2^{t-2} - 5,$$

$\mu(13.2^t - 5) = 5$ for all $t > 2$. This leads to the iterated homomorphism

$$(\widetilde{Sq^0_*})^{t-2}_{(5,13.2^t-5)} : (Q\mathscr{P}_5)_{13.2^t-5} \to (Q\mathscr{P}_5)_{13.2^2-5}$$

is an isomorphism of $\mathbb{Z}/2(GL_5)$-modules for any $t \geq 2$. From this we need only to study the structure of $(Q\mathscr{P}_5)_{13.2^t-5}$ for $0 \leq t \leq 2$. The result when $t = 0$ was computed by Tín [60]. We remark that for $t \in \{1, 2\}$, since $(\widetilde{Sq^0_*})_{(5,13.2^t-5)} : (Q\mathscr{P}_5)_{13.2^t-5} \to (Q\mathscr{P}_5)_{13.2^{t-1}-5}$ is an epimorphism, we need only to determine the kernel of $(\widetilde{Sq^0_*})_{(5,13.2^t-5)}$. To study this space, we combine our recent results in [43] with previous results by Kameko [18], Mothebe [25,26], Singer [49], Sum [54], and Tín [60,62].

Recently, Sum [56] has proved some properties of $\mathcal{A}_2$-generators for $\mathscr{P}_d$. Then, he made a conjecture on the relation between the admissible monomials for the polynomial algebras (see Section 3). The conjecture helps us to reduce remarkably in computing the hit problem. From the results of Peterson [34], Kameko [18] and Sum [54], this conjecture holds true for $d \leq 4$. Sum proved in [56] that the conjecture is true in the case $d = 5$ and the degree n of the form (1.1) for $(r; s) = (5; 10)$ and $t \geq 0$. Based upon the proof of Theorem 1.1, and previous results of the present author and Sum (see [36–39,43,44,57]), the conjecture also satisfies for $d = 5$ and in degrees of the form (1.1) for $(r; s) = (4; 0),\ (5; 6),\ (5; 8)$ and $(3; s)$ for $s = 1,\ t \geq 0$, and $s = 2^{m+u} + 2^m - 2,\ m \geq 0,\ u > 0,\ t \geq 6$.

It is well known that $\mathbb{Z}/2 \otimes_{GL_d} P_{\mathcal{A}_2} H_n(B(\mathbb{Z}/2)^{\times d})$ is dual to $(Q\mathscr{P}_d)^{GL_d}_n$, the subspace of $Q\mathscr{P}_d$ generated all GL_d-invariants of degree n. Computation of the GL_d-invariants is very difficult, particularly for values of d as large as $d = 5$. The understanding of special cases should be a helpful step toward the solution of the general problem. Now, applying Theorem 1.1 for $t = 1$, we get the following, which is our second main result.

Theorem 1.2. *There exists uniquely a non-zero class in* $(Q\mathscr{P}_5)_{13.2^1-5}$ *invariant under the usual action of* GL_5. *This implies that* $(Q\mathscr{P}_5)^{GL_5}_{13.2^1-5}$ *is one-dimensional.*

Note that the Kameko operation

$$(\widetilde{Sq^0_*})_{(5,13.2^1-5)} : (Q\mathscr{P}_5)_{13.2^1-5} \to (Q\mathscr{P}_5)_{13.2^0-5}$$

is an epimorphism of $\mathbb{Z}/2(GL_5)$-modules. So, in order to prove Theorem 1.2, we describe the $\mathbb{Z}/2$-vector space structure of $(Q\mathscr{P}_5)^{GL_5}_{13.2^0-5}$. Combining this with a basis of $(QP_5)_{21}$ in Theorem 1.1, we explicitly compute all GL_5-invariants of $Q\mathscr{P}_5$ in degree $13.2^1 - 5$.

The (mod 2) lambda algebra Λ (see Bousfield et al. [7]) is one of the tools to compute the groups $\mathrm{Ext}^{d,*}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$. Λ is defined as a differential, bigraded, associative algebra with unit over $\mathbb{Z}/2$, is generated by $\lambda_i \in \Lambda^{1,i}$, satisfying the Adem relations

$$\lambda_i\lambda_{2i+d+1} = \sum_{j\geq 0}\binom{d-j-1}{j}\lambda_{i+d-j}\lambda_{2i+1+j} \quad (i \geq 0,\ d \geq 0) \tag{1.2}$$

and the differential

$$\partial(\lambda_{d-1}) = \sum_{j\geq 1}\binom{d-j-1}{j}\lambda_{d-j-1}\lambda_{j-1} \quad (d \geq 1), \tag{1.3}$$

where $\binom{d-j-1}{j}$ denotes the binomial coefficient reduced modulo 2. Furthermore, we have

$$H^{d,*}(\Lambda, \partial) = \mathrm{Ext}^{d,d+*}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2).$$

For non-negative integers $j_1, \ldots, j_d$, a monomial $\lambda_{j_1}\cdots\lambda_{j_d} \in \Lambda$ is called *the monomial of the length d.* We shall write $\lambda_J,\ J = (j_1, \ldots, j_d)$ for $\prod_{1\leq k\leq d}\lambda_{j_k}$ and refer to $\ell(J) = d$ as the length of J. Note that the algebra Λ is not commutative and that the bigrading of a monomial indexed by J may be written (d, n), where the homological degree d, as above, is the length of J, and $n = \sum_{1\leq k\leq d} j_k$. A monomial λ_J is called *admissible* if $j_k \leq 2j_{k+1}$ for all $1 \leq k \leq d-1$. By the relations (1.2), the $\mathbb{Z}/2$-vector subspace

$$\Lambda^{d,*} = \langle\{\lambda_J \mid J = (j_1, \ldots, j_d),\ j_k \geq 0, 1 \leq k \leq d\}\rangle$$

of Λ has an additive basis consisting of all admissible monomials of the length d. Recall that the dual of $\mathscr{P}_d$ is isomorphic to $\Gamma(a_1, \ldots, a_d)$, the divided power algebra generated by $a_1, \ldots, a_d$, where $a_t = a_t^{(1)}$ is dual to x_t with respect to the basis of $\mathscr{P}_d$ consisting of all monomials in $x_1, \ldots, x_d$. In other words, $H_*(B(\mathbb{Z}/2)^{\times d}) = H_*((\mathbb{Z}/2)^{\times d}) \cong \Gamma(a_1, \ldots, a_d)$. We note that the algebra $H_*(B(\mathbb{Z}/2)^{\times d})$ has a right $\mathcal{A}_2$-module structure. The right action of $\mathcal{A}_2$ on this algebra is given by $(a_t^{(j)})Sq^k = \binom{j-k}{k}a_t^{(j-k)}$ and Cartan's formula. In [10], Cho'n and Hà have established a homomorphism $\psi_d : H_*(B(\mathbb{Z}/2)^{\times d}) \longrightarrow \Lambda^{d,*}$, which is considered as a presentation in the $\mathbb{Z}/2$-lambda algebra of Singer's transfer of rank d and determined by the following inductive formula:

$$\psi_d(a^J) = \begin{cases} \lambda_{j_1} & \text{if } \ell(J) = 1, \\ \sum_{t\geq j_d} \psi_{d-1}(\prod_{1\leq k\leq d-1} a_k^{(j_k)} Sq^{t-j_d})\lambda_t & \text{if } \ell(J) > 1, \end{cases}$$

for any $a^J := \prod_{1\leq k\leq d} a_k^{(j_k)} \in H_*(B(\mathbb{Z}/2)^{\times d})$ and $J := (j_1, j_2, \ldots, j_d)$. Note that ψ_d is not an $\mathcal{A}_2$-homomorphism. The authors showed in [10] that if $a^J \in P_{\mathcal{A}_2}H_*(B(\mathbb{Z}/2)^{\times d})$, then $\psi_d(a^J)$ is a cycle in $\Lambda^{d,*}$ and $Tr_d([a^J]) = [\psi_d(a^J)]$. Applying this event and Theorem 1.2 into the investigation of the Singer transfer of rank 5, we obtain the following theorem, which is our third main result.

Theorem 1.3. *The fifth transfer homomorphism*

$$Tr_5 : \mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2} H_{13.2^1-5}(B(\mathbb{Z}/2)^{\times 5}) \to \mathrm{Ext}_{\mathcal{A}_2}^{5,5+(13.2^1-5)}(\mathbb{Z}/2, \mathbb{Z}/2)$$

is an isomorphism.

As it is known, there exists an endomorphism Sq^0 of the lambda algebra Λ, determined by $Sq^0(\lambda_J = \prod_{1\leq k\leq d}\lambda_{j_k}) = \prod_{1\leq k\leq d}\lambda_{2j_k+1}$, where λ_J is not necessarily admissible. It respects the relations in (1.2) and commutes the differential in (1.3) (see also [20]). Then, Sq^0 induces the classical squaring operation in the Ext groups

$$Sq^0 : H^{d,*}(\Lambda, \partial) = \mathrm{Ext}_{\mathcal{A}_2}^{d,d+*}(\mathbb{Z}/2, \mathbb{Z}/2) \to H^{d,d+2*}(\Lambda, \partial) = \mathrm{Ext}_{\mathcal{A}_2}^{d,2(d+*)}(\mathbb{Z}/2, \mathbb{Z}/2).$$

Note that Sq^0 is not the identity map (see [21]). As above mentioned, the structure of the groups $\mathrm{Ext}_{\mathcal{A}_2}^{d,d+*}(\mathbb{Z}/2, \mathbb{Z}/2)$ has intensively been studied by many authors, but remains very mysterious in general. In what follows, $(Sq^0)^t : \mathrm{Ext}_{\mathcal{A}_2}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}_2}^{*,*}(\mathbb{Z}/2, \mathbb{Z}/2)$ denotes the composite $Sq^0 \cdots Sq^0$ (t times of Sq^0) if $t > 1$, is Sq^0 if $t = 1$, and is the identity map if $t = 0$. A family $\{a_t : t \geq 0\} \subset \mathrm{Ext}_{\mathcal{A}_2}^{d,d+*}(\mathbb{Z}/2, \mathbb{Z}/2)$ is called a Sq^0-family if $a_t = (Sq^0)^t(a_0)$ for $t \geq 0$. We now return to the internal degree $13.2^t - 5$ in Theorem 1.1. It has been shown (see Tangora [59], Lin [20], Chen [9]) that

$$\mathrm{Ext}_{\mathcal{A}_2}^{5,5+(13.2^t-5)}(\mathbb{Z}/2, \mathbb{Z}/2) = \begin{cases} 0 & \text{if } t = 0, \\ \langle h_{t+1}f_{t-1}\rangle & \text{if } t \geq 1, \end{cases}$$

and that $h_{t+1}f_{t-1} = h_t g_t \neq 0$, where $h_t = (Sq^0)^t(h_0)$ is the Adams element in $\mathrm{Ext}_{\mathcal{A}_2}^{1,2^t}(\mathbb{Z}/2, \mathbb{Z}/2)$, $g_t = (Sq^0)^{t-1}(g_1)$ and $f_{t-1} = (Sq^0)^{t-1}(f_0)$ are the elements non-zero in $\mathrm{Ext}_{\mathcal{A}_2}^{4,12.2^t}(\mathbb{Z}/2, \mathbb{Z}/2)$ and $\mathrm{Ext}_{\mathcal{A}_2}^{4,11.2^t}(\mathbb{Z}/2, \mathbb{Z}/2)$, respectively, for any $t \geq 1$. (Note that by Lin [20], the groups $\mathrm{Ext}_{\mathcal{A}_2}^{4,4+*}(\mathbb{Z}/2, \mathbb{Z}/2)$ contains seven Sq^0-families of indecomposable elements, namely

$$\{d_t\},\ \{e_t\},\ \{f_t\},\ \{g_{t+1}\},\ \{p_t\},\ \{D_3(t)\},\ \{p'_t\}\ (t \geq 0).)$$

As well known, the transfer homomorphism Tr_1 detects the family $\{h_t \,|\, t \geq 1\}$ (see Singer [48]). According to Nam [28], the family $\{f_{t-1} \,|\, t \geq 1\}$ was detected by Tr_4. By Singer [48], $\bigoplus_{d\geq 0} Tr_d$ is a homomorphism of $\mathbb{Z}/2$-algebras. These data show that $h_{t+1}f_{t-1}$ is in the image of Tr_5 for all $t \geq 1$. In Sect. 5, we give another direct proof of this event for the case $t = 1$. More specifically, we proved $h_2f_0 \in \operatorname{Im}(Tr_5)$ by using Theorem 1.2 and a representation of Tr_5 over the lambda algebra.

As above shown, to prove Theorem 1.2, we need to determine all GL_5-invariants of $(Q\mathscr{P}_5)_{13.2^0-5}$. Applying Theorem 1.1 for $t = 0$ with a basis of $Q\mathscr{P}_5$ in degree $13.2^0 - 5$ (see [60]), we showed that $(Q\mathscr{P}_5)^{GL_5}_{13.2^0-5}$ is zero (see Theorem 4.1.1 in Section 4). Combining this with the fact that

$$\operatorname{Ext}^{5,5+(13.2^0-5)}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2) = 0,$$

we claim that Tr_5 is a trivial isomorphism when acting on the space $\mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2}H_{13.2^0-5}(B(\mathbb{Z}/2)^{\times 5})$. As an immediate consequence from this and Theorem 1.3, we get:

Corollary 1.4. *Singer's conjecture for* Tr_5 *satisfies in the bidegrees* $(5, 5 + 8)$ *and* $(5, 5 + 21)$.

To end this introduction, we will discuss whether Tr_5 is an isomorphism or not in the bidegree $(5, 5 + (13.2^t - 5))$ for $t \geq 2$. Since the iterated Kameko homomorphism

$$((\widetilde{Sq^0_*})_{(5,13.2^t-5)})^{t-2} : (Q\mathscr{P}_5)_{13.2^t-5} \to (Q\mathscr{P}_5)_{13.2^2-5}$$

is an GL_5-module isomorphism for all $t \geq 2$, to examine Singer's conjecture for Tr_5 in the above bidegree, we need only to determine all GL_5-invariants of $(Q\mathscr{P}_5)_{13.2^t-5}$ for $t = 2$. Recall that Kameko's map

$$(\widetilde{Sq^0_*})_{(5,13.2^2-5)} : (Q\mathscr{P}_5)_{13.2^2-5} \to (Q\mathscr{P}_5)_{13.2^1-5}$$

is an epimorphism of GL_5-modules and that the element

$$h_3f_1 \in \operatorname{Ext}^{5,5+(13.2^2-5)}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$$

is in the image of Tr_5. So, by Theorem 1.2, we deduce

$$1 \leq \dim\left(\mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2}H_{13.2^2-5}(B(\mathbb{Z}/2)^{\times 5})\right) \leq \dim(\operatorname{Ker}(\widetilde{Sq^0_*})_{(5,13.2^2-5)})^{GL_5} + 1.$$

Furthermore, all elements of $\mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2}H_{13.2^2-5}(B(\mathbb{Z}/2)^{\times 5})$ are of the form

$$(\gamma[\varphi(u_0)] + [v])^*,$$

where $\gamma \in \mathbb{Z}/2$, and the mapping $\varphi : \mathscr{P}_5 \to \mathscr{P}_5$ determined by setting $\varphi(u) = x_1x_2x_3x_4x_5u^2$ for any $u \in \mathscr{P}_5$, $v \in (\mathscr{P}_5)_{13.2^2-5}$ such that $[v]$ belongs to $\operatorname{Ker}(\widetilde{Sq^0_*})_{(5,13.2^2-5)}$, and $u_0 \in (\mathscr{P}_5)_{13.2^1-5}$. Based on Theorem 1.2, $[u_0]$ is the only non-zero element in $(Q\mathscr{P}_5)^{GL_5}_{13.2^1-5}$. Direct calculating the elements $(\gamma[\varphi(u_0)] + [v])^*$ is a hard work. By using techniques of the hit problem of five variables, we will describe explicitly all these elements in the near future. From these data with the fact that $h_{t+1}f_{t-1} \in \operatorname{Im}(Tr_5)$ for $t \geq 1$, we conclude that if $(\operatorname{Ker}(\widetilde{Sq^0_*})_{(5,13.2^2-5)})^{GL_5}$ is zero, then Tr_5 is an isomorphism in the bidegree

$(5, 5 + (13.2^t - 5))$ for every $t \geq 2$. This means that Singer's conjecture for Tr_5 also satisfies in this bidegree. However, it will be much more interesting if $(\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,13.2^2-5)})^{GL_5}$ is non-trivial and $\dim(Q\mathscr{P}_5)^{GL_5}_{13.2^2-5} \neq 1$.

The structure of the paper is as follows. First, some background is reviewed in Section 2. In the next section, we present Singer's criterion on $\mathcal{A}_2$-decomposable and Sum's conjecture related to the minimal set of generators for $\mathcal{A}_2$-modules $\mathscr{P}_d$. Then, the $\mathcal{A}_2$-generators for $\mathscr{P}_5$ in degree $13.2^t - 5$ are described explicitly by proving Theorem 1.1. In Section 4, we prove Theorem 1.2 by using the admissible monomial bases of $(Q\mathscr{P}_5)_{13.2^0-5}$ and $(Q\mathscr{P}_5)_{13.2^1-5}$. Based upon Theorem 1.2 and a representation in the lambda algebra of the fifth Singer transfer, the proof of Theorem 1.3 is presented in Section 5. All the admissible monomials of degree $13.2^t - 5$ in $\mathscr{P}_5$ are provided in Section 6 of the online version [40].

Acknowledgments. This research is supported financially by the National Foundation for Science and Technology Development (NAFOSTED) of Viet Nam under Grant No. 101.04-2017.05.

The author is very grateful to the anonymous referees for the careful reading of the manuscript and their insightful comments and detailed suggestions, which have led me to improve considerably this work.

I would also like to thank Robert Bruner, Phan Hoàng Cho'n, Mbakiso Fix Mothebe for illuminating conversations on the hit problem and the Ext groups, Hadi Zare for sending a copy of [72], and Lê Minh Hà for pointing out an error in the original manuscript.

Finally, the author would like to give my deepest sincere thanks to Professor Nguyễn Sum for many helpful discussions.

2. Preliminaries

This section starts with a recollection of the Kameko squaring operation and some information related to the Peterson hit problem.

2.1. Kameko's squaring operation

Recall that the polynomial algebra $\mathscr{P}_d = \mathbb{Z}/2[x_1, \ldots, x_d]$ is an unstable left module on the ring $\mathcal{A}_2$. Let $GL_d := GL(d, \mathbb{Z}/2)$ denote the general linear group of rank d over the field $\mathbb{Z}/2$. An usual left action of this group on $\mathscr{P}_d$ is given by $(w(f))(x_1, x_2, \ldots, x_d) = f(w(x_1), w(x_2), \ldots, w(x_d))$, where $w = (w_{ij}) \in GL_d$ and $w(x_j) = \sum_{1\leq i\leq d} w_{ij}x_i$, $1 \leq j \leq d$. Thus, $\mathscr{P}_d$ (resp. $(\mathscr{P}_d)^*$) has also a left (resp. right) GL_d-module structure. Furthermore, since the two actions of $\mathcal{A}_2$ and GL_d upon $\mathscr{P}_d$ (resp. $(\mathscr{P}_d)^*$) commute with each other, there is an inherited action of GL_d on $Q\mathscr{P}_d$ (resp. $(Q\mathscr{P}_d)^* = P_{\mathcal{A}_2}H_*(B(\mathbb{Z}/2)^{\times d})$).

We knew that the homological algebra $\{H_n(B(\mathbb{Z}/2)^{\times d}) \,|\, n \geq 0\}$ is dual to $\mathscr{P}_d$. Moreover, it is isomorphic to $\Gamma(a_1, \ldots, a_d)$, the divided power algebra generated by $a_1, \ldots, a_d$, each of degree one, where $a_j = a_j^{(1)}$ is dual to x_j. Here the duality is taken with respect to the basis of $\mathscr{P}_d$ consisting of all monomials in $x_1, \ldots, x_d$. We now denote by $P_{\mathcal{A}_2}H_n(B(\mathbb{Z}/2)^{\times d})$ the primitive subspace consisting of all elements in $H_n(B(\mathbb{Z}/2)^{\times d})$, which are annihilated by every Steenrod's operation $Sq^k,\ k > 0$. So, it is dual to $(Q\mathscr{P}_d)_n$. By Kameko [18], we have the monomorphism

$$\begin{array}{rl} \overline{Sq}^0 : P_{\mathcal{A}_2}H_n(B(\mathbb{Z}/2)^{\times d}) & \longrightarrow P_{\mathcal{A}_2}H_{d+2n}(B(\mathbb{Z}/2)^{\times d}) \\ \prod_{1\leq t\leq d} a_t^{(s_t)} & \longmapsto \prod_{1\leq t\leq d} a_t^{(2s_t+1)} \end{array}$$

where $\prod_{1\leq t\leq d} a_t^{(s_t)}$ is dual to $\prod_{1\leq t\leq d} x_t^{s_t}$. Further, $Sq_*^{2k+1}\overline{Sq}^0 = 0$, and $Sq_*^{2k}\overline{Sq}^0 = \overline{Sq}^0 Sq_*^k$ for any $k \geq 0$, where Sq_*^k denotes the dual Steenrod operation. Note that $\overline{Sq}^0$ is also an GL_d-module homomorphism (see [8], [15]). Then, $\overline{Sq}^0$ induces Kameko's squaring operation in the dual of the spaces $(Q\mathscr{P}_d)_*^{GL_d}$,

$$\widetilde{Sq^0} = id_{\mathbb{Z}/2} \otimes_{GL_d} \overline{Sq}^0 : \mathbb{Z}/2 \otimes_{GL_d} P_{\mathcal{A}_2}H_n(B(\mathbb{Z}/2)^{\times d}) \to \mathbb{Z}/2 \otimes_{GL_d} P_{\mathcal{A}_2}H_{d+2n}(B(\mathbb{Z}/2)^{\times d}),$$

which commutes with the classical squaring operation

$$Sq^0 : \mathrm{Ext}_{\mathcal{A}_2}^{d,d+n}(\mathbb{Z}/2, \mathbb{Z}/2) \to \mathrm{Ext}_{\mathcal{A}_2}^{d,2d+2n}(\mathbb{Z}/2, \mathbb{Z}/2)$$

through the d-th Singer transfer (see [4], [6], [23]). In other words, the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{Z}/2 \otimes_{GL_d} P_{\mathcal{A}_2}H_n(B(\mathbb{Z}/2)^{\times d}) & \xrightarrow{Tr_d} & \mathrm{Ext}_{\mathcal{A}_2}^{d,d+n}(\mathbb{Z}/2, \mathbb{Z}/2) \\ \Big\downarrow \widetilde{Sq^0} & & \Big\downarrow Sq^0 \\ \mathbb{Z}/2 \otimes_{GL_d} P_{\mathcal{A}_2}H_{d+2n}(B(\mathbb{Z}/2)^{\times d}) & \xrightarrow{Tr_d} & \mathrm{Ext}_{\mathcal{A}_2}^{d,2d+2n}(\mathbb{Z}/2, \mathbb{Z}/2). \end{array}$$

The dual homomorphism $\widetilde{Sq^0_*} : (Q\mathscr{P}_d)_{d+2n}^{GL_d} \to (Q\mathscr{P}_d)_n^{GL_d}$ of $\widetilde{Sq^0}$ is induced by the homomorphism $(\widetilde{Sq^0_*})_{(d,d+2n)} : (Q\mathscr{P}_d)_{d+2n} \to (Q\mathscr{P}_d)_n$. The latter is given by the $\mathbb{Z}/2$-linear map

$$\begin{array}{rl} \delta : (\mathscr{P}_d)_{d+2n} & \longrightarrow (\mathscr{P}_d)_n \\ x_1^{t_1}x_2^{t_2}\cdots x_d^{t_d} & \longmapsto \begin{cases} x_1^{\frac{t_1-1}{2}} x_2^{\frac{t_2-1}{2}} \cdots x_d^{\frac{t_d-1}{2}} & \text{if } t_1, \ldots, t_d \text{ odd}, \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Denote by $\alpha(n)$ the number of 1's in dyadic expansion of n. We consider the arithmetic function (see [18], [58]):

$$\begin{aligned} \mu(n) &= \min\{u \in \mathbb{N} : \alpha(n+u) \leq u\} \\ &= \min\{u \in \mathbb{N} : n = \sum_{1\leqslant j\leqslant u} (2^{s_j} - 1),\ s_j > 0,\ 1 \leqslant j \leqslant u\}. \end{aligned}$$

From the above data, $(\widetilde{Sq^0_*})_{(d,d+2n)}$ is a $\mathbb{Z}/2(GL_d)$-module epimorphism. However, in particular, if $\mu(d+2n) = d$, then it is an isomorphism. According to

Hưng [15, Theorem 1.5], if Singer's conjecture for the d-th algebraic transfer is true, then Tr_d does not detect the non-zero elements $u \in \mathrm{Ext}_{\mathcal{A}_2}^{d,d+n}(\mathbb{Z}/2, \mathbb{Z}/2)$ such that $Sq^0(u) = 0$ and $\mu(2n+d) = d$. In this case, u is called *critical.* This leads us to the study of the kernel of $\widetilde{Sq^0}$. Recall that $\overline{Sq}^0$ is a monomorphism, but the squaring operation $\widetilde{Sq^0} = id_{\mathbb{Z}/2} \otimes_{GL_d} \overline{Sq}^0$ is not a monomorphism in general. Indeed, by using a computer calculation, Hưng provided a counter-example in [15] that $\widetilde{Sq^0}$ is not a monomorphism when acting on $\mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2} H_{15}(B(\mathbb{Z}/2)^{\times 5})$. This was confirmed again by the works of Sum [53, 58]. Thereafter, Hưng [15] conjectured that $\widetilde{Sq^0}$ *is a monomorphism in positive degrees* n *of* $\mathbb{Z}/2 \otimes_{GL_d} P_{\mathcal{A}_2} H_n(B(\mathbb{Z}/2)^{\times d})$ *if and only if* $d \leq 4$. By Boardman [6], and Singer [48], the conjecture satisfies for $d \leq 3$. We hope that it can be verified for $d = 4$ by using the dual of $\widetilde{Sq^0}$ and the results on the hit problem in [56].

Thus, in addition to approaching Singer's conjecture based on techniques of the hit problem, we can use the relationship between the algebraic transfer and critical elements to verify this conjecture. However, finding critical elements is difficult.

2.2. On the hit problem of Peterson

To study the hit problem, we need some relevant notations and concepts. For a natural number n, denote by $\alpha_t(n)$ the t-th coefficients in dyadic expansion of n. This means $\alpha(n) = \sum_{t\geq 0} \alpha_t(n)$. Further, n can be represented as follows: $n = \sum_{t\geq 0} \alpha_t(n)2^t$, where $\alpha_t(n) \in \{0, 1\},\ t = 0, 1, \ldots$. Consider the monomial $x = x_1^{u_1} x_2^{u_2} \cdots x_d^{u_d} \in \mathscr{P}_d$, we define two sequences associated with x by $\omega(x) := (\sum_{1\leq j\leq d} \alpha_0(u_j),\ \sum_{1\leq j\leq d} \alpha_1(u_j), \ldots,\ \sum_{1\leq j\leq d} \alpha_{t-1}(u_j), \ldots)$ and $(u_1, u_2, \ldots, u_d)$, which are called the *weight vector* and the *exponent vector* of x, respectively. From now on, we shall write $\omega_t(x)$ for $\sum_{1\leq j\leq d} \alpha_{t-1}(u_j),\ t = 1, 2, \ldots$.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_t, \ldots)$ be a sequence of non-negative integers. Then, the sequence ω are called *the weight vector* if $\omega_t = 0$ for $t \gg 0$. We define $\deg(\omega) = \sum_{t\geq 1} 2^{t-1}\omega_t$. The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

Recall that a homogeneous element $f \in \mathscr{P}_d$ is called $\mathcal{A}_2$-*decomposable* (or "*hit*") if it is in the image of positive degree elements of $\mathcal{A}_2$. This means that f belongs to $\mathcal{A}_2^+ \mathscr{P}_d$.

The equivalence relations on $\mathscr{P}_d$ (see [18, 38]). For a weight vector ω, we denote two subspaces associated with ω:

$$\mathscr{P}_d(\omega) = \langle \{x \in \mathscr{P}_d \,|\, \deg(x) = \deg(\omega),\ \omega(x) \leq \omega\} \rangle,$$

$$\mathscr{P}_d^-(\omega) = \langle \{x \in \mathscr{P}_d \,|\, \deg(x) = \deg(\omega),\ \omega(x) < \omega\} \rangle.$$

Let us now consider the homogeneous polynomials f, and g in $\mathscr{P}_d$ with $\deg(f) = \deg(g)$. We define the following binary relations "$\equiv$" and "$\equiv_\omega$" on $\mathscr{P}_d$:

(i) $f \equiv g$ if and only if $f = g$ modulo $(\mathcal{A}_2^+\mathscr{P}_d)$. In particular, if $f \equiv 0$, then f is $\mathcal{A}_2$-decomposable.

(ii) $f \equiv_\omega g$ if and only if $f, g \in \mathscr{P}_d(\omega)$ and $f = g$ modulo $((\mathcal{A}_2^+\mathscr{P}_d \cap \mathscr{P}_d(\omega)) + \mathscr{P}_d^-(\omega))$.

It is easily seen that these binary relations are equivalence ones. Let $Q\mathscr{P}_d(\omega)$ denote the quotient of $\mathscr{P}_d(\omega)$ by the equivalence relation "$\equiv_\omega$". Then, we have the $\mathbb{Z}/2$-quotient space

$$Q\mathscr{P}_d(\omega) = \mathscr{P}_d(\omega)/((\mathcal{A}_2^+\mathscr{P}_d \cap \mathscr{P}_d(\omega)) + \mathscr{P}_d^-(\omega)).$$

By Sum [58], $Q\mathscr{P}_d(\omega)$ is also a GL_d-module. The following events are shown in [38]. However, to make the paper self-contained, we will present again them in detail.

$$\dim((Q\mathscr{P}_d)_n) = \sum_{\deg \omega = n} \dim(Q\mathscr{P}_d(\omega)),$$

$$\dim((Q\mathscr{P}_d)_n^{GL_d}) \leq \sum_{\deg(\omega) = n} \dim(Q\mathscr{P}_d(\omega)^{GL_d}).$$

Indeed, by Walker and Wood [67], we have a filtration of $Q\mathscr{P}_d$:

$$\begin{aligned}\{0\} \subseteq \cdots \subseteq \mathscr{P}_d^-(\omega)/((\mathcal{A}_2^+\mathscr{P}_d) \cap \mathscr{P}_d^-(\omega))\\ \subseteq \mathscr{P}_d(\omega)/((\mathcal{A}_2^+\mathscr{P}_d) \cap \mathscr{P}_d(\omega)) \subseteq \cdots \subseteq \mathscr{P}_d/(\mathcal{A}_2^+\mathscr{P}_d) = Q\mathscr{P}_d.\end{aligned}$$

This is not only a filtration of $Q\mathscr{P}_d$ as a vector space, but also as a GL_d-module. The inclusion of $\mathscr{P}_d^-(\omega)$ into $\mathscr{P}_d(\omega)$ induces the monomorphism

$$\mathscr{P}_d^-(\omega)/((\mathcal{A}_2^+\mathscr{P}_d) \cap \mathscr{P}_d^-(\omega)) \to \mathscr{P}_d(\omega)/((\mathcal{A}_2^+\mathscr{P}_d) \cap \mathscr{P}_d(\omega))$$

and the following sequence is short exact:

$$\begin{aligned}0 \to \mathscr{P}_d^-(\omega)/((\mathcal{A}_2^+\mathscr{P}_d) \cap \mathscr{P}_d^-(\omega)) \to \mathscr{P}_d(\omega)/((\mathcal{A}_2^+\mathscr{P}_d) \cap \mathscr{P}_d(\omega))\\ \to \mathscr{P}_d(\omega)/(((\mathcal{A}_2^+\mathscr{P}_d) \cap \mathscr{P}_d(\omega)) + \mathscr{P}_d^-(\omega)) \to 0.\end{aligned}$$

From this, we get

$$Q\mathscr{P}_d(\omega) \cong (\mathscr{P}_d(\omega)/((\mathcal{A}_2^+\mathscr{P}_d) \cap \mathscr{P}_d(\omega)))/(\mathscr{P}_d^-(\omega)/((\mathcal{A}_2^+\mathscr{P}_d) \cap \mathscr{P}_d^-(\omega))).$$

This isomorphism is also an isomorphism of GL_d-modules. Taking these events with the filtration of $Q\mathscr{P}_d$ into account, we have immediate the above claims.

The linear order on $\mathscr{P}_d$ (see [18]). Let $u = x_1^{a_1}x_2^{a_2}\cdots x_d^{a_d}$ and $v = x_1^{b_1}x_2^{b_2}\cdots x_d^{b_d}$ be monomials of the same degree in $\mathscr{P}_d$. We write $a,\ b$ for the exponent vectors of $u,\ v$, respectively. We say that $a < b$ if there is a positive integer m such that $a_j = b_j$ for all $j < m$ and $a_m < b_m$, and that $u < v$ if and only if one of the following holds:

(i) $\omega(u) < \omega(v)$;

(ii) $\omega(u) = \omega(v)$ and $a < b$.

The inadmissible monomial (see [18]). A monomial $u \in \mathscr{P}_d$ is said to be *inadmissible* if there exist monomials $x_1, x_2, \ldots, x_k$ such that $x_j < u$ for $1 \leq j \leq k$ and $u \equiv \sum_{1\leq i\leq k} x_i$. Then, u is said to be *admissible* if it is not inadmissible.

Obviously, the set of all the admissible monomials of degree n in $\mathscr{P}_d$ is *a minimal set of $\mathcal{A}_2$-generators for $\mathscr{P}_d$ in degree n*. So, $(Q\mathscr{P}_d)_n$ is a $\mathbb{Z}/2$-vector space with a basis consisting of all the classes represent by the admissible monomials of degree n in $\mathscr{P}_d$.

The strictly inadmissible monomial (see [18]). A monomial $u \in \mathscr{P}_d$ is said to be *strictly inadmissible* if and only if there exist monomials $x_1, x_2, \ldots, x_k$ such that $x_j < u$ for $1 \leq j \leq k$ and $u = \sum_{1\leq j\leq k} x_j + \sum_{1\leq t\leq 2^s-1} Sq^t(y_t)$, where $s = \max\{i \in \mathbb{Z} : \omega_i(u) > 0\}$ and suitable polynomials $y_t \in \mathscr{P}_d$.

Note that every the strictly inadmissible monomial is inadmissible but the converses not generally true (see a counter-example in [38]). The following result is used to study $Q\mathscr{P}_5$ in the next section.

Theorem 2.2.1 (see [18])**.** *Let x, y and u be monomials in $\mathscr{P}_d$ such that $\omega_i(x) = 0$ for $i > r > 0, \omega_t(u) \neq 0$ and $\omega_i(u) = 0$ for $i > t > 0$. Then, if u is inadmissible, then xu^{2^r} is also inadmissible. Furthermore, if u is strictly inadmissible, then uy^{2^t} is also strictly inadmissible.*

Let $\mathscr{P}_d^0$ and $\mathscr{P}_d^+$ denote the $\mathcal{A}_2$-submodules of $\mathscr{P}_d$ spanned all the monomials $x_1^{t_1}x_2^{t_2}\cdots x_d^{t_d}$ such that $t_1t_2\cdots t_d = 0$, and $t_1t_2\cdots t_d > 0$, respectively. Denote by $Q\mathscr{P}_d^0 := \mathbb{Z}/2 \otimes_{\mathcal{A}_2} \mathscr{P}_d^0$, and by $Q\mathscr{P}_d^+ := \mathbb{Z}/2 \otimes_{\mathcal{A}_2} \mathscr{P}_d^+$. Then, we can see that $Q\mathscr{P}_d = Q\mathscr{P}_d^0 \bigoplus Q\mathscr{P}_d^+$. We end this section by establishing a formula below on the dimension of $Q\mathscr{P}_d^0$ in degree n, which will be used in the next section. Note that this formula is similar to the one of [25].

Let $\mathcal{J} = (j_1, j_2, \ldots, j_r)$, where $1 \leq j_1 < \cdots < j_r \leq d$, $1 \leq r \leq d-1$, and let $r := \ell(\mathcal{J})$ be the length of $\mathcal{J}$. We denote $\mathscr{P}_{\mathcal{J}} = \langle\{x_{j_1}^{t_1}x_{j_2}^{t_2}\cdots x_{j_r}^{t_r} \mid t_s \in \mathbb{N}, s = 1, 2, \ldots, r\}\rangle \subset \mathscr{P}_d$. Then, $\mathscr{P}_{\mathcal{J}}$ is an $\mathcal{A}_2$-submodule of $\mathscr{P}_d$. Further, it is isomorphic to $\mathscr{P}_r$. By a simple computation, we get

$$Q\mathscr{P}_d^0 = \bigoplus_{1\leq r\leq d-1} \bigoplus_{\ell(\mathcal{J})=r} Q\mathscr{P}_{\mathcal{J}}^+,$$

where $\mathscr{P}_{\mathcal{J}}^+ = \langle\{x_{j_1}^{t_1}x_{j_2}^{t_2}\cdots x_{j_r}^{t_r} \in \mathscr{P}_{\mathcal{J}} \mid t_1t_2\cdots t_r > 0,\, 1 \leq r \leq d-1\}\rangle$. It can be seen that $\dim(Q\mathscr{P}_{\mathcal{J}}^+)_n = \dim(Q\mathscr{P}_r^+)_n$ for all n, and that $\binom{d}{r}$ is the number of the sequences $\mathcal{J}$ of length r. Therefore, we claim

$$\dim(Q\mathscr{P}_d^0)_n = \sum_{1\leq r\leq d-1} \binom{d}{r} \dim(Q\mathscr{P}_r^+)_n.$$

3. Generators of the $\mathcal{A}_2$-module $\mathscr{P}_5$ in the generic degree $5(2^t-1)+8.2^t$

In this section we study the structure of $Q\mathscr{P}_5$ in degree $5(2^t-1)+8.2^t$ for t a positive integer. More explicitly, we will prove Theorem 1.1 as given at the beginning. We first review some homomorphisms and related results, Sum's conjecture [56] and Singer's criterion on $\mathcal{A}_2$-decomposable [48].

3.1. Singer's criterion on $\mathcal{A}_2$-decomposable

Definition 3.1.1. A monomial $z = x_1^{t_1}x_2^{t_2}\cdots x_d^{t_d}$ in $\mathscr{P}_d$ is called a *spike* if $t_j = 2^{a_j} - 1$ for a_j a non-negative integer and $1 \leq j \leq d$. If z is a spike with $a_1 > a_2 > \cdots > a_{r-1} \geq a_r > 0$ and $a_j = 0$ for $j > r$, then it is called a *minimal spike*.

Proposition 3.1.2 (see [38, 43]). *All the spikes in $\mathscr{P}_d$ are admissible and their weight vectors are weakly decreasing. Furthermore, if a weight vector $\omega = (\omega_1, \omega_2, \ldots)$ is weakly decreasing and $\omega_1 \leq d$, then there is a spike z in $\mathscr{P}_d$ such that $\omega(z) = \omega$.*

We refer the reader to [38] for the detailed proofs of the proposition. Singer showed in [49] that if $\mu(n) \leq d$, then there exists uniquely a minimal spike of degree n in $\mathscr{P}_d$. Further, we have the following, which is one of the important keys for examining the hit monomials in generic degrees.

Theorem 3.1.3 (Singer [49]). *Suppose that $X \in \mathscr{P}_d$ is a monomial of degree n, where $\mu(X) \leq d$. Let z be the minimal spike of degree n in $\mathscr{P}_d$. If $\omega(X) < \omega(z)$, then X is $\mathcal{A}_2$-decomposable.*

3.2. Some homomorphisms and Sum's conjecture

For $1 \leq k \leq d$, we define the map $\rho_{(k,\,d)} : \mathscr{P}_{d-1} \to \mathscr{P}_d$ of $\mathbb{Z}/2$-algebras by setting

$$\rho_{(k,\,d)}(x_j) = \begin{cases} x_j & \text{if } 1 \leq j < k, \\ x_{j+1} & \text{if } k \leq j < d. \end{cases}$$

We consider the following set

$$\mathcal{N}_d := \{(k; \mathscr{K}) \mid \mathscr{K} = (k_1, k_2, \ldots, k_r), 1 \leqslant k < k_1 < k_2 < \cdots < k_r \leq d,\ 0 \leq r < d\},$$

where by convention, $\mathscr{K} = \emptyset$, if $r = 0$. Denote by $r = \ell(\mathscr{K})$ the length of $\mathscr{K}$. For any $(k; \mathscr{K}) \in \mathcal{N}_d$, we have the projections (see [54]) $\pi_{(k;\mathscr{K})} : \mathscr{P}_d \to \mathscr{P}_{d-1}$, which are determined by

$$\pi_{(k;\mathscr{K})}(x_j) = \begin{cases} x_j & \text{if } 1 \leq j < k, \\ \sum_{p\in\mathscr{K}} x_{p-1} & \text{if } j = k, \\ x_{j-1} & \text{if } k < j \leq d. \end{cases}$$

Note that $\rho_{(k,\,d)}$ and $\pi_{(k;\mathscr{K})}$ are also the homomorphisms of the $\mathcal{A}_2$-modules. In particular, we have $\pi_{(k;\emptyset)}(x_k) = 0$ for $1 \leq k \leq d$ and $\pi_{(k;\mathscr{K})}(\rho_{(k,\,d)}(u)) = u$ for any $u \in \mathscr{P}_{d-1}$.

Proposition 3.2.1 (see [43]). *If x is a monomial in $\mathscr{P}_d$, then $\pi_{(k;\mathscr{K})}(x) \in \mathscr{P}_{d-1}(\omega(x))$.*

This result implies that if ω is a weight vector and $x \in \mathscr{P}_d(\omega(x))$, then $\pi_{(k;\mathscr{K})}(x) \in \mathscr{P}_{d-1}(\omega)$. Furthermore, $\pi_{(k;\mathscr{K})}$ passes to a homomorphism from $Q\mathscr{P}_d(\omega)$ to $Q\mathscr{P}_{d-1}(\omega)$.

Let $(k;\mathscr{K}) \in \mathcal{N}_d,\ 0 < r < d$, and let

$$x_{(\mathscr{K},\,u)} = x_{k_u}^{2^{r-1}+2^{r-2}+\cdots+2^{r-u}} \prod_{u<m\leq r} x_{k_m}^{2^{r-m}}$$

for $1 \leq u \leq r,\ x_{(\emptyset,1)} = 1$. In [54], Sum defined a $\mathbb{Z}/2$-linear transformation $\phi_{(k;\mathscr{K})} : \mathscr{P}_{d-1} \to \mathscr{P}_d$, which is determined by

$$\phi_{(k;\mathscr{K})}(x) = \begin{cases} \rho_{(k,\,d)}(x) & \text{if } \mathscr{K} = \emptyset, \\ \dfrac{x_k^{2^r-1}\rho_{(k,\,d)}(x)}{x_{(\mathscr{K},\,u)}} & \text{if there exists } u \text{ such that,} \\ & t_{k_1-1} = t_{k_2-1} = \cdots = t_{k_{(u-1)}-1} = 2^r - 1, \\ & t_{k_u-1} > 2^r - 1, \\ & \alpha_{r-m}(t_{k_u-1}) = 1,\ \forall m,\ 1 \leq m \leq u, \\ & \alpha_{r-m}(t_{k_m-1}) = 1,\ \forall m,\ u < m \leq r, \\ 0 & \text{otherwise,} \end{cases}$$

for any $x = x_1^{t_1}x_2^{t_2}\cdots x_{d-1}^{t_{d-1}}$ in $\mathscr{P}_{d-1}$. Note that $\phi_{(k;\mathscr{K})}$ is not an $\mathcal{A}_2$-homomorphism in general. Moreover, for each $x \in \mathscr{P}_{d-1}$, if $\phi_{(k;\mathscr{K})}(x) \neq 0$, then $\omega(\phi_{(k;\mathscr{K})}(x)) = \omega(x)$.

From now on, we adopt the following notations: For a natural number d, we consider $\Gamma_d = \{1, 2, \ldots, d\},\ X_{(S,\,d)} = X_{(\{s_1,s_2,\ldots,s_r\},\,d)} = \prod_{s\in\Gamma_d\setminus S} x_s$, where $S = \{s_1, s_2, \ldots, s_r\} \subseteq \Gamma_d$. In particular, $X_{(\Gamma_d,d)} = 1,\ X_{(\emptyset,d)} = x_1x_2\cdots x_d$, $X_{(\{s\},d)} = x_1\cdots\hat{x}_s\cdots x_d,\ 1 \leq s \leq d$. Now consider $X = x_1^{t_1}x_2^{t_2}\cdots x_d^{t_d} \in \mathscr{P}_d$ and let $S_j(X) = \{s \in \Gamma_d :\ \alpha_j(t_s) = 0\}$ for $j \geq 0$. Then, by a simple computation, we get $X = \prod_{j\geq 0} X_{(S_j(X),\,d)}^{2^j}$.

The following examples on the map $\phi_{(k;\mathscr{K})}$ can be found in [54]. However, we present them in more detail.

(i) Let $\mathscr{K} = (j)$ and $1 \leq k < j \leq d$. Then, for any the monomial $x = x_1^{a_1}x_2^{a_2}\cdots x_{d-1}^{a_{d-1}} \in \mathscr{P}_{d-1}$ and $\alpha_0(a_{j-1}) = 1$, we conclude $\phi_{(k;\mathscr{K})}(x) = \frac{x_k\rho_{(k,\,d)}(x)}{x_j}$.

(ii) Let m be a positive integer and let $x = Y^{2^m-1}y^{2^m}$ with $y = x_1^{b_1}x_2^{b_2}\cdots x_{d-1}^{b_{d-1}}$ and $Y = X_{(\{d\},d)} = x_1x_2\cdots x_{d-1} \in \mathscr{P}_{d-1}$. Then if $m > r = \ell(\mathscr{K})$ and $u = 1$ then

$$\begin{aligned} \phi_{(k;\mathscr{K})}(x) &= \phi_{(k;\mathscr{K})}(Y^{2^m-1})(\rho_{(k,d)}(y))^{2^m} \\ &= x_k^{2^r-1} \prod_{1\leqslant t\leqslant r} x_{k_t}^{2^m-2^{r-t}-1} X_{(\{k,k_1,\ldots,k_r\},d)}^{2^m-1}(\rho_{(k,d)}(y))^{2^m}. \end{aligned}$$

Indeed, since $\rho_{(k,d)}$ is a $\mathbb{Z}/2$-algebras homomorphism,

$$\rho_{(k,d)}(x) = \rho_{(k,d)}(X^{2^m-1})(\rho_{(k,d)}(y))^{2^m},\ 1 \leq k \leq d.$$

Since $Y^{2^m-1} = x_1^{2^m-1}\cdots x_{d-1}^{2^m-1}$ and $2^d-1 > 2^r-1$, for each $(k;\mathscr{K})$, $\mathscr{K} = (k_1,k_2,\ldots,k_r)$ and $u=1$, we have

$$\begin{aligned}
&\phi_{(k;\mathscr{K})}(Y^{2^m-1})\\
&= \frac{x_k^{2^r-1}x_2^{2^m-1}\cdots x_{k_1}^{2^m-1}\cdots x_{k_r}^{2^m-1}\cdots x_d^{2^m-1}}{x_{k_1}^{2^{r-1}}x_{k_2}^{2^{r-2}}\cdots x_{k_r}^{2^{r-r}}}\\
&= x_k^{2^r-1}x_{k_1}^{2^m-2^{r-1}-1}\cdots x_{k_2}^{2^m-2^{r-2}-1}\cdots x_{k_r}^{2^m-2^{r-r}-1}X_{(\{k,k_1,\ldots,k_r\},d)}^{2^m-1}\\
&= x_k^{2^r-1}\prod_{1\leqslant t\leqslant r}x_{k_t}^{2^m-2^{r-t}-1}X_{(\{k,k_1,\ldots,k_r\},d)}^{2^m-1}.
\end{aligned}$$

Then, one gets

$$\begin{aligned}
\phi_{(k;\mathscr{K})}(x) &= \phi_{(k;\mathscr{K})}(Y^{2^m-1}y^{2^m})\\
&= \frac{x_k^{2^r-1}\rho_{(k,d)}(X^{2^m-1}y^{2^m})}{x_{(\mathscr{K};1)}}\\
&= \left(\frac{x_k^{2^r-1}\rho_{(k,d)}(Y^{2^m-1})}{x_{(\mathscr{K};1)}}\right)(\rho_{(k,d)}(y))^{2^m}\\
&= \phi_{(k;\mathscr{K})}(X^{2^m-1})(\rho_{(k,d)}(y))^{2^m}\\
&= x_k^{2^r-1}\prod_{1\leqslant t\leqslant r}x_{k_t}^{2^m-2^{r-t}-1}X_{(\{k,k_1,\ldots,k_r\},d)}^{2^m-1}(\rho_{(k,d)}(y))^{2^m}.
\end{aligned}$$

Now, if $m=r$, $b_{j-1}=0$, $j=k_1,k_2,\ldots,k_{u-1}$ and $b_{k_u-1}>0$, then for each $(k;\mathscr{K})$ and $1\le u\le r=m$, we deduce

$$\phi_{(k;\mathscr{K})}(x) = \phi_{(k_u;\{k_{u+1},\ldots,k_m\})}(Y^{2^m-1})(\rho_{(k,d)}(y))^{2^m}.$$

Indeed, we have

$$\begin{aligned}
&Y^{2^m-1}y^{2^m}\\
&= (x_1\cdots x_{d-1})^{2^m-1}(x_1^{b_1}\cdots \hat{x}_{k_1}^{b_{k_1}}\cdots \hat{x}_{k_{(u-1)}-1}^{b_{k_{(u-1)}-1}}x_{k_u-1}^{b_{k_u-1}}\cdots x_{d-1}^{b_{d-1}})^{2^m}\\
&= (x_1\cdots x_{k_1}\cdots x_{k_{u-1}-1}x_{k_u-1}\cdots x_{d-1})^{2^m-1}\\
&\quad\times (x_1^{b_1}\cdots \hat{x}_{k_1}^{b_{k_1}}\cdots \hat{x}_{k_{(u-1)}-1}^{b_{k_{(u-1)}-1}}x_{k_u-1}^{b_{k_u-1}}\cdots x_{d-1}^{b_{d-1}})^{2^m}.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\phi_{(k;\mathscr{K})}(x)\\
&= \frac{x_k^{2^m-1}\rho_{(k,d)}(Y^{2^m-1})}{x_{k_u}^{2^{m-1}+2^{m-2}+\cdots+2^{m-u}}\prod_{u<t\leqslant r}x_{k_t}^{2^{m-t}}}(\rho_{(k,d)}(y))^{2^m}
\end{aligned}$$

$$= \frac{x_k^{2^m-1}x_2^{2^m-1}\cdots x_{k_{u-1}}^{2^m-1}x_{k_u}^{2^m-1}\prod\limits_{u+1\leqslant t\leqslant m}x_{k_t}^{2^m-1}\cdots x_d^{2^m-1}}{x_{k_u}^{2^{m-1}+2^{m-2}+\cdots+2^{m-u}}\prod\limits_{u<t\leqslant m}x_{k_t}^{2^{m-t}}}(\rho_{(k,d)}(y))^{2^m}$$

$$= x_k^{2^m-1}x_2^{2^m-1}\cdots x_{k_{u-1}}^{2^m-1}x_{k_u}^{2^{m-(u+1)}+\cdots+2^{m-m}}\cdots x_d^{2^m-1}$$

$$\prod_{u+1\leqslant t\leqslant m}x_{k_t}^{2^m-2^{m-t}-1}\cdots x_d^{2^m-1}(\rho_{(k,d)}(y))^{2^m}$$

$$= \left(x_{k_u}^{2^{m-(u+1)}+\cdots+2^{m-m}}\prod_{u+1\leqslant t\leqslant m}x_{k_t}^{2^m-2^{m-t}-1}X_{(\{k_u,k_{u+1},\ldots,k_m\},d)}^{2^m-1}\right)(\rho_{(k,d)}(y))^{2^m}$$

$$= \phi_{(k_u;\{k_{u+1},\ldots,k_m\})}(Y^{2^m-1})(\rho_{(k,d)}(y))^{2^m}.$$

We end this subsection by reviewing Sum's conjecture [56] on the relation between the admissible monomials for the polynomial algebras.

For a subset $\mathcal{U}\subset\mathscr{P}_{d-1}$, we denote

$$\overline{\Phi}^0(\mathcal{U}) = \bigcup_{1\leq k\leq d}\phi_{(k;\emptyset)}(\mathcal{U}) = \bigcup_{1\leq k\leq d}\rho_{(k,\,d)}(\mathcal{U}),$$

$$\overline{\Phi}^+(\mathcal{U}) = \bigcup_{(k;\mathscr{K})\in\mathcal{N}_d,\;0<\ell(\mathscr{K})<d}\phi_{(k;\mathscr{K})}(\mathcal{U})\setminus\mathscr{P}_d^0,$$

$$\overline{\Phi}(\mathcal{U}) = \overline{\Phi}^0(\mathcal{U})\bigcup\overline{\Phi}^+(\mathcal{U}).$$

Since $\rho_{(k,\,d)}$ is a homomorphism of the $\mathcal{A}_2$-modules, if $\mathcal{U}$ is a minimal set of generators for the $\mathcal{A}_2$-module $\mathscr{P}_{d-1}$ in degree n, then $\overline{\Phi}^0(\mathcal{U})$ is also a minimal set of generators for the $\mathcal{A}_2$-module $\mathscr{P}_d^0$ in degree n.

Now, for a polynomial $f\in\mathscr{P}_d$, we denote by $[f]$ the classes in $Q\mathscr{P}_d$ represented by f. If ω is a weight vector and $f\in\mathscr{P}_d(\omega)$, then denote by $[f]_\omega$ the classes in $Q\mathscr{P}_d(\omega)$ represented by f. For a subset $\mathscr{B}\subset\mathscr{P}_d$, we denote $[\mathscr{B}] = \{[f] : f\in\mathscr{B}\}$. If $\mathscr{B}\subset\mathscr{P}_d(\omega)$, then we set $[\mathscr{B}]_\omega = \{[f]_\omega : f\in\mathscr{B}\}$.

Denote by $\mathscr{B}_d(n)$ the set of all admissible monomials of degree n in $\mathscr{P}_d$. Thus when we write $x\in\mathscr{B}_d(n)$ we mean that it is an admissible monomial of degree n. We set

$$\mathscr{B}_d^0(n) := \mathscr{B}_d(n)\cap(\mathscr{P}_d^0)_n,\ \mathscr{B}_d^+(n) := \mathscr{B}_d(n)\cap(\mathscr{P}_d^+)_n.$$

If ω is a weight vector of degree n, we set

$$\mathscr{B}_d(\omega) := \mathscr{B}_d(n)\cap\mathscr{P}_d(\omega),\ \mathscr{B}_d^0(\omega) := \mathscr{B}_d(\omega)\cap(\mathscr{P}_d^0)_n,\ \mathscr{B}_d^+(\omega) := \mathscr{B}_d(\omega)\cap(\mathscr{P}_d^+)_n.$$

Then, $[\mathscr{B}_d(\omega)]_\omega$, $[\mathscr{B}_d^0(\omega)]_\omega$ and $[\mathscr{B}_d^+(\omega)]_\omega$ are respectively the bases of the $\mathbb{Z}/2$-vector spaces

$$Q\mathscr{P}_d(\omega),\ Q\mathscr{P}_d^0(\omega) := Q\mathscr{P}_d(\omega)\cap(Q\mathscr{P}_d^0)_n \text{ and } Q\mathscr{P}_d^+(\omega) := Q\mathscr{P}_d(\omega)\cap(Q\mathscr{P}_d^+)_n.$$

Throughout this paper, to prove a certain subset of $Q\mathscr{P}_d$ is linearly independent, we use a result in Sum [54] combining with Theorem 3.1.3 (Singer's criterion on the $\mathcal{A}_2$-decomposable) and Proposition 3.2.1. More precisely, let $\mathscr{B}$ be a finite subset of $\mathscr{P}_d$ consisting of some monomials of degree n. Denote by $|\mathscr{B}|$ the cardinal of $\mathscr{B}$. To prove the set $[\mathscr{B}]$ is linearly independent in $(Q\mathscr{P}_d)_n$, we denote the elements of $\mathscr{B}$ by $\mathcal{Y}_{n,\,i},\ 1 \le i \le m = |\mathscr{B}|$ and assume that there is a linear relation

$$\mathcal{S} = \sum_{1\le i\le m} \gamma_i \mathcal{Y}_{n,\,i} = 0 \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_d + \mathscr{P}_d^-(\omega)),$$

with $\gamma_i \in \mathbb{Z}/2$ for all i, $1 \le i \le m$. For $(k;\mathscr{K}) \in \mathcal{N}_d$, we explicitly compute $\pi_{(k;\mathscr{K})}(\mathcal{S})$ in terms of the admissible monomials in $\mathscr{P}_{d-1}$ (modulo$(\mathcal{A}_2^+\mathscr{P}_{d-1} + \mathscr{P}_{d-1}^-(\omega))$). Computing from some relations $\pi_{(k;\mathscr{K})}(\mathcal{S}) = 0$ modulo$(\mathcal{A}_2^+\mathscr{P}_{d-1} + \mathscr{P}_{d-1}^-(\omega))$ with $(k;\mathscr{K}) \in \mathcal{N}_d$, we obtain $\gamma_i = 0$ for all i.

In [56], Sum made the following conjecture, which plays an important role in studying the minimal set of $\mathcal{A}_2$-module $\mathscr{P}_d$ in certain generic degree.

Conjecture 3.2.2 (Sum [56])**.** *If ω is a weight vector, then $\overline{\Phi}(\mathscr{B}_{d-1}(\omega)) \subseteq \mathscr{B}_d(\omega)$.*

Obviously, if this conjecture is true, then $\overline{\Phi}(\mathscr{B}_{d-1}(n)) \subseteq \mathscr{B}_d(n)$ for any positive integer n. In other words, if $x \in \mathscr{B}_{d-1}(n)$, then $\phi_{(k;\mathscr{K})}(x) \in \mathscr{B}_d(n)$. By previous results of Peterson [34], Kameko [18] and Sum [54], the conjecture is true for $d \le 4$. In particular, we have the remark below.

Remark 3.2.3. Consider the spike monomial $Y = X_{(\{d\},d)} = x_1x_2\cdots x_{d-1} \in \mathscr{P}_{d-1}$. Let m be a positive integer such that $m > r = \ell(\mathscr{K})$. Then, from the above calculations, we have

$$\phi_{(k;\mathscr{K})}(Y^{2^m-1}) = x_k^{2^r-1}\prod_{1\le t\le r} x_{i_t}^{2^m-2^{r-t}-1} X^{2^m-1}_{(\{k,k_1,\ldots,k_r\},d)}.$$

It is easy to see that $\omega(Y) = \underbrace{(d-1, d-1, \ldots, d-1)}_{m \text{ times of } (d-1)}$. Based on the results in [44,57], the set $\{\phi_{(k;\mathscr{K})}(Y^{2^m-1}) : (k;\mathscr{K}) \in \mathcal{N}_d\}$ is a basis of $Q\mathscr{P}_d(\omega(Y))$. Note that this also holds true for $m \le r$ (see Sum [57]). By Proposition 3.1.2, Y^{2^m-1} is admissible. These data imply that Sum's conjecture is true for the weight vector $\omega(Y)$, where d is an arbitrary positive integer.

In [56], Sum showed that Conjecture 3.2.2 is true for $d = 5$ and any weight vector of generic degree of (1.1) with $r = d = 5$, $s = 10$ and $t \ge 0$. In the next subsection, we will show that this conjecture is also satisfying for $d = 5$ and in generic degree of Theorem 1.1.

3.3. Proof of Theorem 1.1

As shown in Section 1, we have $\mu(13.2^t - 5) = 5$ for every $t > 2$, hence the inverse function $\widetilde{\phi} : (Q\mathscr{P}_5)_{13.2^{t-1}-5} \to (Q\mathscr{P}_5)_{13.2^t-5}$ of $(\widetilde{Sq^0_*})_{(5,13.2^t-5)}$ defined by $\widetilde{\phi}([u]) = [X_{(\emptyset,5)}u^2]$ for all $[u] \in (Q\mathscr{P}_5)_{13.2^{t-1}-5},\ t > 2$. On the other hand, since the iterated Kameko squaring operation

$$(\widetilde{Sq^0_*})^{t-2}_{(5,13.2^t-5)} : (Q\mathscr{P}_5)_{13.2^t-5} \to (Q\mathscr{P}_5)_{13.2^2-5}$$

is a $\mathbb{Z}/2$-vector space isomorphism for every $t \geq 2$, a basis of $Q\mathscr{P}_5$ in degree $13.2^t - 5$ is the set

$$[\mathscr{B}_5(13.2^t - 5)] = \widetilde{\phi}^{t-2}([\mathscr{B}_5(13.2^2 - 5)])$$

for $t > 2$. Thus, we need only to find the minimal set of $\mathbb{Z}/2$-generators for $(Q\mathscr{P}_5)_{13.2^t-5}$ with $t \in \{0, 1, 2\}$. It has been determined by Tín [60] for $t = 0$. Note that our methods of studying $Q\mathscr{P}_5$ in this paper are different from the ones of Tín.

3.3.1. *The case* $\mathbf{t = 1}$. Consider Kameko's homomorphism

$$(\widetilde{Sq^0_*})_{(5,21)} : (Q\mathscr{P}_5)_{21} \longrightarrow (Q\mathscr{P}_5)_8.$$

We know that it is an epimorphism of $\mathbb{Z}/2$-vector spaces, hence $(Q\mathscr{P}_5)_{21} \cong \mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)} \bigoplus (Q\mathscr{P}_5)_8$. We note that $\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)}$ is isomorphic to

$$(Q\mathscr{P}^0_5)_{21} \bigoplus (\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)} \cap (Q\mathscr{P}^+_5)_{21}).$$

From the calculations of $Q\mathscr{P}_d$ in degree 21 for $1 \leq d \leq 4$ (see [18], [34], [54]) and $Q\mathscr{P}_5$ in degree 8 (see [60]), we have

$$|\mathscr{B}^+_1(21)| = 0,\ |\mathscr{B}^+_2(21)| = 0,\ |\mathscr{B}^+_3(21)| = 7,\ |\mathscr{B}^+_4(21)| = 66,\ |\mathscr{B}_5(8)| = 174.$$

Note that $(Q\mathscr{P}_3)_{21} \cong (Q\mathscr{P}_3)_3$ and $\mathscr{B}_3(21) = \mathscr{B}^+_3(21) = \widetilde{\varphi}^2(\mathscr{B}_3(3))$ with the $\mathbb{Z}/2$-linear map $\widetilde{\varphi} : \mathscr{P}_3 \to \mathscr{P}_3$, determined by $\widetilde{\varphi}(u) = X_{(\emptyset,\,3)}u^2,\ \forall u \in \mathscr{P}_3$. As well known,

$$(Q\mathscr{P}^0_5)_{21} = \bigoplus_{1\leq r\leq 4} \bigoplus_{1\leq j\leq \binom{5}{r}} (Q\mathscr{P}^+_r)_{21},$$

hence $\dim(Q\mathscr{P}^0_5)_{21} = \binom{5}{3} \cdot 7 + \binom{5}{4} \cdot 66 = 400$. Moreover, a direct computation shows that

$$\mathscr{B}^0_5(21) = \overline{\Phi}^0(\mathscr{B}_4(21)) = \{\mathcal{Y}_{21,\,i} : 1 \leq i \leq 400\},$$

where the monomials $\mathcal{Y}_{21,\,i}$, $1 \leq i \leq 400$, are listed in Section 6.2 of the online version [40].

Proposition 3.3.1. *The set* $\{[\mathcal{Y}_{21,i}] : 401 \leq i \leq 666\}$ *is the basis of the* $\mathbb{Z}/2$-*vector space* $Ker(\widetilde{Sq^0_*})_{(5,21)} \cap (Q\mathscr{P}^+_5)_{21}$. *Here the monomials* $\mathcal{Y}_i := \mathcal{Y}_{21,i},\ 401 \leq i \leq 666$*, which are determined in Section 6.3 of* [40].

Combining Proposition 3.3.1 and the above data, we deduce that the $\mathbb{Z}/2$-vector space $(Q\mathscr{P}_5)_{21}$ is 840-dimensional. This completes the proof of the theorem for the case $t = 1$.

We now need to some results for the proof of Proposition 3.3.1. First, we have the following lemma.

Lemma 3.3.2. *If $u \in \mathscr{B}_5(21)$ and $[u] \in \mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)}$, then the weight vector of u is either $\omega(u) = (3,3,1,1)$ or $\omega(u) = (3,3,3)$.*

Proof. Note that $x_1^{15}x_2^3x_3^3 \in (\mathscr{P}_5)_{21}$ is the minimal spike, and that by Proposition 3.1.2, it is an admissible monomial. Moreover, $\omega(x_1^{15}x_2^3x_3^3) = (3,3,1,1)$. Since $[u] \neq [0]$, by Theorem 3.1.3, we get either $\omega_1(u) = 3$ or $\omega_1(u) = 5$. If $\omega_1(u) = 5$, then $u = X_{(\emptyset,\,5)}y^2$ with y a monomial of degree 8 in $\mathscr{P}_5$. Since u is admissible, by Theorem 2.2.1, one gets $y \in \mathscr{B}_5(8)$. So $(\widetilde{Sq^0_*})_{(5,21)}([u]) = [y] \neq [0]$. This contradicts the fact that $[u] \in \mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)}$. Hence, $\omega_1(u) = 3$. Then, we have $u = X_{(\{i,j\},\,5)}y_1^2$ with $1 \leq i < j \leq 5$ and $y_1 \in \mathscr{B}_5(9)$. Since y_1 is admissible, according to a result in [62], we have either $\omega(y_1) = (3,1,1)$ or $\omega(y_1) = (3,3)$. The lemma is proved. $\square$

As an immediate consequence, we see that the dimension of $(\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)} \cap (Q\mathscr{P}_5^+)_{21})$ is equal to the sum of dimensions of $Q\mathscr{P}_5^+(3,3,1,1)$ and $Q\mathscr{P}_5^+(3,3,3)$. This leads us to determine the subspaces $Q\mathscr{P}_5^+(\omega)$, where the weight vectors ω are $(3,3,1,1)$ and $(3,3,3)$.

The following lemma is an immediate corollary from a result in [54].

Lemma 3.3.3. *The following monomials are strictly inadmissible:*

(i) $x_i^2x_jx_k^3x_\ell^7$, $x_i^6x_jx_k^3x_\ell^3$, $x_i^2x_j^5x_k^3x_\ell^3$, $x_i^2x_jx_k^3x_\ell^3$, *where* $1 \leq i < j \leq 5$, $1 \leq k, \ell \leq 5$, $k \neq \ell$, *and* $k, \ell \neq i, j$;

(ii) $x_i^3x_j^4x_k^3x_\ell^3$, $1 \leq i < j < k < \ell \leq 5$;

(iii) $\rho_{(k,\,5)}(X)$, $1 \leq k \leq 5$, *where X is one of the following monomials*:
$x_1^3x_2^4x_3^7x_4^7$, $x_1^3x_2^7x_3^4x_4^7$, $x_1^3x_2^7x_3^7x_4^4$, $x_1^7x_2^3x_3^4x_4^7$,
$x_1^7x_2^3x_3^7x_4^4$, $x_1^7x_2^7x_3^3x_4^4$, $x_1^7x_2^8x_3^3x_4^3$.

Lemma 3.3.4. *The following monomials are strictly inadmissible:*

(i) $x_i^2x_j^2x_kx_\ell x_m^7$, $x_i^2x_j^6x_kx_\ell x_m^3$, $x_i^6x_j^2x_kx_\ell x_m^3$, $x_i^2x_j^2x_k^5x_\ell x_m^3$, $x_i^2x_j^4x_kx_\ell^3x_m^3$, $x_i^4x_j^2x_kx_\ell^3x_m^3$, $x_i^2x_jx_k^4x_\ell^3x_m^3$, *where (i,j,k,ℓ,m) is a permutation of* $(1,2,3,4,5)$;

(ii) $x_1^3x_2^2x_3x_4x_5^6$ $x_1^3x_2^6x_3x_4x_5^2$, $x_1^3x_2^2x_3^5x_4^2x_5$, $x_1^3x_2^2x_3x_4^5x_5^2$,
$x_1^3x_2^2x_3^5x_4x_5^2$, $x_1^3x_2^5x_3^2x_4^2x_5$, $x_1^3x_2^2x_3^2x_4x_5$, $x_1^3x_2^2x_3x_4^2x_5$,
$x_1^3x_2^2x_3x_4x_5^2$.

Proof. Consider the monomials $X = x_i^2x_j^2x_kx_\ell x_m^7$ and $Y = x_1^3x_2^2x_3x_4x_5^6$. We prove that these monomials are strictly inadmissible. The others can be proved by a similar computation. Obviously, $\omega(X) = \omega(Y) = (3,3,1)$. By a direct computation using the Cartan formula, we obtain

$$\begin{aligned} X &= Sq^2(x_ix_jx_kx_\ell x_m^7) + Sq^4(x_ix_jx_kx_\ell x_m^5) + x_ix_jx_k^2x_\ell^2x_m^7 \\ &\quad + x_ix_j^2x_kx_\ell^2x_m^7 + x_ix_j^2x_k^2x_\ell x_m^7 \ \text{modulo}\,(\mathscr{P}_5^-(3,3,1)); \\ Y &= Sq^1(x_1^3x_2x_3x_4x_5^6) + x_1^3x_2x_3^2x_4x_5^6 + x_1^3x_2x_3x_4^2x_5^6 \ \text{modulo}\,(\mathscr{P}_5^-(3,3,1)). \end{aligned}$$

These equalities show that X and Y are strictly inadmissible. The lemma follows. $\square$

Lemma 3.3.5. *The following monomials are strictly inadmissible*:

$X_1 = x_1x_2^6x_3^8x_4^3x_5^3$, $X_2 = x_1x_2^3x_3^6x_4^6x_5^5$, $X_3 = x_1^3x_2^5x_3^2x_4^6x_5^5$, $X_4 = x_1x_2^6x_3^7x_4^6x_5$,
$X_5 = x_1x_2^7x_3^6x_4^6x_5$, $X_6 = x_1^7x_2x_3^6x_4^6x_5$, $X_7 = x_1x_2^2x_3^6x_4^5x_5^7$, $X_8 = x_1x_2^2x_3^6x_4^7x_5^5$,
$X_9 = x_1x_2^6x_3^2x_4^5x_5^7$, $X_{10} = x_1x_2^6x_3^2x_4^7x_5^5$, $X_{11} = x_1x_2^2x_3^7x_4^6x_5^5$, $X_{12} = x_1x_2^6x_3^7x_4^2x_5^5$,
$X_{13} = x_1x_2^7x_3^2x_4^6x_5^5$, $X_{14} = x_1x_2^7x_3^6x_4^2x_5^5$, $X_{15} = x_1^7x_2x_3^2x_4^6x_5^5$, $X_{16} = x_1^7x_2x_3^6x_4^2x_5^5$,
$X_{17} = x_1x_2^6x_3^3x_4^4x_5^7$, $X_{18} = x_1x_2^6x_3^3x_4^7x_5^4$, $X_{19} = x_1x_2^6x_3^7x_4^3x_5^4$, $X_{20} = x_1x_2^7x_3^6x_4^3x_5^4$,
$X_{21} = x_1^3x_2^4x_3x_4^6x_5^7$, $X_{22} = x_1^3x_2^4x_3x_4^7x_5^6$, $X_{23} = x_1^3x_2^4x_3^7x_4x_5^6$, $X_{24} = x_1^3x_2^7x_3^4x_4x_5^6$,
$X_{25} = x_1^7x_2x_3^6x_4^3x_5^4$, $X_{26} = x_1^7x_2^3x_3^4x_4x_5^6$, $X_{27} = x_1x_2^6x_3^6x_4^3x_5^5$, $X_{28} = x_1x_2^6x_3^3x_4^6x_5^5$,
$X_{29} = x_1x_2^6x_3^6x_4^7x_5$, $X_{30} = x_1^3x_2x_3^6x_4^6x_5^5$, $X_{31} = x_1^3x_2^5x_3^6x_4^6x_5$, $X_{32} = x_1^3x_2^4x_3^5x_4^2x_5^7$,
$X_{33} = x_1^3x_2^4x_3^5x_4^7x_5^2$, $X_{34} = x_1^3x_2^4x_3^7x_4^5x_5^2$, $X_{35} = x_1^3x_2^5x_3^4x_4^2x_5^7$, $X_{36} = x_1^3x_2^5x_3^4x_4^7x_5^2$,
$X_{37} = x_1^3x_2^7x_3^4x_4^5x_5^2$, $X_{38} = x_1^3x_2^5x_3^7x_4^4x_5^2$, $X_{39} = x_1^3x_2^7x_3^5x_4^4x_5^2$, $X_{40} = x_1^7x_2^3x_3^4x_4^5x_5^2$,
$X_{41} = x_1^7x_2^3x_3^5x_4^4x_5^2$, $X_{42} = x_1x_2^6x_3^6x_4x_5^7$, $X_{43} = x_1^3x_2^5x_3^6x_4^2x_5^5$, $X_{44} = x_1^3x_2^5x_3^6x_4^5x_5^2$,
$X_{45} = x_1^3x_2^5x_3^5x_4^2x_5^6$, $X_{46} = x_1^3x_2^5x_3^5x_4^6x_5^2$, $X_{47} = x_1^3x_2^4x_3^4x_4^3x_5^7$, $X_{48} = x_1^3x_2^4x_3^4x_4^7x_5^3$,
$X_{49} = x_1^3x_2^4x_3^3x_4^4x_5^7$, $X_{50} = x_1^3x_2^4x_3^7x_4^4x_5^3$, $X_{51} = x_1^3x_2^4x_3^3x_4^7x_5^4$, $X_{52} = x_1^3x_2^4x_3^7x_4^3x_5^4$,
$X_{53} = x_1^3x_2^7x_3^4x_4^4x_5^3$, $X_{54} = x_1^3x_2^7x_3^4x_4^3x_5^4$, $X_{55} = x_1^7x_2^3x_3^4x_4^4x_5^3$, $X_{56} = x_1^7x_2^3x_3^4x_4^3x_5^4$,
$X_{57} = x_1^3x_2^4x_3^5x_4^6x_5^3$, $X_{58} = x_1^3x_2^4x_3^5x_4^3x_5^6$, $X_{59} = x_1^3x_2^5x_3^4x_4^6x_5^3$, $X_{60} = x_1^3x_2^5x_3^6x_4^4x_5^3$,
$X_{61} = x_1^3x_2^5x_3^4x_4^3x_5^6$.

Proof. It is easily seen that $\omega(X_1) = (3,3,1,1)$ and $\omega(X_j) = (3,3,3)$ for $j = 2,3,\ldots,61$. We prove the lemma for the monomials $X_1 = x_1x_2^6x_3^8x_4^3x_5^3$, $X_2 = x_1x_2^3x_3^6x_4^6x_5^5$ and $X_3 = x_1^3x_2^5x_3^2x_4^6x_5^5$. The others can be proven by a similar computation. By a direct computation, we have

$$\begin{aligned}X_1 = {}& x_1x_2^3x_3^3x_4^6x_5^8 + x_1x_2^3x_3^3x_4^8x_5^6 + x_1x_2^3x_3^6x_4^3x_5^8 + x_1x_2^3x_3^6x_4^8x_5^3 + x_1x_2^3x_3^8x_4^3x_5^6 \\ & + x_1x_2^3x_3^8x_4^6x_5^3 + x_1x_2^4x_3^3x_4^3x_5^{10} + x_1x_2^4x_3^3x_4^{10}x_5^3 + x_1x_2^4x_3^{10}x_4^3x_5^3 \\ & + x_1x_2^6x_3^3x_4^3x_5^8 + x_1x_2^6x_3^3x_4^8x_5^3 \\ & + Sq^1(A_1) + Sq^2(A_2) + Sq^4(A_4) \text{ modulo}\,(\mathscr{P}_5^-(3,3,1,1)),\end{aligned}$$

where

$$\begin{aligned}A_1 &= x_1^2x_2^3x_3^5x_4^5x_5^5 + x_1^2x_2^5x_3^3x_4^5x_5^5 + x_1^2x_2^5x_3^5x_4^3x_5^5 + x_1^2x_2^5x_3^5x_4^5x_5^3, \\ A_2 &= x_1x_2^3x_3^3x_4^6x_5^6 + x_1x_2^3x_3^5x_4^5x_5^5 + x_1x_2^3x_3^6x_4^3x_5^6 + x_1x_2^3x_3^6x_4^6x_5^3 + x_1x_2^5x_3^3x_4^5x_5^5 \\ &\quad + x_1x_2^5x_3^5x_4^3x_5^5 + x_1x_2^5x_3^5x_4^5x_5^3 + x_1x_2^6x_3^3x_4^3x_5^6 + x_1x_2^6x_3^3x_4^6x_5^3 + x_1x_2^6x_3^6x_4^3x_5^3, \\ A_4 &= x_1x_2^4x_3^3x_4^3x_5^6 + x_1x_2^4x_3^3x_4^6x_5^3 + x_1x_2^4x_3^6x_4^3x_5^3.\end{aligned}$$

This relation implies that X_1 is strictly inadmissible. By a similar technique, we obtain

$$\begin{aligned}X_2 = Sq^1(B_1) + Sq^2(B_2) + Sq^8(B_8) + x_1x_2^3x_3^5x_4^6x_5^6 \\ + x_1x_2^3x_3^6x_4^5x_5^6 \text{ modulo}\,(\mathscr{P}_5^-(3,3,3)),\end{aligned}$$

where

$$\begin{aligned}B_1 &= x_1^2x_2^5x_3^3x_4^5x_5^5 + x_1^2x_2^5x_3^5x_4^3x_5^5 + x_1^2x_2^5x_3^5x_4^5x_5^3, \\ B_2 &= x_1x_2^3x_3^3x_4^6x_5^6 + x_1x_2^3x_3^5x_4^5x_5^5 + x_1x_2^3x_3^6x_4^3x_5^6 + x_1x_2^6x_3^3x_4^3x_5^6 \\ &\quad + x_1x_2^6x_3^3x_4^6x_5^3 + x_1x_2^6x_3^6x_4^3x_5^3,\end{aligned}$$

$$B_8 = x_1x_2^3x_3^3x_4^3x_5^3,$$
$$X_3 = Sq^1(C_1) + Sq^2(C_2) + Sq^8(C_8) + x_1^3x_2^5x_3x_4^6x_5^6 + x_1^3x_2^5x_3^2x_4^5x_5^6 \text{ modulo} (\mathscr{P}_5^-(3,3,3)),$$

where

$$C_1 = x_1^3x_2^5x_3x_4^5x_5^6 + x_1^3x_2^5x_3x_4^6x_5^5 + x_1^5x_2^3x_3^2x_4^5x_5^5 + x_1^5x_2^5x_3^2x_4^3x_5^5 + x_1^5x_2^5x_3^2x_4^5x_5^3,$$
$$C_2 = x_1^3x_2^3x_3x_4^6x_5^6 + x_1^3x_2^5x_3x_4^5x_5^5 + x_1^3x_2^6x_3x_4^3x_5^6 + x_1^3x_2^6x_3x_4^6x_5^3 + x_1^6x_2^3x_3x_4^3x_5^6 + x_1^6x_2^3x_3x_4^6x_5^3 + x_1^6x_2^6x_3x_4^3x_5^3,$$
$$C_8 = x_1^3x_2^3x_3x_4^3x_5^3.$$

The lemma is proved. □

Now we denote by $\mathcal{C}$ the set of the following monomials:

$$\begin{array}{llll} x_1^3x_2^{12}x_3x_4^2x_5^3 & x_1^3x_2^{12}x_3x_4^3x_5^2, & x_1^3x_2^{12}x_3^3x_4x_5^2, & x_1^3x_2^4x_3x_4^2x_5^{11}, \\ x_1^3x_2^4x_3x_4^{11}x_5^2, & x_1^3x_2^4x_3^{11}x_4x_5^2, & x_1^7x_2^8x_3x_4^2x_5^3, & x_1^7x_2^8x_3x_4^3x_5^2, \\ x_1^7x_2^8x_3^3x_4x_5^2, & x_1^3x_2^4x_3x_4^3x_5^{10}, & x_1^3x_2^4x_3x_4^{10}x_5^3, & x_1^3x_2^4x_3^3x_4x_5^{10}, \\ x_1^3x_2^4x_3^3x_4^9x_5^2, & x_1^3x_2^4x_3^9x_4^2x_5^3, & x_1^3x_2^4x_3^9x_4^3x_5^2. & \end{array}$$

A direct computation shows that $\overline{\Phi}^+(\mathscr{B}_4(3,3,1,1)) \cup \mathcal{C}$ is the set of 196 monomials: $\mathcal{Y}_j := \mathcal{Y}_{21,j}$, $401 \leq j \leq 596$ (see Section 6.3 of the online version [40]).

Proposition 3.3.6. *Under the above notations, the $\mathbb{Z}/2$-vector space*

$$Q\mathscr{P}_5^+(3,3,1,1)$$

is spanned by the set

$$[\overline{\Phi}^+(\mathscr{B}_4(3,3,1,1)) \cup \mathcal{C}].$$

Proof. Let X be an admissible monomial in $\mathscr{P}_5$ such that $\omega(X) = (3,3,1,1)$. Then $X = X_{(\{k,\ell\},5)}Y^2$ with $1 \leq k < \ell \leq 5$ and Y a monomial of degree 9 in $\mathscr{P}_5$. Since X is admissible, according to Theorem 2.2.1, $Y \in \mathscr{B}_5(3,1,1)$.

A direct computation shows that if $z \in \mathscr{B}_5^+(3,1,1)$, $1 \leq k < \ell \leq 5$, and $X_{(\{k,\ell\},5)}z^2 \neq \mathcal{Y}_j$, $\forall j$, $401 \leq j \leq 596$, then there exists a monomial w which is given in one of Lemmas 3.3.3-3.3.5 such that $X_{(\{k,\ell\},5)}z^2 = wz_1^{2^a}$ with a monomial $z_1 \in \mathscr{P}_5$ and $a = \max\{m \in \mathbb{Z} : \omega_m(w) > 0\}$. By Theorem 2.2.1, $X_{(\{k,\ell\},5)}z^2$ is inadmissible. Since $X = X_{(\{k,\ell\},5)}Y^2$ with $Y \in \mathscr{B}_5(3,1,1)$ and X is admissible, one can see that $X = \mathcal{Y}_j$, $401 \leq j \leq 596$. The lemma follows. □

By a direct computation, we see that

$$\overline{\Phi}^+(\mathscr{B}_4(3,3,3)) \cup \{x_1^3x_2^4x_3x_4^5x_5^6,\ x_1^3x_2^5x_3x_4^6x_5^4,\ x_1^3x_2^5x_3^6x_4^3x_5^4\}$$

is the set consisting of 70 monomials: $\mathcal{Y}_t := \mathcal{Y}_{21,t}$, $597 \leq t \leq 666$ (see Section 6.3 of the online version [40]).

Proposition 3.3.7. *The $\mathbb{Z}/2$-vector space $Q\mathscr{P}_5^+(3,3,3)$ is spanned by the set* $\{[\mathcal{Y}_t] : 597 \leq t \leq 666\}$.

Proof. Let u be an admissible monomial in $\mathscr{P}_5$ such that $\omega(u) = (3,3,3)$. Then $u = x_ix_jx_\ell y^2$ with $1 \le i < j < \ell \le 5$ and $y \in \mathscr{B}_5(3,3)$.

By a direct computation, we can verify that for any $X \in \mathscr{B}_5(3,3)$, $1 \le i < j < \ell \le 5$, such that $x_ix_jx_\ell X^2 \neq \mathcal{Y}_t$, $\forall t$, $597 \le t \le 666$, there is a monomial z which is given in one of Lemmas 3.3.3-3.3.5 such that $x_ix_jx_\ell X^2 = zw^{2^b}$ with suitable monomial $w \in \mathscr{P}_5$ and $b = \max\{r \in \mathbb{Z} : \omega_r(z) > 0\}$. Then, according to Theorem 2.2.1, $x_ix_jx_\ell X^2$ nadmissible. Since $u = x_ix_jx_\ell y^2$ is admissible and $y \in \mathscr{B}_5(3,3)$, one gets $u = \mathcal{Y}_t$, for some t. This proves the proposition. □

Proof of Proposition 3.3.1. From Propositions 3.3.6 and 3.3.7, the space $\mathrm{Ker}((\widetilde{Sq^0_*})_{(5,21)}) \cap (Q\mathscr{P}_5^+)_{21}$ is spanned by the set $\{[\mathcal{Y}_i := \mathcal{Y}_{21,i}] : 401 \le i \leqslant 666\}$. Furthermore, this set is linearly independent in $(Q\mathscr{P}_5)_{21}$. Indeed, suppose there is a linear relation

$$\mathcal{S} = \sum_{401 \le i \leqslant 666} \gamma_i \mathcal{Y}_i = 0 \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5),$$

where $\gamma_i \in \mathbb{Z}/2$. Based on Theorem 3.1.3 and Proposition 3.2.1, for $(k;\mathscr{K}) \in \mathcal{N}_5$, we explicitly compute $\pi_{(k;\mathscr{K})}(\mathcal{S})$ in terms of a given minimal set of $\mathcal{A}_2$-generators in $\mathscr{P}_4$ (modulo$(\mathcal{A}_2^+\mathscr{P}_4)$). Computing directly from the relations $\pi_{(k;p)}(\mathcal{S}) \equiv 0$, $1 \le k < p \le 5$, $\pi_{(1,(2;j))}(\mathcal{S}) \equiv 0$, $j = 3,4,5$, and $\pi_{(1,(3;4))}(\mathcal{S}) \equiv 0$, we obtain $\gamma_i = 0$, $\forall i$, $401 \le i \le 666$. This finishes the proof. □

3.3.2. *The case* $\boldsymbol{t = 2}$. Note that $13.2^2 - 5 = 47$ and $\mu(47) = 3 < 5$. Since Kameko's operation

$$(\widetilde{Sq^0_*})_{(5,47)} : (Q\mathscr{P}_5)_{47} \to (Q\mathscr{P}_5)_{21}$$

is an epimorphism of the $\mathbb{Z}/2$-vector spaces, hence

$$(Q\mathscr{P}_5)_{47} \cong \mathrm{Ker}((\widetilde{Sq^0_*})_{(5,47)}) \bigoplus (Q\mathscr{P}_5)_{21}.$$

This implies that we need only to determine $\mathrm{Ker}((\widetilde{Sq^0_*})_{(5,47)})$.

Remark 3.3.8. If $Y \in \mathscr{B}_5(47)$ and $[Y] \in \mathrm{Ker}((\widetilde{Sq^0_*})_{(5,47)})$, then $\omega_1(Y) = 3$.

Indeed, we see that $z = x_1^{31}x_2^{15}x_3$ is the minimal spike of degree 47 in $\mathscr{P}_5$. By Proposition 3.1.2, $z \in \mathscr{B}_5(47)$. Since $[Y] \neq 0$, by Theorem 3.1.3, either $\omega_1(Y) = 3$ or $\omega_1(Y) = 5$. If $\omega_1(Y) = 5$, then $Y = X_{(\emptyset,5)}Z^2$ with Z a monomial of degree 21 in $\mathscr{P}_5$. Since Y is admissible, by Theorem 2.2.1, $Z \in \mathscr{B}_5(21)$. So, we have $(\widetilde{Sq^0_*})_{(5,47)}([Y]) = [Z] \neq [0]$. This contradicts the face that $[Y] \in \mathrm{Ker}((\widetilde{Sq^0_*})_{(5,47)})$; hence we get $\omega_1(Y) = 3$.

From Remark 3.3.8, we have $Y = x_kx_\ell x_m g^2$ with $1 \le k < \ell < m \le 5$ and $g \in \mathscr{B}_5(22)$. Thus, to determine $\mathrm{Ker}((\widetilde{Sq^0_*})_{(5,47)})$, we need to compute all the admissible monomials of degree 22 in the $\mathcal{A}_2$-module $\mathscr{P}_5$.

Computation of $(Q\mathscr{P}_5)_{22}$

We consider the following weight vectors:

$$\begin{aligned}\omega_{(1)} &= (2,2,2,1),\\ \omega_{(2)} &= (2,4,1,1),\\ \omega_{(3)} &= (2,4,3),\\ \omega_{(4)} &= (4,3,1,1),\\ \omega_{(5)} &= (4,3,3).\end{aligned}$$

It is easy to see that $\deg\omega_{(i)} = 22,\ 1 \leq i \leq 5$. By Proposition 3.1.2, $x_1^{15}x_2^7$ is the minimal spike in $\mathscr{B}_5(22)$ and $\omega(x_1^{15}x_2^7) = \omega_{(1)}$. Let u be an admissible monomial of degree 22 in $\mathscr{P}_5$. Then $[u] \neq [0]$ and by Theorem 3.1.3, either $\omega_1(u) = 2$ or $\omega_1(u) = 4$. Since $u \in \mathscr{B}_5(22)$, by Theorem 2.2.1, if $\omega_1(u) = 2$, then $u = X_{(\{i,j,k\},\,5)}y^2$ with $y \in \mathscr{B}_5(10)$ and $1 \leqslant i < j < k \leq 5$. According to Tín [61], $\omega(y)$ is one of the sequences $(2,2,1),\ (4,1,1)$, and $(4,3)$. If $\omega_1(u) = 4$, then $u = X_{(\{i\},\,5)}y_1^2$ with y_1 a monomial of degree 9 in $\mathscr{P}_5$ and $1 \leq i \leq 5$. By Tín [61], either $\omega(y_1) = (3,1,1)$ or $\omega(y_1) = (3,3)$. Hence, we have the following.

Remark 3.3.9. If $u \in \mathscr{B}_5(22)$, then $\omega(u)$ is one of the sequences $\omega_{(t)},\ 1 \leq t \leq 5$.

As it is known, $(Q\mathscr{P}_5)_{22} = (Q\mathscr{P}_5^0)_{22} \bigoplus (Q\mathscr{P}_5^+)_{22}$. By Sum [54], $Q\mathscr{P}_5^+$ has dimension 72 in degree 22. Then, based on the results in [34] and [18] with fact that $(Q\mathscr{P}_5^0)_{22} = \bigoplus_{1\leq s\leq 4}\bigoplus_{1\leq u\leq\binom{5}{u}}(Q\mathscr{P}_s^+)_{22}$, we deduce that $\dim(Q\mathscr{P}_5^0)_{22} = \binom{5}{2}.2 + \binom{5}{3}.8 + \binom{5}{4}.72 = 460$, and that

$$\mathscr{B}_5^0(22) = \overline{\Phi}^0(\mathscr{B}_4(22)) = \{\mathcal{Y}_{22,\,t} : 1 \leq t \leq 460\},$$

where the monomials $\mathcal{Y}_{22,\,t},\ 1 \leq t \leq 460$, are determined in Section 6.4 of the online version [40].

Next, we compute $(Q\mathscr{P}_5^+)_{22}$. For $r,\ k \in \mathbb{N}$ and $1 \leq k \leq 5$, we denote

$$\overline{\mathscr{B}}(k,22) := \big\{x_k^{2^r-1}\rho_{(k,\,5)}(x) \in (\mathscr{P}_5)_{22} : x \in \mathscr{B}_4(23-2^r),\ \alpha(27-2^r) \leq 4\big\}.$$

By Mothebe and Uys [26], $\overline{\mathscr{B}}(5,22) \subseteq \mathscr{B}_5(22),\ 1 \leq k \leq 5$. We set

$$\overline{\mathscr{B}}(k,\omega_{(t)}) := \overline{\mathscr{B}}(k,22) \cap \mathscr{P}_5(\omega_{(t)}),\ \overline{\mathscr{B}}^+(k,\omega_{(t)}) := \overline{\mathscr{B}}(k,\omega_{(t)}) \cap (\mathscr{P}_5^+)_{22},$$

for all $1 \leq t,\ k \leq 5$. By a simple computation, we find that

$$\overline{\Phi}^+(\mathscr{B}_4(\omega_{(1)})\bigcup\big(\bigcup_{1\leq k\leq 5}\overline{\mathscr{B}}^+(k,\omega_{(1)})\big)$$

is the set of 31 admissible monomials: $\mathcal{Y}_{22,\,i},\ 461 \leq i \leq 491$ (see Section 6.5 of the online version [40]).

Denote by $\mathcal{D}$ is the set of the following monomials:

$y_{22,\,492} = x_1x_2x_3^6x_4^6x_5^8$, $y_{22,\,493} = x_1x_2x_3^6x_4^{10}x_5^4$, $y_{22,\,494} = x_1x_2^2x_3^3x_4^4x_5^{12}$,
$y_{22,\,495} = x_1x_2^2x_3^3x_4^{12}x_5^4$, $y_{22,\,496} = x_1x_2^2x_3^4x_4^9x_5^6$, $y_{22,\,497} = x_1x_2^2x_3^5x_4^8x_5^6$,
$y_{22,\,498} = x_1x_2^3x_3^2x_4^4x_5^{12}$, $y_{22,\,499} = x_1x_2^3x_3^2x_4^{12}x_5^4$, $y_{22,\,500} = x_1x_2^3x_3^4x_4^8x_5^6$,
$y_{22,\,501} = x_1x_2^3x_3^6x_4^4x_5^8$, $y_{22,\,502} = x_1x_2^3x_3^6x_4^8x_5^4$, $y_{22,\,503} = x_1^3x_2x_3^2x_4^4x_5^{12}$,
$y_{22,\,504} = x_1^3x_2x_3^2x_4^{12}x_5^4$, $y_{22,\,505} = x_1^3x_2x_3^4x_4^8x_5^6$, $y_{22,\,506} = x_1^3x_2x_3^6x_4^4x_5^8$,
$y_{22,\,507} = x_1^3x_2x_3^6x_4^8x_5^4$, $y_{22,\,508} = x_1^3x_2^5x_3^2x_4^4x_5^8$, $y_{22,\,509} = x_1^3x_2^5x_3^2x_4^8x_5^4$,
$y_{22,\,510} = x_1^3x_2^5x_3^8x_4^2x_5^4$.

Proposition 3.3.10. $\mathscr{B}_5^+(\omega_{(1)}) = \overline{\Phi}^+(\mathscr{B}_4(\omega_{(1)})\bigcup\big(\bigcup_{1\le k\le 5}\overline{\mathscr{B}}^+(k,\omega_{(1)})\big)\bigcup\mathcal{D}$.

In order to prove the proposition, we need some lemmas.

Lemma 3.3.11. *The following monomials are strictly inadmissible*:

(i) $x_1^2x_jx_k^2x_l^3x_m^6,\ x_1^6x_jx_kx_l^2x_m^4,\ l<m,\ x_1^2x_jx_kx_l^4x_m^6$.
Here (j,k,l,m) *is a permutation of* $(2,3,4,5)$;

(ii) $x_1x_2^2x_3^4x_4^6x_5$, $x_1x_2^2x_3^6x_4x_5^4$, $x_1x_2^2x_3^6x_4^4x_5$, $x_1x_2^6x_3^2x_4x_5^4$,
$x_1x_2^6x_3^2x_4^4x_5$, $x_1^3x_2^3x_3^4x_4^2x_5^2$, $x_1^3x_2^4x_3^2x_4^2x_5^3$, $x_1^3x_2^4x_3^2x_4^3x_5^2$,
$x_1^3x_2^4x_3^3x_4^2x_5^2$, $x_1^3x_2^2x_3x_4^2x_5^6$, $x_1^3x_2^2x_3x_4^6x_5^2$, $x_1^3x_2^2x_3^2x_4x_5^6$,
$x_1^3x_2^2x_3^2x_4^6x_5$, $x_1^3x_2^2x_3^6x_4x_5^2$, $x_1^3x_2^2x_3^6x_4^2x_5$, $x_1^3x_2^6x_3x_4^2x_5^2$,
$x_1^3x_2^6x_3^2x_4x_5^2$, $x_1^3x_2^6x_3^2x_4^2x_5$, $x_1^2x_2^3x_3^3x_4^3x_5^3$, $x_1^3x_2^2x_3^3x_4^3x_5^3$,
$x_1^3x_2^3x_3^2x_4^3x_5^3$, $x_1^3x_2^3x_3^3x_4^2x_5^3$, $x_1^3x_2^3x_3^3x_4^3x_5^2$.

Proof. We prove the lemma for the monomials $u = x_1^2x_jx_k^2x_l^3x_m^6$, and $v = x_1x_2^2x_3^4x_4^6x_5$. The others can be proved by a similar computation. We have $\omega(u) = (2,4,1)$ and $\omega(v) = (2,2,2)$. By a simple computation, one gets

$$u = x_1x_j^2x_k^2x_l^3x_m^6 + Sq^1(x_1x_jx_k^2x_l^3x_m^6) \text{ modulo}(\mathscr{P}_5^-(2,4,1)),$$
$$v = x_1x_2x_3^4x_4^6x_5^2 + x_1x_2^2x_3^4x_4^5x_5^2 + Sq^1(f_1) + Sq^2(f_2) \text{ modulo}(\mathscr{P}_5^-(2,2,2)),$$

where $f_1 = x_1^2x_2x_3^4x_4^5x_5$ and $f_2 = x_1x_2x_3^4x_4^5x_5$. Hence, u and v are strictly inadmissible. The lemma follows. □

The following lemma can easily be proved by a direct computation.

Lemma 3.3.12. *If* (i,j,k,l,m) *is a permutation of* $(1,2,3,4,5)$, *then the following monomials are strictly inadmissible*:

(i) $x_i^6x_jx_k^7,\ x_i^2x_j^5x_k^7,\ x_i^3x_j^4x_k^7,\ x_i^2x_jx_k^2x_l^2x_m^7$,
$x_i^2x_j^5x_k^2x_l^2x_m^3, x_i^2x_j^4x_k^2x_l^3x_m^3, x_i^2x_jx_k^2x_l^2x_m^3,\ i<j$;

(ii) $x_ix_j^6x_k^3x_l^2x_m^2,\ j<k$,
$x_i^3x_j^6x_k^5,\ x_i^6x_j^3x_k^5,\ x_i^2x_jx_kx_l^3x_m^3,\ x_i^2x_jx_kx_l^2,\ i<j<k$;

(iii) $x_i^2x_jx_k^4x_l^3x_m^4,\ x_i^2x_j^4x_kx_l^3x_m^4,\ i<j<k,\ l<m$.

Lemma 3.3.13. *The following monomials are strictly inadmissible*:

$$\begin{array}{llll}
x_1x_2^2x_3^2x_4^7x_5^{10}, & x_1x_2^2x_3^4x_4^3x_5^{12}, & x_1x_2^2x_3^4x_4^{11}x_5^4, & x_1x_2^2x_3^7x_4^2x_5^{10},\\
x_1x_2^2x_3^7x_4^8x_5^4, & x_1x_2^2x_3^7x_4^{10}x_5^2, & x_1x_2^2x_3^{12}x_4^3x_5^4, & x_1x_2^6x_3x_4^6x_5^8,\\
x_1x_2^6x_3x_4^{10}x_5^4, & x_1x_2^6x_3^3x_4^6x_5^6, & x_1x_2^6x_3^6x_4^3x_5^6, & x_1x_2^6x_3^6x_4^6x_5^3,\\
x_1x_2^6x_3^9x_4^2x_5^4, & x_1x_2^7x_3^2x_4^2x_5^{10}, & x_1x_2^7x_3^2x_4^8x_5^4, & x_1x_2^7x_3^2x_4^{10}x_5^2,\\
x_1x_2^7x_3^8x_4^2x_5^4, & x_1x_2^7x_3^{10}x_4^2x_5^2, & x_1^3x_2^3x_3^4x_4^4x_5^8, & x_1^3x_2^3x_3^4x_4^8x_5^4,\\
x_1^3x_2^4x_3x_4^2x_5^{12}, & x_1^3x_2^4x_3x_4^{10}x_5^4, & x_1^3x_2^4x_3^4x_4^4x_5^7, & x_1^3x_2^4x_3^4x_4^5x_5^6,\\
x_1^3x_2^4x_3^4x_4^7x_5^4, & x_1^3x_2^4x_3^5x_4^4x_5^6, & x_1^3x_2^4x_3^5x_4^6x_5^4, & x_1^3x_2^4x_3^7x_4^4x_5^4,\\
x_1^3x_2^4x_3^9x_4^2x_5^4, & x_1^3x_2^5x_3^4x_4^4x_5^6, & x_1^3x_2^5x_3^4x_4^6x_5^4, & x_1^3x_2^5x_3^6x_4^4x_5^4,\\
x_1^3x_2^7x_3^4x_4^4x_5^4, & x_1^3x_2^{12}x_3x_4^2x_5^4, & x_1^7x_2x_3^2x_4^2x_5^{10}, & x_1^7x_2x_3^2x_4^8x_5^4,\\
x_1^7x_2x_3^2x_4^{10}x_5^2, & x_1^7x_2x_3^8x_4^2x_5^4, & x_1^7x_2x_3^{10}x_4^2x_5^2, & x_1^7x_2^3x_3^4x_4^4x_5^4,\\
x_1^7x_2^8x_3x_4^2x_5^4, & x_1^7x_2^9x_3^2x_4^2x_5^2. & &
\end{array}$$

Proof. We prove the lemma for the monomials $x = x_1x_2^2x_3^4x_4^3x_5^{12}$, and $y = x_1^3x_2^3x_3^4x_4^4x_5^8$. The others can be proved by a similar computation. By a direct computation using the Cartan formula, we have

$$\begin{aligned}
x &= x_1x_2^2x_3^3x_4^4x_5^{12} + x_1x_2^2x_3^4x_4^2x_5^{13} + x_1x_2^2x_3^2x_4^4x_5^{13} + x_1x_2x_3^6x_4^6x_5^8 + x_1x_2x_3^6x_4^4x_5^{10}\\
&\quad + x_1x_2x_3^4x_4^6x_5^{10} + Sq^1(f_1) + Sq^2(f_2) + Sq^4(f_4) \text{ modulo}(\mathscr{P}_5^-(\omega_{(1)})),
\end{aligned}$$

where

$$\begin{aligned}
f_1 &= x_1x_2^4x_3^3x_4^4x_5^9 + x_1x_2^4x_3^3x_4^5x_5^8 + x_1x_2^4x_3^4x_4^3x_5^9 + x_1x_2^4x_3^4x_4^5x_5^7\\
&\quad + x_1x_2^4x_3^5x_4^3x_5^8 + x_1x_2^4x_3^5x_4^4x_5^7 + x_1^2x_2x_3^4x_4^5x_5^9 + x_1^2x_2x_3^5x_4^4x_5^9\\
&\quad + x_1^2x_2x_3^5x_4^5x_5^8 + x_1^4x_2^4x_3^3x_4^3x_5^7,\\
f_2 &= x_1x_2x_3^4x_4^5x_5^9 + x_1x_2x_3^5x_4^4x_5^9 + x_1x_2x_3^5x_4^5x_5^8 + x_1x_2^2x_3^3x_4^4x_5^{10}\\
&\quad + x_1x_2^2x_3^3x_4^6x_5^8 + x_1x_2^2x_3^4x_4^3x_5^{10} + x_1x_2^2x_3^4x_4^6x_5^7 + x_1x_2^2x_3^6x_4^3x_5^8\\
&\quad + x_1x_2^2x_3^6x_4^4x_5^7 + x_1x_2^4x_3^2x_4^2x_5^{11} + x_1^2x_2^4x_3^3x_4^3x_5^8 + x_1^2x_2^4x_3^3x_4^4x_5^7\\
&\quad + x_1^2x_2^4x_3^4x_4^3x_5^7,\\
f_4 &= x_1x_2^2x_3^2x_4^2x_5^{11} + x_1x_2^2x_3^4x_4^4x_5^7.
\end{aligned}$$

The above equalities show that x is strictly inadmissible. By a similar computation, we obtain

$$\begin{aligned}
y &= x_1^2x_2x_3^2x_4^{13}x_5^4 + x_1^2x_2x_3^3x_4^{12}x_5^4 + x_1^2x_2x_3^4x_4^7x_5^8 + x_1^2x_2x_3^4x_4^{13}x_5^2 + x_1^2x_2x_3^5x_4^6x_5^8\\
&\quad + x_1^2x_2x_3^6x_4^9x_5^4 + x_1^2x_2x_3^8x_4^7x_5^4 + x_1^2x_2x_3^{12}x_4^5x_5^2 + x_1^2x_2^3x_3^5x_4^8x_5^4 + x_1^2x_2^3x_3^8x_4^5x_5^4\\
&\quad + x_1^2x_2^5x_3^2x_4^5x_5^8 + x_1^2x_2^5x_3^2x_4^9x_5^4 + x_1^2x_2^5x_3^3x_4^8x_5^4 + x_1^2x_2^5x_3^4x_4^3x_5^8 + x_1^2x_2^5x_3^4x_4^9x_5^2\\
&\quad + x_1^2x_2^5x_3^8x_4^5x_5^2 + x_1^3x_2x_3^2x_4^{12}x_5^4 + x_1^3x_2x_3^4x_4^6x_5^8 + x_1^3x_2x_3^4x_4^{10}x_5^4\\
&\quad + x_1^3x_2x_3^4x_4^{12}x_5^2 + x_1^3x_2x_3^4x_4^{12}x_5^2 + x_1^3x_2^2x_3^4x_4^9x_5^4 + x_1^3x_2^2x_3^8x_4^5x_5^4\\
&\quad + Sq^1(g_1) + Sq^2(g_2) + Sq^4(g_4) + Sq^8(x_1^3x_2^3x_3^2x_4^4x_5^2) \text{ modulo}(\mathscr{P}_5^-(\omega_{(1)})),
\end{aligned}$$

where

$$g_1 = x_1^3x_2^3x_3^2x_4^9x_5^4 + x_1^3x_2^3x_3^4x_4^9x_5^2 + x_1^3x_2^3x_3^2x_4^5x_5^8 + x_1^3x_2^3x_3^8x_4^5x_5^2 + x_1^5x_2x_3^2x_4^9x_5^4$$

$$+ x_1^3x_2x_3^4x_4^9x_5^4 + x_1^5x_2x_3^4x_4^9x_5^2 + x_1^5x_2x_3^8x_4^5x_5^2 + x_1^3x_2x_3^8x_4^5x_5^4 + x_1^5x_2x_3^6x_4^5x_5^4$$
$$+ x_1^3x_2^4x_3^5x_4^5x_5^4 + x_1^5x_2x_3^5x_4^6x_5^4 + x_1^5x_2x_3^3x_4^8x_5^4 + x_1^3x_2^3x_3^3x_4^8x_5^4 + x_1^5x_2x_3^2x_4^5x_5^8$$
$$+ x_1^5x_2x_3^4x_4^3x_5^8 + x_1^3x_2^3x_3^4x_4^3x_5^8,$$
$$g_2 = x_1^5x_2^3x_3^2x_4^8x_5^2 + x_1^5x_2^3x_3^2x_4^6x_5^4 + x_1^5x_2^3x_3^4x_4^6x_5^2 + x_1^2x_2^3x_3^2x_4^9x_5^4 + x_1^3x_2^2x_3^2x_4^9x_5^4$$
$$+ x_1^3x_2x_3^2x_4^{10}x_5^4 + x_1^6x_2x_3^2x_4^7x_5^4 + x_1^2x_2^3x_3^4x_4^9x_5^2 + x_1^3x_2^2x_3^4x_4^9x_5^2 + x_1^3x_2x_3^4x_4^{10}x_5^2$$
$$+ x_1^6x_2x_3^4x_4^7x_5^2 + x_1^2x_2^3x_3^8x_4^5x_5^2 + x_1^2x_2^3x_3^2x_4^5x_5^8 + x_1^3x_2^2x_3^8x_4^5x_5^2 + x_1^3x_2^2x_3^2x_4^5x_5^8$$
$$+ x_1^3x_2x_3^8x_4^6x_5^2 + x_1^6x_2x_3^6x_4^5x_5^2 + x_1^3x_2^2x_3^6x_4^5x_5^4 + x_1^3x_2^2x_3^5x_4^6x_5^4 + x_1^3x_2^2x_3^3x_4^8x_5^4$$
$$+ x_1^2x_2^3x_3^3x_4^8x_5^4 + x_1^6x_2x_3^2x_4^3x_5^8 + x_1^3x_2x_3^2x_4^6x_5^8 + x_1^3x_2^2x_3^4x_4^3x_5^8 + x_1^6x_2x_3^3x_4^6x_5^4$$
$$+ x_1^2x_2^3x_3^4x_4^3x_5^8 + x_1^2x_2x_3^{10}x_4^5x_5^2 + x_1^2x_2x_3^2x_4^7x_5^8 + x_1^2x_2x_3^2x_4^{11}x_5^4 + x_1^2x_2x_3^3x_4^6x_5^8$$
$$+ x_1^2x_2x_3^3x_4^{10}x_5^4 + x_1^2x_2x_3^4x_4^{11}x_5^2 + x_1^2x_2x_3^6x_4^9x_5^2 + x_1^2x_2x_3^8x_4^7x_5^2,$$
$$g_4 = x_1^3x_2^3x_3^2x_4^6x_5^4 + x_1^3x_2^3x_3^4x_4^6x_5^2 + x_1^4x_2x_3^2x_4^7x_5^4$$
$$+ x_1^4x_2x_3^4x_4^7x_5^2 + x_1^4x_2x_3^6x_4^5x_5^2 + x_1^4x_2x_3^3x_4^6x_5^4.$$

The above relations imply that y is also strictly inadmissible. The lemma is proved. □

Proof of Proposition 3.3.10. We denote by $\mathcal{Y}_t := \mathcal{Y}_{22,\,t}$, $461 \leq t \leq 510$ the admissible monomials in $\mathscr{B}_5^+(\omega_{(1)})$ (see Section 6.5 of the online version [40]). For $x \in \mathscr{B}_5^+(\omega_{(1)})$, we have $x = X_{\{i,j,k\}}y^2$ with y a monomial of degree 10 in $\mathscr{P}_5$, and $1 \leq i < j < k \leq 5$. Since x is admissible, by Theorem 2.2.1, $y \in \mathscr{B}_5(2,2,1)$.

Let $y_1 \in \mathscr{B}_5(2,2,1)$ such that $X_{(\{i,j,k\},\,5)}y_1^2 \in \mathscr{P}_5^+$. By a direct computation using a result in [62], we see that if $X_{(\{i,j,k\},\,5)}y_1^2 \neq \mathcal{Y}_t$, for all t, $461 \leq t \leq 510$, then there is a monomial w which is given in one of Lemmas 3.3.12-3.3.13 such that $X_{(\{i,j,k\},\,5)}y_1^2 = wz^{2^u}$ with suitable monomial $z \in \mathscr{P}_5$ and $u = \max\{j \in \mathbb{Z} : \omega_j(w) > 0\}$. By Theorem 2.2.1, $X_{(\{i,j,k\},\,5)}y_1^2$ is inadmissible. Since $x = X_{(\{i,j,k\},\,5)}y^2$ and x is admissible, one gets $x = \mathcal{Y}_t$. This implies $Q\mathscr{P}_5^+(\omega_{(1)})$ is spanned by the set $\{[\mathcal{Y}_t := \mathcal{Y}_{22,\,t}]_{\omega_{(1)}} : 461 \leq t \leq 510\}$.

We now prove the set $\{[\mathcal{Y}_t]_{\omega_{(1)}} : 461 \leq t \leq 510\}$ is linearly independent in $Q\mathscr{P}_5(\omega_{(1)})$. Suppose there is a linear relation

$$\mathcal{S} = \sum_{461 \leq t \leq 510} \gamma_t \mathcal{Y}_t \equiv_{\omega_{(1)}} 0, \tag{3.1}$$

where $\gamma_t \in \mathbb{Z}/2$. From a result in [54], $\dim Q\mathscr{P}_4^+(\omega_{(1)}) = 26$, with the basis $\{[u_j]_{\omega_{(1)}} : 1 \leq j \leq 26\}$, where

$u_1.\ x_1x_2x_3^6x_4^{14}$, $u_2.\ x_1x_2x_3^{14}x_4^6$, $u_3.\ x_1x_2^2x_3^4x_4^{15}$, $u_4.\ x_1x_2^2x_3^5x_4^{14}$, $u_5.\ x_1x_2^2x_3^7x_4^{12}$,
$u_6.\ x_1x_2^2x_3^{12}x_4^7$, $u_7.\ x_1x_2^2x_3^{13}x_4^6$, $u_8.\ x_1x_2^2x_3^{15}x_4^4$, $u_9.\ x_1x_2^3x_3^4x_4^{14}$, $u_{10}.\ x_1x_2^3x_3^6x_4^{12}$,
$u_{11}.\ x_1x_2^3x_3^{12}x_4^6$, $u_{12}.\ x_1x_2^3x_3^{14}x_4^4$, $u_{13}.\ x_1x_2^6x_3x_4^{14}$, $u_{14}.\ x_1x_2^7x_3^2x_4^{12}$, $u_{15}.\ x_1x_2^{14}x_3x_4^6$,
$u_{16}.\ x_1x_2^{15}x_3^2x_4^4$, $u_{17}.\ x_1^3x_2x_3^4x_4^{14}$, $u_{18}.\ x_1^3x_2x_3^6x_4^{12}$, $u_{19}.\ x_1^3x_2x_3^{12}x_4^6$, $u_{20}.\ x_1^3x_2x_3^{14}x_4^4$,
$u_{21}.\ x_1^3x_2^5x_3^2x_4^{12}$, $u_{22}.\ x_1^3x_2^5x_3^6x_4^8$, $u_{23}.\ x_1^3x_2^5x_3^{10}x_4^4$, $u_{24}.\ x_1^3x_2^{13}x_3^2x_4^4$, $u_{25}.\ x_1^7x_2x_3^2x_4^{12}$,
$u_{26}.\ x_1^{15}x_2x_3^2x_4^4$.

Consider the homomorphism $\pi_{(1;2)} : \mathscr{P}_5 \to \mathscr{P}_4$. By a direct computation using Theorem 3.1.3 and Proposition 3.2.1, we have

$$\begin{aligned}
&\pi_{(1;2)}(\mathcal{S})\\
\equiv_{\omega_{(1)}}\ & (\gamma_{473} + \gamma_{475} + \gamma_{481} + \gamma_{483})u_1 + \gamma_{477}u_6 + \gamma_{478}u_7\\
&+ (\gamma_{476} + \gamma_{477} + \gamma_{478})u_4 + \gamma_{483}(u_9 + u_{14}) + \gamma_{481}(u_{10} + u_{11})\\
&+ (\gamma_{475} + \gamma_{476} + \gamma_{477} + \gamma_{478})u_{13} + (\gamma_{470} + \gamma_{481} + \gamma_{484})u_{18}\\
&+ (\gamma_{471} + \gamma_{478} + \gamma_{481} + \gamma_{484})u_{19} + \gamma_{472}u_{20} + (\gamma_{479} + \gamma_{484})u_{21}\\
&+ (\gamma_{480} + \gamma_{481} + \gamma_{484})u_{22} + \gamma_{482}u_{23} + \gamma_{485}u_{24} + \gamma_{495}u_{25} + \gamma_{497}u_{26}\\
&+ (\gamma_{474} + \gamma_{478} + \gamma_{481} + \gamma_{484})u_2 + \gamma_{476}u_5 + (\gamma_{469} + \gamma_{475})u_{17} + \gamma_{484}u_{15}\\
\equiv_{\omega_{(1)}}\ & 0.
\end{aligned}$$

This relation implies

$$\begin{aligned}
\gamma_{469} &= \gamma_{470} = \gamma_{471} = \gamma_{472} = \gamma_{473} = \gamma_{474} = \gamma_{475} = \gamma_{476}\\
&= \gamma_{477} = \gamma_{478} = \gamma_{479} = \gamma_{480} = \gamma_{481} = \gamma_{482} = \gamma_{483}\\
&= \gamma_{484} = \gamma_{485} = \gamma_{495} = \gamma_{497} = 0.
\end{aligned} \tag{3.2}$$

Substituting (3.2) into the relation (3.1), we have

$$\sum_{461\leq t\leq 468}\gamma_t\mathcal{Y}_t + \sum_{486\leqslant t\leqslant 494}\gamma_t\mathcal{Y}_t + \gamma_{496}\mathcal{Y}_{36} + \sum_{498\leq t\leq 510}\gamma_t\mathcal{Y}_t \equiv_{\omega_{(1)}} 0. \tag{3.3}$$

Applying the homomorphisms $\pi_{(1;3)}, \pi_{(1;4)} : \mathscr{P}_5 \to \mathscr{P}_4$ to (3.3), we get

$$\begin{cases}
\gamma_t = 0,\ t \in \mathbb{J},\\
\gamma_{464} = \gamma_{487} = \gamma_{499} = \gamma_{508}, \gamma_{468} = \gamma_{494} = \gamma_{506} = \gamma_{509}, \gamma_{498} = \gamma_{502},\\
\gamma_{467} + \gamma_{468} + \gamma_{503} = \gamma_{467} + \gamma_{468} + \gamma_{505} = 0,\\
\gamma_{462} + \gamma_{468} + \gamma_{502} + \gamma_{508} = \gamma_{463} + \gamma_{468} + \gamma_{502} + \gamma_{508} = 0,\\
\gamma_{466} + \gamma_{501} + \gamma_{502} + \gamma_{508} = \gamma_{466} + \gamma_{468} + \gamma_{504} + \gamma_{507} + \gamma_{508} = 0.
\end{cases} \tag{3.4}$$

Here $\mathbb{J} = \{461, 465, 486, 488, 489, 490, 491, 492, 493, 496, 500\}$. Then, combining (3.2), (3.4), and the relation $\pi_{(1;5)}(\mathcal{S}) \equiv_{\omega_{(1)}} 0$, we obtain $\gamma_t = 0$ for $461 \leq t \leq 510$. The proposition is proved. □

Using a similar technique as mentioned in the proof of Proposition 3.3.10, we obtain:

Proposition 3.3.14. (I) $\mathscr{B}_5^+(\omega_{(2)}) = \mathscr{B}^+(5, \omega_{(2)}) \cup \mathcal{E}$, *where* $\mathcal{E}$ *is the set of the following monomials*:

$x_1x_2^2x_3^2x_4^3x_5^{14},\ x_1x_2^2x_3^3x_4^2x_5^{14},\ x_1x_2^2x_3^3x_4^6x_5^{10},\ x_1x_2^2x_3^3x_4^{14}x_5^2,\ x_1x_2^3x_3^2x_4^2x_5^{14},$
$x_1x_2^3x_3^2x_4^6x_5^{10},\ x_1x_2^3x_3^2x_4^{14}x_5^2,\ x_1x_2^3x_3^6x_4^2x_5^{10},\ x_1x_2^3x_3^6x_4^{10}x_5^2,\ x_1x_2^3x_3^{14}x_4^2x_5^2,$
$x_1^3x_2x_3^2x_4^2x_5^{14},\ x_1^3x_2x_3^2x_4^6x_5^{10},\ x_1^3x_2x_3^2x_4^{14}x_5^2,\ x_1^3x_2x_3^6x_4^2x_5^{10},\ x_1^3x_2x_3^6x_4^{10}x_5^2,$
$x_1^3x_2x_3^{14}x_4^2x_5^2,\ x_1^3x_2^5x_3^2x_4^2x_5^{10},\ x_1^3x_2^5x_3^2x_4^{10}x_5^2,\ x_1^3x_2^5x_3^{10}x_4^2x_5^2,\ x_1^3x_2^{13}x_3^2x_4^2x_5^2,$

(II) $\mathscr{B}_5^+(\omega_{(3)}) = \{x_1x_2^3x_3^6x_4^6x_5^6,\ x_1^3x_2x_3^6x_4^6x_5^6,\ x_1^3x_2^5x_3^2x_4^6x_5^6,\ x_1^3x_2^5x_3^6x_4^2x_5^6,$ $x_1^3x_2^5x_3^6x_4^6x_5^2\}$,

(III) $\mathscr{B}_5^+(\omega_{(4)}) = \overline{\Phi}^+(\mathscr{B}_4(\omega_{(4)}) \bigcup \big(\bigcup_{1\le k\le 5}\overline{\mathscr{B}}^+(k,\omega_{(4)})\big)$,

(IV) $\mathscr{B}_5^+(\omega_{(5)}) = \overline{\Phi}^+(\mathscr{B}_4(\omega_{(5)}) \bigcup \big(\bigcup_{1\le k\le 5}\overline{\mathscr{B}}^+(k,\omega_{(5)})\big)$.

A direct computation shows: $|\mathscr{B}_5^+(\omega_{(2)})| = 25$, $|\mathscr{B}_5^+(\omega_{(4)})| = 300$ and $|\mathscr{B}_5^+(\omega_{(5)})| = 125$. On the other hand, by Remark 3.3.9, we have $(Q\mathscr{P}_5)_{22}^+ \cong \bigoplus_{1\le j\le 5} Q\mathscr{P}_5^+(\omega_{(j)})$. Combining this with the above results, we obtain:

Corollary 3.3.15. *$(Q\mathscr{P}_5^+)_{22}$ is the $\mathbb{Z}/2$-vector space of dimension* 505 *with a basis consisting of all the classes represented by the monomials $\mathcal{Y}_{22,\,t}$, $1\le t\le$* 505, *which are described in Section 6.5 of the online version* [40].

Structure of the kernel of Kameko's map $(\widetilde{Sq^0_*})_{(5,47)}$

The following weight vectors that have the same degrees are 47:

$$\overline{\omega}_{(1)} = (3,2,2,2,1),\ \overline{\omega}_{(2)} = (3,2,4,1,1),\ \overline{\omega}_{(3)} = (3,2,4,3),$$
$$\overline{\omega}_{(4)} = (3,4,3,1,1),\ \overline{\omega}_{(5)} = (3,4,3,3).$$

From Remarks 3.3.8 and 3.3.9, we conclude that if $X\in\mathscr{B}_5(47)$ and $[X]$ belongs to the kernel of $(\widetilde{Sq^0_*})_{(5,47)}$, then the weight vector of X is one of the above sequences $\overline{\omega}_{(k)}$, $1\le k\le 5$. This implies that the dimension of $\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,47)}$ is equal to the sum of the dimensions of $Q\mathscr{P}_5^0$ and $Q\mathscr{P}_5^+(\overline{\omega}_{(k)})$ in degree 47 for all $1\le k\le 5$. Since $(Q\mathscr{P}_5^0)_{47}$ is isomorphic to $\bigoplus_{1\le t\le 4}\bigoplus_{1\le \ell\le \binom{5}{t}}(Q\mathscr{P}_t^+)_{47}$, by a direct computation using a result in [18], [34] and [54], we claim

$$\dim(Q\mathscr{P}_5^0)_{47} = \binom{5}{3}\cdot 14 + \binom{5}{4}\cdot 84 = 560.$$

Furthermore, $\mathscr{B}_5^0(47) = \mathscr{B}_5^0(\overline{\omega}_{(1)}) = \overline{\Phi}^0(\mathscr{B}_4(47)) = \{\mathcal{Y}_{47,\,i} : 1\le i\le 560\}$, where the monomials $\mathcal{Y}_{47,\,i}\in\mathscr{B}_5^0(47)$ are explicitly described in Section 6.6. of the online version [40].

We now determine the $\mathbb{Z}/2$-subspaces $Q\mathscr{P}_5^+(\overline{\omega}_{(k)})$ for $k=1,2,\ldots,5$.

Lemma 3.3.16. *The following monomials are strictly inadmissible*:

I) $X_1 = x_1^3x_2^{12}x_kx_\ell^3x_m^{12},\ X_2 = x_1^3x_2^4x_k^3x_\ell^8x_m^{13},\ X_3 = x_1x_2^{14}x_k^2x_\ell x_m^{13},$
$X_5 = x_1^3x_2^{14}x_k^{12}x_\ell x_m,\ X_6 = x_1^7x_2^{10}x_k^{12}x_\ell x_m,\ X_7 = x_1^3x_2^2x_k^{12}x_\ell x_m^{13},$
$X_8 = x_1^3x_2^{12}x_k^2x_\ell x_m^{13},\ X_9 = x_1^{15}x_2^2x_kx_\ell^4x_m^9,\ X_{10} = x_1^{15}x_2^2x_kx_\ell^5x_m^8,$
$X_{11} = x_1^7x_2^2x_kx_\ell^8x_m^{13},\ X_{12} = x_1^7x_2^2x_kx_\ell^9x_m^{12},\ X_{13} = x_1^{15}x_2^2x_k^{12}x_\ell x_m,$
$X_{14} = x_1^3x_2^4x_kx_\ell^8x_m^{15},\ X_{15} = x_1^3x_2^4x_kx_\ell^9x_m^{14},\ X_{16} = x_1^3x_2^{14}x_kx_\ell^4x_m^9,$
$X_{17} = x_1^3x_2^5x_kx_\ell^8x_m^{14},\ X_{18} = x_1^3x_2^6x_kx_\ell^8x_m^{13},\ X_{19} = x_1^3x_2^6x_kx_\ell^9x_m^{12},$

$X_{20} = x_1^7x_2^{10}x_kx_\ell^4x_m^9,\ X_{21} = x_1^7x_2^{10}x_kx_\ell^5x_m^8,\ X_{22} = x_1^3x_2^2x_k^5x_\ell^8x_m^{13},$
$X_{23} = x_1^3x_2^2x_k^5x_\ell^9x_m^{12},\ X_{24} = x_1^7x_2^2x_k^5x_\ell^8x_m^9,\ X_{25} = x_1x_2^2x_k^{14}x_\ell x_m^{13},$
$X_{26} = x_1^3x_2^6x_k^5x_\ell^8x_m^9,\ X_{27} = x_1^3x_2^{12}x_k^{14}x_\ell x_m.$
Here (k,ℓ,m) *is a permutation of* $(3,4,5)$;

II) $X_{27} = x_i^3x_j^2x_k^{13}x_\ell^4x_m^9,\ j<k,\ X_{28} = x_ix_j^2x_k^6x_\ell^9x_m^{13},\ X_{29} = x_i^3x_j^4x_k^6x_\ell^9x_m^9,$
where (i,j,k,ℓ,m) *is a permutation of* $(1,2,3,4,5)$.

Proof. It is easy to see that $\omega(X_t) = \omega^* := (3,2,2,2)$ for $1 \le t \le 29$. We prove the lemma for the monomials $X_1 = x_1^3x_2^{12}x_kx_\ell^3x_m^{12}$ and $X_2 = x_1^3x_2^4x_k^3x_\ell^8x_m^{13}$, where (k,ℓ,m) is a permutation of $(3,4,5)$. The others can be proved by a similar technique. We have

$$\begin{aligned}X_1 &= x_1^2x_2^{11}x_kx_\ell^5x_m^{12} + x_1^2x_2^{11}x_k^4x_\ell^5x_m^9 + x_1^2x_2^{11}x_k^8x_\ell^5x_m^5 + x_1^2x_2^{13}x_kx_\ell^3x_m^{12}\\
&+ x_1^2x_2^{13}x_k^4x_\ell^3x_m^9 + x_1^2x_2^{13}x_k^8x_\ell^3x_m^5 + x_1^3x_2^3x_k^8x_\ell^5x_m^{12} + x_1^3x_2^5x_k^8x_\ell^5x_m^{10}\\
&+ x_1^3x_2^5x_k^8x_\ell^6x_m^9 + x_1^3x_2^7x_k^4x_\ell^8x_m^9 + x_1^3x_2^7x_k^8x_\ell^5x_m^8 + x_1^3x_2^7x_k^8x_\ell^8x_m^5\\
&+ x_1^3x_2^9x_k^2x_\ell^5x_m^{12} + x_1^3x_2^9x_k^4x_\ell^3x_m^{12} + x_1^3x_2^9x_k^4x_\ell^5x_m^{10} + x_1^3x_2^9x_k^4x_\ell^6x_m^9\\
&+ x_1^3x_2^9x_k^8x_\ell^5x_m^6 + x_1^3x_2^9x_k^8x_\ell^6x_m^5 + x_1^3x_2^{11}x_kx_\ell^4x_m^{12}\\
&+ Sq^1(g_1) + Sq^2(g_2) + Sq^4(g_4) + Sq^8(x_1^6x_2^5x_k^4x_\ell^3x_m^5) \text{ modulo}(\mathscr{P}_5^-(\omega^*)),\end{aligned}$$

where

$$\begin{aligned}g_1 &= x_1^3x_2^7x_kx_\ell^3x_m^{16} + x_1^3x_2^{11}x_kx_\ell^3x_m^{12} + x_1^5x_2^3x_k^8x_\ell^5x_m^9 + x_1^5x_2^7x_k^4x_\ell^5x_m^9\\
&+ x_1^5x_2^7x_k^8x_\ell^5x_m^5,\\
g_2 &= x_1^2x_2^{11}x_kx_\ell^3x_m^{12} + x_1^2x_2^{11}x_k^4x_\ell^3x_m^9 + x_1^2x_2^{11}x_k^8x_\ell^3x_m^5 + x_1^3x_2^3x_k^8x_\ell^5x_m^{10}\\
&+ x_1^3x_2^3x_k^8x_\ell^6x_m^9 + x_1^3x_2^7x_k^4x_\ell^5x_m^{12} + x_1^3x_2^7x_k^4x_\ell^6x_m^9 + x_1^3x_2^7x_k^8x_\ell^5x_m^6\\
&+ x_1^3x_2^7x_k^8x_\ell^6x_m^5 + x_1^5x_2^7x_k^2x_\ell^3x_m^{12} + x_1^6x_2^3x_k^8x_\ell^3x_m^9 + x_1^6x_2^7x_k^4x_\ell^3x_m^9\\
&+ x_1^6x_2^7x_k^8x_\ell^3x_m^5,\\
g_4 &= x_1^3x_2^7x_k^2x_\ell^3x_m^{12} + x_1^4x_2^7x_k^4x_\ell^3x_m^9 + x_1^4x_2^7x_k^8x_\ell^3x_m^5 + x_1^{10}x_2^5x_k^4x_\ell^3x_m^5.\end{aligned}$$

This equality implies that X_1 is strictly inadmissible.

Next, we show that X_2 is also strictly inadmissible. Indeed, using Cartan's formula, we obtain

$$\begin{aligned}X_2 &= x_1^2x_2x_k^3x_\ell^{12}x_m^{13} + x_1^2x_2x_k^5x_\ell^9x_m^{14} + x_1^2x_2x_k^5x_\ell^{10}x_m^{13} + x_1^2x_2x_k^6x_\ell^9x_m^{13}\\
&+ x_1^2x_2x_k^{10}x_\ell^5x_m^{13} + x_1^2x_2x_k^{12}x_\ell^5x_m^{11} + x_1^2x_2^5x_kx_\ell^9x_m^{14} + x_1^2x_2^5x_kx_\ell^{10}x_m^{13}\\
&+ x_1^2x_2^5x_k^8x_\ell^3x_m^{13} + x_1^2x_2^5x_k^8x_\ell^5x_m^{11} + x_1^3x_2x_k^4x_\ell^9x_m^{14} + x_1^3x_2x_k^4x_\ell^{10}x_m^{13}\\
&+ x_1^3x_2x_k^6x_\ell^8x_m^{13} + x_1^3x_2x_k^8x_\ell^6x_m^{13} + x_1^3x_2^2x_k^4x_\ell^9x_m^{13} + x_1^3x_2^2x_k^5x_\ell^8x_m^{13}\\
&+ x_1^3x_2^3x_k^8x_\ell^4x_m^{13} + x_1^3x_2^3x_k^8x_\ell^5x_m^{12} + x_1^3x_2^4x_kx_\ell^9x_m^{14} + x_1^3x_2^4x_kx_\ell^{10}x_m^{13}\\
&+ x_1^3x_2^4x_k^2x_\ell^9x_m^{13}\\
&+ Sq^1(Z_1) + Sq^2(Z_2) + Sq^4(Z_4) + Sq^8(x_1^3x_2^4x_k^4x_\ell^5x_m^7) \text{ modulo}(\mathscr{P}_5^-(\omega^*)),\end{aligned}$$

where

$$\begin{aligned}
Z_1 = {} & x_1^3x_2x_k^3x_\ell^5x_m^{18} + x_1^3x_2x_k^3x_\ell^6x_m^{17} + x_1^3x_2x_k^3x_\ell^9x_m^{14} + x_1^3x_2x_k^3x_\ell^{10}x_m^{13}\\
& + x_1^3x_2x_k^6x_\ell^9x_m^{11} + x_1^3x_2x_k^{10}x_\ell^5x_m^{11} + x_1^3x_2^3x_kx_\ell^6x_m^{17} + x_1^3x_2^3x_kx_\ell^{10}x_m^{13}\\
& + x_1^3x_2^3x_kx_\ell^{12}x_m^{11} + x_1^3x_2^3x_kx_\ell^{16}x_m^7 + x_1^3x_2^3x_k^4x_\ell^3x_m^{17} + x_1^3x_2^3x_k^8x_\ell^3x_m^{13}\\
& + x_1^3x_2^3x_k^8x_\ell^5x_m^{11} + x_1^3x_2^3x_k^8x_\ell^9x_m^7 + x_1^3x_2^4x_k^5x_\ell^5x_m^{13} + x_1^5x_2^5x_kx_\ell^5x_m^{14},\\
Z_2 = {} & x_1^2x_2x_k^3x_\ell^9x_m^{14} + x_1^2x_2x_k^3x_\ell^{10}x_m^{13} + x_1^2x_2x_k^6x_\ell^9x_m^{11} + x_1^2x_2x_k^{10}x_\ell^5x_m^{11}\\
& + x_1^2x_2^3x_kx_\ell^9x_m^{14} + x_1^2x_2^3x_kx_\ell^{10}x_m^{13} + x_1^2x_2^3x_kx_\ell^{12}x_m^{11} + x_1^2x_2^3x_k^8x_\ell^3x_m^{13}\\
& + x_1^2x_2^3x_k^8x_\ell^5x_m^{11} + x_1^2x_2^3x_k^8x_\ell^9x_m^7 + x_1^3x_2^5x_kx_\ell^6x_m^{14} + x_1^3x_2^5x_k^2x_\ell^5x_m^{14}\\
& + x_1^3x_2^6x_kx_\ell^5x_m^{14} + x_1^5x_2x_k^6x_\ell^6x_m^{11} + x_1^5x_2^2x_k^2x_\ell^9x_m^{11} + x_1^5x_2^2x_k^3x_\ell^5x_m^{14}\\
& + x_1^5x_2^2x_k^3x_\ell^6x_m^{13} + x_1^5x_2^2x_k^6x_\ell^5x_m^{11} + x_1^5x_2^2x_k^8x_\ell^3x_m^{11} + x_1^5x_2^3x_kx_\ell^6x_m^{14}\\
& + x_1^5x_2^3x_k^2x_\ell^6x_m^{13} + x_1^5x_2^3x_k^2x_\ell^8x_m^{11} + x_1^5x_2^3x_k^2x_\ell^{10}x_m^9 + x_1^5x_2^3x_k^2x_\ell^{12}x_m^7\\
& + x_1^5x_2^3x_k^4x_\ell^3x_m^{14} + x_1^5x_2^3x_k^4x_\ell^6x_m^{11} + x_1^5x_2^3x_k^4x_\ell^{10}x_m^7 + x_1^6x_2^3x_kx_\ell^5x_m^{14},\\
Z_4 = {} & x_1^3x_2x_k^6x_\ell^5x_m^{12} + x_1^3x_2x_k^6x_\ell^6x_m^{11} + x_1^3x_2^2x_k^2x_\ell^9x_m^{11} + x_1^3x_2^2x_k^3x_\ell^6x_m^{13}\\
& + x_1^3x_2^2x_k^6x_\ell^5x_m^{11} + x_1^3x_2^2x_k^8x_\ell^3x_m^{11} + x_1^3x_2^3x_kx_\ell^6x_m^{14} + x_1^3x_2^3x_k^2x_\ell^6x_m^{13}\\
& + x_1^3x_2^3x_k^2x_\ell^8x_m^{11} + x_1^3x_2^3x_k^2x_\ell^{10}x_m^9 + x_1^3x_2^3x_k^2x_\ell^{12}x_m^7 + x_1^3x_2^3x_k^4x_\ell^3x_m^{14}\\
& + x_1^3x_2^3x_k^4x_\ell^6x_m^{11} + x_1^3x_2^3x_k^4x_\ell^{10}x_m^7 + x_1^3x_2^4x_kx_\ell^5x_m^{14} + x_1^3x_2^8x_k^4x_\ell^5x_m^7\\
& + x_1^4x_2^3x_kx_\ell^5x_m^{14}.
\end{aligned}$$

The above relations imply that X_2 is strictly inadmissible. The lemma follows. □

Lemma 3.3.17. *The following monomials are strictly inadmissible*:

$Y_1 = x_1^3x_2^4x_3^7x_4^8x_5^9$,	$Y_2 = x_1^3x_2^7x_3^8x_4^5x_5^8$,	$Y_3 = x_1x_2^6x_3^{10}x_4x_5^{13}$,	$Y_4 = x_1x_2^6x_3^{10}x_4^{13}x_5$,
$Y_5 = x_1x_2^6x_3^{11}x_4^{12}x_5$,	$Y_6 = x_1x_2^7x_3^{10}x_4^{12}x_5$,	$Y_7 = x_1^7x_2x_3^{10}x_4^{12}x_5$,	$Y_8 = x_1x_2^2x_3^2x_4^{13}x_5^{13}$,
$Y_9 = x_1x_2^2x_3^{14}x_4^5x_5^9$,	$Y_{10} = x_1x_2^{14}x_3^2x_4^5x_5^9$,	$Y_{11} = x_1^3x_2^3x_3^{12}x_4^{12}x_5$,	$Y_{12} = x_1^3x_2^{15}x_3^4x_4^8x_5$,
$Y_{13} = x_1^3x_2^{15}x_3^4x_4x_5^8$,	$Y_{14} = x_1^{15}x_2^3x_3^4x_4^8x_5$,	$Y_{15} = x_1^{15}x_2^3x_3^4x_4x_5^8$,	$Y_{16} = x_1x_2^{14}x_3^3x_4^4x_5^9$,
$Y_{17} = x_1^3x_2^4x_3^{10}x_4x_5^{13}$,	$Y_{18} = x_1^3x_2^4x_3^{10}x_4^{13}x_5$,	$Y_{19} = x_1^3x_2^4x_3^{11}x_4^{12}x_5$,	$Y_{20} = x_1x_2^{14}x_3^3x_4^5x_5^8$,
$Y_{21} = x_1^3x_2^{14}x_3^5x_4^8x_5$,	$Y_{22} = x_1^3x_2^{14}x_3x_4^5x_5^8$,	$Y_{23} = x_1^3x_2^{14}x_3^5x_4x_5^8$,	$Y_{24} = x_1x_2^6x_3^3x_4^{13}x_5^8$,
$Y_{25} = x_1x_2^{14}x_3^{14}x_4x_5$,	$Y_{26} = x_1^3x_2^7x_3^8x_4^{12}x_5$,	$Y_{27} = x_1^3x_2^7x_3^{12}x_4^8x_5$,	$Y_{28} = x_1^3x_2^7x_3^{12}x_4x_5^8$,
$Y_{29} = x_1^7x_2^3x_3^8x_4^{12}x_5$,	$Y_{30} = x_1^7x_2^3x_3^{12}x_4^8x_5$,	$Y_{31} = x_1^7x_2^3x_3^{12}x_4x_5^8$,	$Y_{32} = x_1x_2^6x_3^{11}x_4^4x_5^9$,
$Y_{33} = x_1^7x_2^{11}x_3^4x_4^8x_5$,	$Y_{34} = x_1^7x_2^{11}x_3^4x_4x_5^8$,	$Y_{35} = x_1x_2^6x_3^{11}x_4^5x_5^8$,	$Y_{36} = x_1x_2^6x_3^{10}x_4^5x_5^9$,
$Y_{37} = x_1^3x_2^5x_3^8x_4^2x_5^{13}$,	$Y_{38} = x_1^3x_2^{12}x_3^2x_4^5x_5^9$,	$Y_{39} = x_1^3x_2^5x_3^9x_4^2x_5^{12}$,	$Y_{40} = x_1^7x_2^2x_3^4x_4^9x_5^9$,
$Y_{41} = x_1^7x_2^2x_3^9x_4^4x_5^9$,	$Y_{42} = x_1^3x_2^4x_3^3x_4^{12}x_5^9$,	$Y_{43} = x_1^3x_2^{12}x_3^3x_4^4x_5^9$,	$Y_{44} = x_1^3x_2^4x_3^9x_4^3x_5^{12}$,
$Y_{45} = x_1^3x_2^{12}x_3^3x_4^5x_5^8$,	$Y_{46} = x_1^3x_2^5x_3^8x_4^3x_5^{12}$,	$Y_{47} = x_1^3x_2^4x_3^{11}x_4^4x_5^9$,	$Y_{48} = x_1^3x_2^4x_3^{11}x_4^5x_5^8$,
$Y_{49} = x_1^3x_2^4x_3^{10}x_4^5x_5^9$,	$Y_{50} = x_1x_2^6x_3^3x_4^{12}x_5^9$,	$Y_{51} = x_1^3x_2^4x_3^7x_4^9x_5^8$,	$Y_{52} = x_1^3x_2^7x_3^8x_4^4x_5^9$,
$Y_{53} = x_1^7x_2^3x_3^8x_4^4x_5^9$,	$Y_{54} = x_1x_2^{14}x_3^3x_4^{12}x_5$,	$Y_{55} = x_1^7x_2^3x_3^8x_4^5x_5^8$.	

Proof. Note that $\omega(Y_j) = \omega^*,\ j = 1, 2, \ldots, 55$. We prove this lemma for the monomials $Y_1 = x_1^3x_2^4x_3^7x_4^8x_5^9$, and $Y_2 = x_1^3x_2^7x_3^8x_4^5x_5^8$. The others can be proved by a similar computation. A direct computation shows:

$$Y_1 = x_1^2x_2x_3^7x_4^8x_5^{13} + x_1^2x_2x_3^{13}x_4x_5^{14} + x_1^2x_2x_3^{13}x_4^8x_5^7 + x_1^2x_2^3x_3^9x_4^4x_5^{13}$$

$$
\begin{aligned}
&+ x_1^2x_2^3x_3^{12}x_4x_5^{13} + x_1^2x_2^5x_3^9x_4^4x_5^{11} + x_1^2x_2^5x_3^9x_4^8x_5^7 + x_1^2x_2^5x_3^{12}x_4x_5^{11}\\
&+ x_1^3x_2x_3^5x_4^8x_5^{14} + x_1^3x_2x_3^7x_4^8x_5^{12} + x_1^3x_2x_3^{12}x_4x_5^{14} + x_1^3x_2x_3^{12}x_4^4x_5^{11}\\
&+ x_1^3x_2^2x_3^9x_4^4x_5^{13} + x_1^3x_2^3x_3^9x_4^4x_5^{12} + x_1^3x_2^3x_3^{12}x_4x_5^{12} + x_1^3x_2^3x_3^{12}x_4^4x_5^9\\
&+ Sq^1(u_1) + Sq^2(u_2) + Sq^4(u_4) + Sq^8(u_8) \text{ modulo}(\mathscr{P}_5^-(\omega^*)),
\end{aligned}
$$

where

$$
\begin{aligned}
u_1 = {}& x_1^3x_2x_3^5x_4^8x_5^{13} + x_1^3x_2x_3^7x_4x_5^{18} + x_1^3x_2x_3^9x_4^4x_5^{13} + x_1^3x_2x_3^{11}x_4x_5^{14}\\
&+ x_1^3x_2^3x_3^9x_4^4x_5^{11} + x_1^3x_2^3x_3^9x_4^8x_5^7 + x_1^3x_2^3x_3^{12}x_4x_5^{11} + x_1^3x_2^3x_3^{16}x_4x_5^7\\
&+ x_1^3x_2^4x_3^9x_4x_5^{13} + x_1^5x_2x_3^5x_4^8x_5^{11} + x_1^5x_2x_3^7x_4^8x_5^9,\\
u_2 = {}& x_1^2x_2x_3^7x_4^8x_5^{11} + x_1^2x_2x_3^{11}x_4x_5^{14} + x_1^2x_2x_3^{11}x_4^8x_5^7 + x_1^2x_2^3x_3^9x_4^4x_5^{11}\\
&+ x_1^2x_2^3x_3^9x_4^8x_5^7 + x_1^2x_2^3x_3^{12}x_4x_5^{11} + x_1^3x_2x_3^6x_4^8x_5^{11} + x_1^3x_2x_3^7x_4^8x_5^{10}\\
&+ x_1^3x_2x_3^{10}x_4^4x_5^{11} + x_1^3x_2^2x_3^5x_4^8x_5^{11} + x_1^3x_2^2x_3^7x_4^8x_5^9 + x_1^3x_2^2x_3^9x_4^4x_5^{11}\\
&+ x_1^5x_2x_3^7x_4^2x_5^{14} + x_1^5x_2^2x_3^7x_4x_5^{14} + x_1^5x_2^2x_3^9x_4^2x_5^{11} + x_1^5x_2^3x_3^{10}x_4^2x_5^9\\
&+ x_1^5x_2^3x_3^{10}x_4^4x_5^7 + x_1^5x_2^3x_3^{12}x_4^2x_5^7 + x_1^6x_2x_3^7x_4^8x_5^7,\\
u_4 = {}& x_1^3x_2x_3^7x_4^2x_5^{14} + x_1^3x_2^2x_3^7x_4x_5^{14} + x_1^3x_2^2x_3^9x_4^2x_5^{11} + x_1^3x_2^2x_3^{10}x_4x_5^{11}\\
&+ x_1^3x_2^3x_3^{10}x_4^2x_5^9 + x_1^3x_2^3x_3^{10}x_4^4x_5^7 + x_1^3x_2^3x_3^{12}x_4^2x_5^7 + x_1^3x_2^8x_3^5x_4^4x_5^7\\
&+ x_1^4x_2x_3^7x_4^8x_5^7 + x_1^{10}x_2x_3^5x_4^4x_5^7,\\
u_8 = {}& x_1^3x_2^4x_3^5x_4^4x_5^7 + x_1^6x_2x_3^5x_4^4x_5^7.
\end{aligned}
$$

By a similar computation, we have

$$
\begin{aligned}
Y_2 = {}& x_1^2x_2^7x_3^8x_4^5x_5^9 + x_1^2x_2^{11}x_3x_4^5x_5^{12} + x_1^2x_2^{11}x_3^8x_4^5x_5^5 + x_1^2x_2^{13}x_3x_4^5x_5^{10}\\
&+ x_1^2x_2^{13}x_3x_4^9x_5^6 + x_1^2x_2^{13}x_3^8x_4^3x_5^5 + x_1^3x_2^3x_3x_4^{12}x_5^{12} + x_1^3x_2^3x_3^4x_4^9x_5^{12}\\
&+ x_1^3x_2^3x_3^8x_4^5x_5^{12} + x_1^3x_2^5x_3^4x_4^9x_5^{10} + x_1^3x_2^5x_3^4x_4^{10}x_5^9 + x_1^3x_2^5x_3^8x_4^5x_5^{10}\\
&+ x_1^3x_2^5x_3^8x_4^6x_5^9 + x_1^3x_2^5x_3^8x_4^9x_5^6 + x_1^3x_2^5x_3^8x_4^{10}x_5^5 + x_1^3x_2^7x_3x_4^8x_5^{12}\\
&+ x_1^3x_2^7x_3x_4^{12}x_5^8 + x_1^3x_2^7x_3^4x_4^9x_5^8 + x_1^3x_2^7x_3^8x_4^4x_5^9\\
&+ Sq^1(v_1) + Sq^2(v_2) + Sq^4(v_4) + Sq^8(v_8) \text{ modulo}(\mathscr{P}_5^-(\omega^*)),
\end{aligned}
$$

where

$$
\begin{aligned}
v_1 = {}& x_1^5x_2^3x_3x_4^9x_5^{12} + x_1^5x_2^3x_3^8x_4^5x_5^9 + x_1^5x_2^7x_3x_4^5x_5^{12}\\
&+ x_1^5x_2^7x_3x_4^9x_5^8 + x_1^5x_2^7x_3^4x_4^5x_5^9 + x_1^5x_2^7x_3^8x_4^5x_5^5,\\
v_2 = {}& x_1^2x_2^7x_3^8x_4^3x_5^9 + x_1^2x_2^{11}x_3x_4^5x_5^{10} + x_1^2x_2^{11}x_3x_4^9x_5^6 + x_1^2x_2^{11}x_3^8x_4^3x_5^5\\
&+ x_1^3x_2^3x_3x_4^{10}x_5^{12} + x_1^3x_2^3x_3^2x_4^9x_5^{12} + x_1^3x_2^3x_3^8x_4^5x_5^{10} + x_1^3x_2^3x_3^8x_4^6x_5^9\\
&+ x_1^3x_2^7x_3x_4^6x_5^{12} + x_1^3x_2^7x_3x_4^{10}x_5^8 + x_1^3x_2^7x_3^2x_4^5x_5^{12} + x_1^3x_2^7x_3^2x_4^9x_5^8\\
&+ x_1^3x_2^7x_3^4x_4^5x_5^{10} + x_1^3x_2^7x_3^4x_4^6x_5^9 + x_1^3x_2^7x_3^8x_4^5x_5^6 + x_1^3x_2^7x_3^8x_4^6x_5^5\\
&+ x_1^6x_2^3x_3x_4^9x_5^{10} + x_1^6x_2^3x_3^8x_4^3x_5^9 + x_1^6x_2^7x_3x_4^5x_5^{10} + x_1^6x_2^7x_3x_4^9x_5^6
\end{aligned}
$$

$$\begin{aligned}
&+ x_1^6x_2^7x_3^4x_4^3x_5^9 + x_1^6x_2^7x_3^8x_4^3x_5^5,\\
v_4 &= x_1^3x_2^5x_3x_4^6x_5^{12} + x_1^3x_2^5x_3^2x_4^5x_5^{12} + x_1^3x_2^5x_3^4x_4^5x_5^{10} + x_1^3x_2^5x_3^4x_4^6x_5^9\\
&+ x_1^3x_2^5x_3^8x_4^5x_5^6 + x_1^3x_2^5x_3^8x_4^6x_5^5 + x_1^3x_2^{11}x_3^4x_4^4x_5^5 + x_1^4x_2^7x_3x_4^5x_5^{10}\\
&+ x_1^4x_2^7x_3x_4^9x_5^6 + x_1^4x_2^7x_3^8x_4^3x_5^5 + x_1^{10}x_2^5x_3x_4^5x_5^6 + x_1^{10}x_2^5x_3^4x_4^3x_5^5,\\
v_8 &= x_1^3x_2^7x_3^4x_4^4x_5^5 + x_1^6x_2^5x_3^1x_4^5x_5^6 + x_1^6x_2^5x_3^4x_4^3x_5^5.
\end{aligned}$$

From the above equalities, we see that Y_1 and Y_2 are strictly inadmissible. The lemma is proved. □

For $s,\ k \in \mathbb{N}$ and $1 \le k \le 5$, we denote

$$\overline{\mathscr{B}}(k,47) := \big\{x_k^{2^s-1}\rho_{(k,5)}(x) \in (\mathscr{P}_5)_{47} : x \in \mathscr{B}_4(45-2^s),\ \alpha(52-2^s) \le 4\big\}.$$

It has been shown (see [26]) that $\overline{\mathscr{B}}(k,47) \subseteq \mathscr{B}_5(47)$ for $k = 1,2,\ldots,5$. We set $\overline{\mathscr{B}}(k,\overline{\omega}_{(1)}) := \overline{\mathscr{B}}(k,47)\cap \mathscr{P}_5(\overline{\omega}_{(1)})$ and $\overline{\mathscr{B}}^+(k,\overline{\omega}_{(1)}) := \overline{\mathscr{B}}(k,\overline{\omega}_{(1)})\cap(\mathscr{P}_5^+)_{47}$. Then, we obtain the following result.

Proposition 3.3.18. *We have*

$$\begin{aligned}
Q\mathscr{P}_5^+(\overline{\omega}_{(1)}) &= \big\langle [\overline{\Phi}^+(\mathscr{B}_4(\overline{\omega}_{(1)}))\bigcup\big(\bigcup_{1\le k\le 5}\overline{\mathscr{B}}^+(k,\overline{\omega}_{(1)})\big)\bigcup \mathcal{F}]_{\overline{\omega}_{(1)}}\big\rangle\\
&= \big\langle [\mathcal{Y}_{47,\,j}]_{\overline{\omega}_{(1)}} : 1 \le j \le 370\big\rangle,
\end{aligned}$$

where the monomials $\mathcal{Y}_{47,\,j},\ j = 1,2,\ldots,370,$ *are determined in Section 6.7 of the online version* [40].

The proof of the proposition is based on Lemmas 3.3.16, 3.3.17 and some below results. The follwing lemma is an immediate consequence of a result in Sum [54].

Lemma 3.3.19. *The following monomials are strictly inadmissible*:

I) $x_i^{15}x_j^2x_kx_\ell^{13},\ x_i^3x_j^{12}x_kx_\ell^{15},\ x_i^3x_j^{14}x_kx_\ell^{13},$
$x_i^7x_j^{10}x_kx_\ell^{13},\ x_i^3x_j^13x_k^8x_\ell^7,\ x_i^3x_j^{12}x_k^9x_\ell^7, 1 \le j < k \le 5,\ 1 \le i,\ell \le 5,$
$i \ne \ell,\ i,\ell \ne j,k;$

II) $x_i^{15}x_j^{14}x_kx_\ell,\ x_i^3x_j^2x_k^{13}x_\ell^{13},\ x_i^{15}x_j^2x_k^5x_\ell^9,\ x_i^{13}x_j^2x_k^7x_\ell^9,\ x_i^{13}x_j^7x_k^2x_\ell^9,$
$x_i^{15}x_j^3x_k^4x_\ell^9, x_i^{15}x_j^3x_k^5x_\ell^8,\ x_i^{14}x_j^3x_k^5x_\ell^9,\ x_i^{13}x_j^3x_k^6x_\ell^9,\ x_i^{13}x_j^6x_k^3x_\ell^9,$
$1 \le j < k < \ell \le 5,\ 1 \le i \le 5,\ i \ne j,k,\ell;$

III) $\rho_{(1,5)}(x_1^2x_2^5x_3^{11}x_4^{13}),\ \rho_{(1,5)}(x_1^6x_2^{11}x_3^5x_4^9),\ \rho_{(1,5)}(x_1^6x_2x_3^{11}x_4^{13}),$
$\rho_{(1,5)}(x_1^6x_2^{11}x_3x_4^{13}),\ \rho_{(1,5)}(x_1^6x_2^{11}x_3^{13}x_4).$

Lemma 3.3.20. *The following monomials are strictly inadmissible*:

$Z_{1} = x_1^{3}x_2^{3}x_3^{12}x_4^{5}x_5^{24}$, $Z_{2} = x_1^{7}x_2x_3^{11}x_4^{20}x_5^{8}$, $Z_{3} = x_1^{7}x_2x_3^{2}x_4^{28}x_5^{9}$, $Z_{4} = x_1x_2^{2}x_3^{7}x_4^{12}x_5^{25}$,
$Z_{5} = x_1x_2^{7}x_3^{2}x_4^{12}x_5^{25}$, $Z_{6} = x_1^{7}x_2x_3^{2}x_4^{12}x_5^{25}$, $Z_{7} = x_1x_2^{6}x_3^{3}x_4^{8}x_5^{29}$, $Z_{8} = x_1x_2^{6}x_3^{3}x_4^{24}x_5^{13}$,
$Z_{9} = x_1^{3}x_2^{7}x_3^{8}x_4x_5^{28}$, $Z_{10} = x_1^{7}x_2^{3}x_3^{8}x_4x_5^{28}$, $Z_{11} = x_1^{3}x_2^{7}x_3^{24}x_4x_5^{12}$, $Z_{12} = x_1^{7}x_2^{3}x_3^{24}x_4x_5^{12}$,
$Z_{13} = x_1x_2^{7}x_3^{26}x_4^{4}x_5^{9}$, $Z_{14} = x_1^{7}x_2x_3^{26}x_4^{4}x_5^{9}$, $Z_{15} = x_1x_2^{7}x_3^{10}x_4^{4}x_5^{25}$, $Z_{16} = x_1^{7}x_2x_3^{10}x_4^{4}x_5^{25}$,
$Z_{17} = x_1x_2^{7}x_3^{26}x_4^{5}x_5^{8}$, $Z_{18} = x_1^{7}x_2x_3^{26}x_4^{5}x_5^{8}$, $Z_{19} = x_1x_2^{7}x_3^{10}x_4^{5}x_5^{24}$, $Z_{20} = x_1^{7}x_2x_3^{10}x_4^{5}x_5^{24}$,
$Z_{21} = x_1x_2^{6}x_3^{11}x_4^{13}x_5^{16}$, $Z_{22} = x_1x_2^{7}x_3^{10}x_4^{21}x_5^{8}$, $Z_{23} = x_1^{7}x_2x_3^{10}x_4^{21}x_5^{8}$, $Z_{24} = x_1x_2^{7}x_3^{11}x_4^{20}x_5^{8}$,
$Z_{25} = x_1x_2^{2}x_3^{7}x_4^{28}x_5^{9}$, $Z_{26} = x_1^{7}x_2^{11}x_3x_4^{20}x_5^{8}$, $Z_{27} = x_1x_2^{7}x_3^{10}x_4^{20}x_5^{9}$, $Z_{28} = x_1^{7}x_2x_3^{10}x_4^{20}x_5^{9}$,
$Z_{29} = x_1x_2^{7}x_3^{10}x_4^{13}x_5^{16}$, $Z_{30} = x_1^{7}x_2x_3^{10}x_4^{13}x_5^{16}$, $Z_{31} = x_1x_2^{7}x_3^{11}x_4^{12}x_5^{16}$, $Z_{32} = x_1^{7}x_2x_3^{11}x_4^{12}x_5^{16}$,
$Z_{33} = x_1^{7}x_2^{11}x_3x_4^{12}x_5^{16}$, $Z_{34} = x_1^{3}x_2^{3}x_3^{4}x_4^{28}x_5^{9}$, $Z_{35} = x_1^{3}x_2^{3}x_3^{28}x_4^{4}x_5^{9}$, $Z_{36} = x_1^{3}x_2^{3}x_3^{4}x_4^{12}x_5^{25}$,
$Z_{37} = x_1^{3}x_2^{3}x_3^{12}x_4^{4}x_5^{25}$, $Z_{38} = x_1^{3}x_2^{3}x_3^{28}x_4^{5}x_5^{8}$, $Z_{39} = x_1x_2^{7}x_3^{2}x_4^{28}x_5^{9}$, $Z_{40} = x_1^{3}x_2^{3}x_3^{12}x_4^{21}x_5^{8}$.

Proof. We have $\omega(Z_u) = \overline{\omega}_{(1)}, \forall u, 1 \leq u \leq 40$. Consider the monomials $Z_1 = x_1^3x_2^3x_3^{12}x_4^5x_5^{24}$ and $Z_2 = x_1^7x_2x_3^{11}x_4^{20}x_5^8$. We prove that these monomials are strictly inadmissible. The others can be proved by a similar technique. Computing the monomials Z_1, Z_2 is long and technical. Indeed, by using Cartan's formula, we get

$$\begin{aligned} Z_1 = &\sum X + Sq^1(\sum \sigma_1) + Sq^2(\sum \sigma_2) \\ &+ Sq^4(\sum \sigma_4) + Sq^8(\sum \sigma_8) \text{ modulo}(\mathscr{P}_5^-(\overline{\omega}_{(1)})), \end{aligned}$$

where

$$\begin{aligned} \sum X = &\ x_1^{2}x_2x_3^{3}x_4^{12}x_5^{29} + x_1^{2}x_2x_3^{3}x_4^{13}x_5^{28} + x_1^{2}x_2x_3^{5}x_4^{12}x_5^{27} + x_1^{2}x_2x_3^{5}x_4^{13}x_5^{26} \\ &+ x_1^{2}x_2x_3^{5}x_4^{24}x_5^{15} + x_1^{2}x_2x_3^{5}x_4^{25}x_5^{14} + x_1^{2}x_2x_3^{7}x_4^{9}x_5^{28} + x_1^{2}x_2x_3^{7}x_4^{13}x_5^{24} \\ &+ x_1^{2}x_2x_3^{12}x_4^{3}x_5^{29} + x_1^{2}x_2x_3^{12}x_4^{5}x_5^{27} + x_1^{2}x_2x_3^{13}x_4^{7}x_5^{24} + x_1^{2}x_2x_3^{13}x_4^{9}x_5^{22} \\ &+ x_1^{2}x_2x_3^{13}x_4^{16}x_5^{15} + x_1^{2}x_2x_3^{13}x_4^{17}x_5^{14} + x_1^{2}x_2x_3^{16}x_4^{13}x_5^{15} + x_1^{2}x_2x_3^{24}x_4^{5}x_5^{15} \\ &+ x_1^{2}x_2^{3}x_3^{5}x_4^{9}x_5^{28} + x_1^{2}x_2^{3}x_3^{5}x_4^{12}x_5^{25} + x_1^{2}x_2^{3}x_3^{9}x_4^{5}x_5^{28} + x_1^{2}x_2^{3}x_3^{12}x_4^{5}x_5^{25} \\ &+ x_1^{2}x_2^{5}x_3^{3}x_4^{9}x_5^{28} + x_1^{2}x_2^{5}x_3^{3}x_4^{12}x_5^{25} + x_1^{2}x_2^{5}x_3^{9}x_4^{5}x_5^{26} + x_1^{2}x_2^{5}x_3^{12}x_4^{3}x_5^{25} \\ &+ x_1^{3}x_2x_3^{3}x_4^{12}x_5^{28} + x_1^{3}x_2x_3^{4}x_4^{12}x_5^{27} + x_1^{3}x_2x_3^{4}x_4^{13}x_5^{26} + x_1^{3}x_2x_3^{4}x_4^{24}x_5^{15} \\ &+ x_1^{3}x_2x_3^{4}x_4^{25}x_5^{14} + x_1^{3}x_2x_3^{7}x_4^{12}x_5^{24} + x_1^{3}x_2x_3^{7}x_4^{12}x_5^{24} + x_1^{3}x_2x_3^{9}x_4^{12}x_5^{22} \\ &+ x_1^{3}x_2x_3^{12}x_4^{3}x_5^{28} + x_1^{3}x_2x_3^{12}x_4^{4}x_5^{27} + x_1^{3}x_2x_3^{12}x_4^{7}x_5^{24} + x_1^{3}x_2x_3^{12}x_4^{9}x_5^{22} \\ &+ x_1^{3}x_2x_3^{12}x_4^{16}x_5^{15} + x_1^{3}x_2x_3^{12}x_4^{17}x_5^{14} + x_1^{3}x_2x_3^{16}x_4^{12}x_5^{15} + x_1^{3}x_2x_3^{24}x_4^{4}x_5^{15} \\ &+ x_1^{3}x_2^{2}x_3^{5}x_4^{9}x_5^{28} + x_1^{3}x_2^{2}x_3^{5}x_4^{12}x_5^{25} + x_1^{3}x_2^{2}x_3^{9}x_4^{12}x_5^{21} + x_1^{3}x_2^{2}x_3^{12}x_4^{5}x_5^{25} \\ &+ x_1^{3}x_2^{2}x_3^{12}x_4^{9}x_5^{21} + x_1^{3}x_2^{3}x_3^{4}x_4^{9}x_5^{28} + x_1^{3}x_2^{3}x_3^{4}x_4^{12}x_5^{25} + x_1^{3}x_2^{3}x_3^{5}x_4^{8}x_5^{28} \\ &+ x_1^{3}x_2^{3}x_3^{5}x_4^{12}x_5^{24}, \end{aligned}$$

$$\begin{aligned} \sum \sigma_1 = &\ x_1^{3}x_2x_3^{3}x_4^{12}x_5^{27} + x_1^{3}x_2x_3^{3}x_4^{13}x_5^{26} + x_1^{3}x_2x_3^{3}x_4^{16}x_5^{23} + x_1^{3}x_2x_3^{3}x_4^{17}x_5^{22} \\ &+ x_1^{3}x_2x_3^{3}x_4^{20}x_5^{19} + x_1^{3}x_2x_3^{3}x_4^{21}x_5^{18} + x_1^{3}x_2x_3^{3}x_4^{24}x_5^{15} + x_1^{3}x_2x_3^{3}x_4^{25}x_5^{14} \\ &+ x_1^{3}x_2x_3^{7}x_4^{9}x_5^{26} + x_1^{3}x_2x_3^{7}x_4^{11}x_5^{24} + x_1^{3}x_2x_3^{7}x_4^{16}x_5^{19} + x_1^{3}x_2x_3^{7}x_4^{17}x_5^{18} \\ &+ x_1^{3}x_2x_3^{9}x_4^{13}x_5^{20} + x_1^{3}x_2x_3^{11}x_4^{7}x_5^{24} + x_1^{3}x_2x_3^{11}x_4^{9}x_5^{22} + x_1^{3}x_2x_3^{11}x_4^{16}x_5^{15} \\ &+ x_1^{3}x_2x_3^{11}x_4^{17}x_5^{14} + x_1^{3}x_2x_3^{12}x_4^{3}x_5^{27} + x_1^{3}x_2x_3^{16}x_4^{3}x_5^{23} + x_1^{3}x_2x_3^{16}x_4^{7}x_5^{19} \end{aligned}$$

$$
\begin{aligned}
&+ x_1^3x_2x_3^{16}x_4^{11}x_5^{15} + x_1^3x_2x_3^{20}x_4^3x_5^{19} + x_1^3x_2x_3^{24}x_4^3x_5^{15} + x_1^3x_2^3x_3^3x_4^5x_5^{32}\\
&+ x_1^3x_2^3x_3^3x_4^9x_5^{28} + x_1^3x_2^3x_3^3x_4^{12}x_5^{25} + x_1^3x_2^3x_3^3x_4^{16}x_5^{21} + x_1^3x_2^3x_3^9x_4^5x_5^{26}\\
&+ x_1^3x_2^3x_3^9x_4^9x_5^{22} + x_1^3x_2^3x_3^{12}x_4^3x_5^{25} + x_1^3x_2^3x_3^{16}x_4^3x_5^{21} + x_1^3x_2^4x_3^9x_4^9x_5^{21},\\
\sum \sigma_2 = {}& x_1^2x_2x_3^3x_4^{12}x_5^{27} + x_1^2x_2x_3^3x_4^{13}x_5^{26} + x_1^2x_2x_3^3x_4^{24}x_5^{15} + x_1^2x_2x_3^3x_4^{25}x_5^{14}\\
&+ x_1^2x_2x_3^7x_4^9x_5^{26} + x_1^2x_2x_3^7x_4^{11}x_5^{24} + x_1^2x_2x_3^{11}x_4^7x_5^{24} + x_1^2x_2x_3^{11}x_4^9x_5^{22}\\
&+ x_1^2x_2x_3^{11}x_4^{16}x_5^{15} + x_1^2x_2x_3^{11}x_4^{17}x_5^{14} + x_1^2x_2x_3^{12}x_4^3x_5^{27} + x_1^2x_2x_3^{16}x_4^{11}x_5^{15}\\
&+ x_1^2x_2x_3^{24}x_4^3x_5^{15} + x_1^2x_2^3x_3^3x_4^9x_5^{28} + x_1^2x_2^3x_3^3x_4^{12}x_5^{25} + x_1^2x_2^3x_3^9x_4^5x_5^{26}\\
&+ x_1^2x_2^3x_3^9x_4^9x_5^{22} + x_1^2x_2^3x_3^{12}x_4^3x_5^{25} + x_1^5x_2x_3^3x_4^{14}x_5^{22} + x_1^5x_2x_3^3x_4^{22}x_5^{14}\\
&+ x_1^5x_2x_3^7x_4^{10}x_5^{22} + x_1^5x_2x_3^7x_4^{18}x_5^{14} + x_1^5x_2^2x_3^3x_4^9x_5^{26} + x_1^5x_2^2x_3^3x_4^{11}x_5^{24}\\
&+ x_1^5x_2^2x_3^3x_4^{12}x_5^{23} + x_1^5x_2^2x_3^3x_4^{13}x_5^{22} + x_1^5x_2^2x_3^3x_4^{20}x_5^{15} + x_1^5x_2^2x_3^3x_4^{21}x_5^{14}\\
&+ x_1^5x_2^2x_3^7x_4^7x_5^{24} + x_1^5x_2^2x_3^7x_4^9x_5^{22} + x_1^5x_2^2x_3^7x_4^{16}x_5^{15} + x_1^5x_2^2x_3^7x_4^{17}x_5^{14}\\
&+ x_1^5x_2^2x_3^9x_4^{10}x_5^{19} + x_1^5x_2^2x_3^9x_4^{11}x_5^{18} + x_1^5x_2^2x_3^{10}x_4^9x_5^{19} + x_1^5x_2^2x_3^{12}x_4^3x_5^{23}\\
&+ x_1^5x_2^2x_3^{16}x_4^7x_5^{15} + x_1^5x_2^2x_3^{20}x_4^3x_5^{15} + x_1^5x_2^3x_3^3x_4^6x_5^{28} + x_1^5x_2^3x_3^3x_4^8x_5^{26}\\
&+ x_1^5x_2^3x_3^3x_4^{10}x_5^{24} + x_1^5x_2^3x_3^3x_4^{12}x_5^{22} + x_1^5x_2^3x_3^5x_4^6x_5^{26} + x_1^5x_2^3x_3^5x_4^{10}x_5^{22}\\
&+ x_1^5x_2^3x_3^{10}x_4^3x_5^{24} + x_1^5x_2^3x_3^{10}x_4^5x_5^{22} + x_1^5x_2^3x_3^{12}x_4^3x_5^{22},\\
\sum \sigma_4 = {}& x_1^3x_2x_3^3x_4^{14}x_5^{22} + x_1^3x_2x_3^3x_4^{22}x_5^{14} + x_1^3x_2x_3^5x_4^{14}x_5^{20} + x_1^3x_2x_3^5x_4^{20}x_5^{14}\\
&+ x_1^3x_2x_3^6x_4^{13}x_5^{20} + x_1^3x_2x_3^7x_4^{10}x_5^{22} + x_1^3x_2x_3^7x_4^{18}x_5^{14} + x_1^3x_2^2x_3^3x_4^9x_5^{26}\\
&+ x_1^3x_2^2x_3^3x_4^{11}x_5^{24} + x_1^3x_2^2x_3^3x_4^{12}x_5^{23} + x_1^3x_2^2x_3^3x_4^{13}x_5^{22} + x_1^3x_2^2x_3^3x_4^{20}x_5^{15}\\
&+ x_1^3x_2^2x_3^3x_4^{21}x_5^{14} + x_1^3x_2^2x_3^7x_4^7x_5^{24} + x_1^3x_2^2x_3^7x_4^9x_5^{22} + x_1^3x_2^2x_3^7x_4^{16}x_5^{15}\\
&+ x_1^3x_2^2x_3^7x_4^{17}x_5^{14} + x_1^3x_2^2x_3^9x_4^{10}x_5^{19} + x_1^3x_2^2x_3^9x_4^{11}x_5^{18} + x_1^3x_2^2x_3^{10}x_4^9x_5^{19}\\
&+ x_1^3x_2^2x_3^{12}x_4^3x_5^{23} + x_1^3x_2^2x_3^{16}x_4^7x_5^{15} + x_1^3x_2^2x_3^{20}x_4^3x_5^{15} + x_1^3x_2^3x_3^3x_4^6x_5^{28}\\
&+ x_1^3x_2^3x_3^3x_4^8x_5^{26} + x_1^3x_2^3x_3^3x_4^{10}x_5^{24} + x_1^3x_2^3x_3^3x_4^{12}x_5^{22} + x_1^3x_2^3x_3^5x_4^6x_5^{26}\\
&+ x_1^3x_2^3x_3^5x_4^{10}x_5^{22} + x_1^3x_2^3x_3^{10}x_4^3x_5^{24} + x_1^3x_2^3x_3^{10}x_4^5x_5^{22} + x_1^3x_2^3x_3^{12}x_4^3x_5^{22}\\
&+ x_1^3x_2^8x_3^5x_4^5x_5^{22} + x_1^3x_2^8x_3^5x_4^7x_5^{20} + x_1^3x_2^8x_3^5x_4^{12}x_5^{15} + x_1^3x_2^8x_3^5x_4^{13}x_5^{14}\\
&+ x_1^3x_2^8x_3^{12}x_4^5x_5^{15},\\
\sum \sigma_8 = {}& x_1^3x_2^4x_3^5x_4^5x_5^{22} + x_1^3x_2^4x_3^5x_4^7x_5^{20} + x_1^3x_2^4x_3^5x_4^{12}x_5^{15} + x_1^3x_2^4x_3^5x_4^{13}x_5^{14}\\
&+ x_1^3x_2^4x_3^{12}x_4^5x_5^{15}.
\end{aligned}
$$

The above relations imply that Z_1 is strictly inadmissible.

Next we prove that Z_2 is also strictly inadmissible strictly inadmissible. A direct computation shows that

$$Z_2 = \sum Y + Sq^1(\sum \beta_1) + Sq^2(\sum \beta_2) + Sq^4(\sum \beta_4)$$

$$+ Sq^8(\sum \beta_8) \text{ modulo}(\mathscr{P}_5^-(\overline{\omega}_{(1)})),$$

where

$$\begin{aligned}
\sum Y = {} & x_1^3x_2x_3^7x_4^{12}x_5^{24} + x_1^3x_2x_3^7x_4^{28}x_5^8 + x_1^3x_2x_3^9x_4^{20}x_5^{14} + x_1^3x_2x_3^9x_4^{28}x_5^6\\
& + x_1^3x_2x_3^{11}x_4^{20}x_5^{12} + x_1^3x_2x_3^{12}x_4^{17}x_5^{14} + x_1^3x_2x_3^{12}x_4^{19}x_5^{12} + x_1^3x_2x_3^{12}x_4^{21}x_5^{10}\\
& + x_1^3x_2x_3^{14}x_4^{17}x_5^{12} + x_1^3x_2x_3^{17}x_4^{12}x_5^{14} + x_1^3x_2x_3^{20}x_4^9x_5^{14} + x_1^3x_2^2x_3^{13}x_4^{13}x_5^{16}\\
& + x_1^3x_2^2x_3^{13}x_4^{21}x_5^8 + x_1^3x_2^4x_3^7x_4^{25}x_5^8 + x_1^3x_2^4x_3^9x_4^{11}x_5^{20} + x_1^3x_2^4x_3^9x_4^{21}x_5^{10}\\
& + x_1^3x_2^4x_3^9x_4^{25}x_5^6 + x_1^3x_2^4x_3^{11}x_4^{12}x_5^{17} + x_1^3x_2^4x_3^{11}x_4^{20}x_5^9 + x_1^3x_2^4x_3^{12}x_4^{11}x_5^{17}\\
& + x_1^3x_2^4x_3^{12}x_4^{19}x_5^9 + x_1^4x_2x_3^7x_4^{11}x_5^{24} + x_1^4x_2x_3^{11}x_4^{17}x_5^{14} + x_1^4x_2x_3^{11}x_4^{19}x_5^{12}\\
& + x_1^4x_2x_3^{11}x_4^{21}x_5^{10} + x_1^4x_2x_3^{17}x_4^{11}x_5^{14} + x_1^4x_2^2x_3^7x_4^{25}x_5^9 + x_1^4x_2^2x_3^{11}x_4^{13}x_5^{17}\\
& + x_1^4x_2^2x_3^{11}x_4^{21}x_5^9 + x_1^4x_2^4x_3^{11}x_4^{11}x_5^{17} + x_1^4x_2^4x_3^{11}x_4^{19}x_5^9 + x_1^5x_2x_3^7x_4^{24}x_5^{10}\\
& + x_1^5x_2x_3^{11}x_4^{12}x_5^{18} + x_1^5x_2x_3^{11}x_4^{14}x_5^{16} + x_1^5x_2x_3^{11}x_4^{20}x_5^{10} + x_1^5x_2x_3^{24}x_4^7x_5^{10}\\
& + x_1^5x_2x_3^{24}x_4^{11}x_5^6 + x_1^5x_2^2x_3^7x_4^{24}x_5^9 + x_1^5x_2^2x_3^{11}x_4^{12}x_5^{17} + x_1^5x_2^2x_3^{11}x_4^{20}x_5^9\\
& + x_1^5x_2^8x_3^7x_4^{17}x_5^{10} + x_1^5x_2^8x_3^{11}x_4^{17}x_5^6 + x_1^7x_2x_3^8x_4^{13}x_5^{18} + x_1^7x_2x_3^8x_4^{14}x_5^{17}\\
& + x_1^7x_2x_3^8x_4^{21}x_5^{10} + x_1^7x_2x_3^8x_4^{22}x_5^9 + x_1^7x_2x_3^9x_4^{12}x_5^{18} + x_1^7x_2x_3^9x_4^{20}x_5^{10}\\
& + x_1^7x_2x_3^{10}x_4^9x_5^{20} + x_1^7x_2x_3^{10}x_4^{12}x_5^{17} + x_1^7x_2x_3^{10}x_4^{13}x_5^{16} + x_1^7x_2x_3^{10}x_4^{17}x_5^{12}\\
& + x_1^7x_2x_3^{10}x_4^{20}x_5^9 + x_1^7x_2x_3^{10}x_4^{21}x_5^8 + x_1^7x_2x_3^{11}x_4^{16}x_5^{12},
\end{aligned}$$

$$\begin{aligned}
\sum \beta_1 = {} & x_1^3x_2x_3^7x_4^{11}x_5^{24} + x_1^3x_2x_3^{11}x_4^{17}x_5^{14} + x_1^3x_2x_3^{11}x_4^{19}x_5^{12} + x_1^3x_2x_3^{11}x_4^{21}x_5^{10}\\
& + x_1^3x_2x_3^{17}x_4^{11}x_5^{14} + x_1^3x_2x_3^{17}x_4^{19}x_5^6 + x_1^3x_2^4x_3^{11}x_4^{11}x_5^{17} + x_1^3x_2^4x_3^{11}x_4^{19}x_5^9\\
& + x_1^7x_2x_3^9x_4^{17}x_5^{12} + x_1^7x_2x_3^{11}x_4^{17}x_5^{10} + x_1^7x_2x_3^{13}x_4^{17}x_5^8 + x_1^7x_2^2x_3^7x_4^{13}x_5^{17}\\
& + x_1^7x_2^2x_3^7x_4^{21}x_5^9 + x_1^7x_2^2x_3^9x_4^{11}x_5^{17} + x_1^7x_2^2x_3^9x_4^{19}x_5^9 + x_1^8x_2x_3^7x_4^{13}x_5^{17}\\
& + x_1^8x_2x_3^7x_4^{21}x_5^9 + x_1^{10}x_2x_3^7x_4^{11}x_5^{17} + x_1^{10}x_2x_3^7x_4^{19}x_5^9,
\end{aligned}$$

$$\begin{aligned}
\sum \beta_2 = {} & x_1^3x_2x_3^{14}x_4^{17}x_5^{10} + x_1^3x_2^2x_3^7x_4^{13}x_5^{20} + x_1^3x_2^2x_3^7x_4^{21}x_5^{12} + x_1^3x_2^2x_3^{11}x_4^{11}x_5^{18}\\
& + x_1^3x_2^2x_3^{11}x_4^{19}x_5^{10} + x_1^5x_2x_3^7x_4^{18}x_5^{14} + x_1^5x_2x_3^7x_4^{26}x_5^6 + x_1^5x_2x_3^{18}x_4^7x_5^{14}\\
& + x_1^5x_2^2x_3^7x_4^{11}x_5^{20} + x_1^5x_2^2x_3^7x_4^{21}x_5^{10} + x_1^5x_2^2x_3^7x_4^{25}x_5^6 + x_1^5x_2^2x_3^{11}x_4^{11}x_5^{16}\\
& + x_1^5x_2^2x_3^{11}x_4^{13}x_5^{14} + x_1^5x_2^2x_3^{11}x_4^{19}x_5^8 + x_1^5x_2^2x_3^{11}x_4^{21}x_5^6 + x_1^5x_2^2x_3^{13}x_4^{11}x_5^{14}\\
& + x_1^5x_2^2x_3^{13}x_4^{19}x_5^6 + x_1^5x_2^4x_3^7x_4^{11}x_5^{18} + x_1^5x_2^4x_3^7x_4^{19}x_5^{10} + x_1^7x_2x_3^7x_4^{13}x_5^{17}\\
& + x_1^7x_2x_3^7x_4^{20}x_5^{10} + x_1^7x_2x_3^7x_4^{21}x_5^9 + x_1^7x_2x_3^9x_4^{11}x_5^{17} + x_1^7x_2x_3^9x_4^{19}x_5^9\\
& + x_1^7x_2x_3^{10}x_4^9x_5^{18} + x_1^7x_2x_3^{10}x_4^{11}x_5^{16} + x_1^7x_2x_3^{10}x_4^{19}x_5^8 + x_1^7x_2x_3^{11}x_4^{18}x_5^8\\
& + x_1^7x_2x_3^{13}x_4^{14}x_5^{10} + x_1^7x_2x_3^{14}x_4^{17}x_5^6 + x_1^7x_2x_3^{18}x_4^7x_5^{12} + x_1^7x_2x_3^{18}x_4^{13}x_5^6\\
& + x_1^7x_2^2x_3^7x_4^{17}x_5^{12} + x_1^7x_2^2x_3^9x_4^{17}x_5^{10} + x_1^7x_2^2x_3^{11}x_4^{17}x_5^8 + x_1^7x_2^2x_3^{13}x_4^{17}x_5^6\\
& + x_1^9x_2x_3^7x_4^{11}x_5^{17} + x_1^9x_2x_3^7x_4^{19}x_5^9,
\end{aligned}$$

$$\begin{aligned}\sum\beta_4 &= x_1^{3}x_2x_3^{7}x_4^{18}x_5^{14} + x_1^{3}x_2x_3^{7}x_4^{22}x_5^{10} + x_1^{3}x_2x_3^{10}x_4^{11}x_5^{18} + x_1^{3}x_2x_3^{10}x_4^{19}x_5^{10}\\
&+ x_1^{3}x_2x_3^{11}x_4^{10}x_5^{18} + x_1^{3}x_2x_3^{11}x_4^{22}x_5^{6} + x_1^{3}x_2x_3^{18}x_4^{7}x_5^{14} + x_1^{3}x_2^{2}x_3^{7}x_4^{11}x_5^{20}\\
&+ x_1^{3}x_2^{2}x_3^{7}x_4^{21}x_5^{10} + x_1^{3}x_2^{2}x_3^{7}x_4^{25}x_5^{6} + x_1^{3}x_2^{2}x_3^{11}x_4^{11}x_5^{16} + x_1^{3}x_2^{2}x_3^{11}x_4^{13}x_5^{14}\\
&+ x_1^{3}x_2^{2}x_3^{11}x_4^{19}x_5^{8} + x_1^{3}x_2^{2}x_3^{11}x_4^{21}x_5^{6} + x_1^{3}x_2^{2}x_3^{13}x_4^{11}x_5^{14} + x_1^{3}x_2^{2}x_3^{13}x_4^{19}x_5^{6}\\
&+ x_1^{4}x_2^{2}x_3^{7}x_4^{13}x_5^{17} + x_1^{4}x_2^{2}x_3^{7}x_4^{21}x_5^{9} + x_1^{5}x_2x_3^{7}x_4^{12}x_5^{18} + x_1^{5}x_2x_3^{7}x_4^{18}x_5^{12}\\
&+ x_1^{5}x_2x_3^{7}x_4^{20}x_5^{10} + x_1^{5}x_2x_3^{11}x_4^{14}x_5^{12} + x_1^{5}x_2x_3^{14}x_4^{17}x_5^{6} + x_1^{5}x_2x_3^{20}x_4^{7}x_5^{10}\\
&+ x_1^{5}x_2x_3^{20}x_4^{11}x_5^{6} + x_1^{5}x_2^{2}x_3^{7}x_4^{11}x_5^{18} + x_1^{5}x_2^{2}x_3^{7}x_4^{12}x_5^{17} + x_1^{5}x_2^{2}x_3^{7}x_4^{19}x_5^{10}\\
&+ x_1^{5}x_2^{2}x_3^{7}x_4^{20}x_5^{9} + x_1^{5}x_2^{4}x_3^{7}x_4^{17}x_5^{10} + x_1^{5}x_2^{4}x_3^{11}x_4^{17}x_5^{6} + x_1^{11}x_2x_3^{7}x_4^{14}x_5^{10}\\
&+ x_1^{11}x_2x_3^{10}x_4^{11}x_5^{10} + x_1^{11}x_2x_3^{18}x_4^{7}x_5^{6} + x_1^{11}x_2^{2}x_3^{7}x_4^{17}x_5^{6},\\
\sum\beta_8 &= x_1^{3}x_2x_3^{7}x_4^{10}x_5^{18} + x_1^{3}x_2x_3^{7}x_4^{22}x_5^{6} + x_1^{3}x_2x_3^{10}x_4^{7}x_5^{18} + x_1^{3}x_2x_3^{10}x_4^{19}x_5^{6}\\
&+ x_1^{7}x_2x_3^{7}x_4^{14}x_5^{10} + x_1^{7}x_2x_3^{10}x_4^{11}x_5^{10} + x_1^{7}x_2x_3^{12}x_4^{9}x_5^{10} + x_1^{7}x_2x_3^{12}x_4^{11}x_5^{8}\\
&+ x_1^{7}x_2x_3^{18}x_4^{7}x_5^{6} + x_1^{7}x_2^{2}x_3^{7}x_4^{17}x_5^{6}.\end{aligned}$$

From the above equalities, we conclude that Z_2 is also strictly inadmissible. □

Lemma 3.3.21. *The following monomials are strictly inadmissible*:

$d_1 = x_1^{3}x_2^{3}x_3^{12}x_4^{13}x_5^{16}$, $d_2 = x_1^{3}x_2^{3}x_3^{13}x_4^{12}x_5^{16}$, $d_3 = x_1^{3}x_2^{3}x_3^{13}x_4^{20}x_5^{8}$, $d_4 = x_1^{3}x_2^{3}x_3^{12}x_4^{20}x_5^{9}$,
$d_5 = x_1^{3}x_2^{7}x_3^{4}x_4^{8}x_5^{25}$, $d_6 = x_1^{3}x_2^{7}x_3^{4}x_4^{25}x_5^{8}$, $d_7 = x_1^{7}x_2^{3}x_3^{4}x_4^{8}x_5^{25}$, $d_8 = x_1^{7}x_2^{3}x_3^{4}x_4^{25}x_5^{8}$,
$d_9 = x_1^{3}x_2^{7}x_3^{4}x_4^{24}x_5^{9}$, $d_{10} = x_1^{3}x_2^{7}x_3^{4}x_4^{9}x_5^{24}$, $d_{11} = x_1^{7}x_2^{3}x_3^{4}x_4^{24}x_5^{9}$, $d_{12} = x_1^{7}x_2^{3}x_3^{4}x_4^{9}x_5^{24}$,
$d_{13} = x_1^{3}x_2^{4}x_3^{11}x_4^{17}x_5^{12}$, $d_{14} = x_1^{3}x_2^{4}x_3^{11}x_4^{16}x_5^{13}$, $d_{15} = x_1^{3}x_2^{4}x_3^{11}x_4^{13}x_5^{16}$, $d_{16} = x_1^{3}x_2^{5}x_3^{6}x_4^{8}x_5^{25}$,
$d_{17} = x_1^{3}x_2^{5}x_3^{6}x_4^{25}x_5^{8}$, $d_{18} = x_1^{3}x_2^{5}x_3^{6}x_4^{24}x_5^{9}$, $d_{19} = x_1^{3}x_2^{5}x_3^{6}x_4^{9}x_5^{24}$, $d_{20} = x_1^{3}x_2^{5}x_3^{7}x_4^{8}x_5^{24}$,
$d_{21} = x_1^{3}x_2^{5}x_3^{7}x_4^{24}x_5^{8}$, $d_{22} = x_1^{3}x_2^{7}x_3^{5}x_4^{8}x_5^{24}$, $d_{23} = x_1^{3}x_2^{7}x_3^{5}x_4^{24}x_5^{8}$, $d_{24} = x_1^{7}x_2^{3}x_3^{5}x_4^{8}x_5^{24}$,
$d_{25} = x_1^{7}x_2^{3}x_3^{5}x_4^{24}x_5^{8}$, $d_{26} = x_1^{3}x_2^{5}x_3^{10}x_4^{17}x_5^{12}$, $d_{27} = x_1^{3}x_2^{5}x_3^{10}x_4^{16}x_5^{13}$, $d_{28} = x_1^{3}x_2^{5}x_3^{11}x_4^{16}x_5^{12}$,
$d_{29} = x_1^{3}x_2^{7}x_3^{8}x_4^{17}x_5^{12}$, $d_{30} = x_1^{7}x_2^{3}x_3^{8}x_4^{17}x_5^{12}$, $d_{31} = x_1^{3}x_2^{7}x_3^{8}x_4^{16}x_5^{13}$, $d_{32} = x_1^{3}x_2^{7}x_3^{8}x_4^{13}x_5^{16}$,
$d_{33} = x_1^{7}x_2^{3}x_3^{8}x_4^{16}x_5^{13}$, $d_{34} = x_1^{7}x_2^{3}x_3^{8}x_4^{13}x_5^{16}$, $d_{35} = x_1^{3}x_2^{7}x_3^{9}x_4^{16}x_5^{12}$, $d_{36} = x_1^{7}x_2^{3}x_3^{9}x_4^{16}x_5^{12}$,
$d_{37} = x_1^{7}x_2^{11}x_3^{17}x_4^{4}x_5^{8}$.

Proof. Note that the weight vector of d_i is $\overline{\omega}_{(1)}$ for $i = 1, 2, \ldots, 37$. We prove the lemma for the monomial $d_1 = x_1^{3}x_2^{3}x_3^{12}x_4^{13}x_5^{16}$. The others can be proved by an argument similar to the proof of Lemma 3.3.20. By a direct computation, we have

$$\begin{aligned}d_1 &= x_1^{2}x_2x_3^{3}x_4^{13}x_5^{28} + x_1^{2}x_2x_3^{3}x_4^{28}x_5^{13} + x_1^{2}x_2x_3^{5}x_4^{13}x_5^{26} + x_1^{2}x_2x_3^{5}x_4^{14}x_5^{25}\\
&+ x_1^{2}x_2x_3^{5}x_4^{25}x_5^{14} + x_1^{2}x_2x_3^{5}x_4^{26}x_5^{13} + x_1^{2}x_2x_3^{10}x_4^{13}x_5^{21} + x_1^{2}x_2x_3^{12}x_4^{7}x_5^{25}\\
&+ x_1^{2}x_2x_3^{12}x_4^{11}x_5^{21} + x_1^{2}x_2x_3^{13}x_4^{14}x_5^{17} + x_1^{2}x_2x_3^{13}x_4^{17}x_5^{14} + x_1^{2}x_2x_3^{18}x_4^{13}x_5^{13}\\
&+ x_1^{2}x_2x_3^{20}x_4^{11}x_5^{13} + x_1^{2}x_2^{3}x_3^{5}x_4^{13}x_5^{24} + x_1^{2}x_2^{3}x_3^{5}x_4^{24}x_5^{13} + x_1^{2}x_2^{3}x_3^{9}x_4^{13}x_5^{20}\\
&+ x_1^{2}x_2^{3}x_3^{12}x_4^{13}x_5^{17} + x_1^{2}x_2^{3}x_3^{16}x_4^{13}x_5^{13} + x_1^{2}x_2^{5}x_3^{3}x_4^{13}x_5^{24} + x_1^{2}x_2^{5}x_3^{3}x_4^{24}x_5^{13}\\
&+ x_1^{2}x_2^{5}x_3^{9}x_4^{13}x_5^{18} + x_1^{2}x_2^{5}x_3^{9}x_4^{17}x_5^{14} + x_1^{2}x_2^{5}x_3^{12}x_4^{11}x_5^{17} + x_1^{2}x_2^{5}x_3^{16}x_4^{11}x_5^{13}\\
&+ x_1^{3}x_2x_3^{4}x_4^{13}x_5^{26} + x_1^{3}x_2x_3^{4}x_4^{14}x_5^{25} + x_1^{3}x_2x_3^{4}x_4^{25}x_5^{14} + x_1^{3}x_2x_3^{4}x_4^{26}x_5^{13}\end{aligned}$$

$$
\begin{aligned}
&+ x_1^3x_2x_3^9x_4^{14}x_5^{20} + x_1^3x_2x_3^9x_4^{20}x_5^{14} + x_1^3x_2x_3^{10}x_4^{12}x_5^{21} + x_1^3x_2x_3^{12}x_4^7x_5^{24}\\
&+ x_1^3x_2x_3^{12}x_4^9x_5^{22} + x_1^3x_2x_3^{12}x_4^{14}x_5^{17} + x_1^3x_2x_3^{12}x_4^{17}x_5^{14} + x_1^3x_2x_3^{18}x_4^{12}x_5^{13}\\
&+ x_1^3x_2x_3^{20}x_4^9x_5^{14} + x_1^3x_2^2x_3^5x_4^{13}x_5^{24} + x_1^3x_2^2x_3^5x_4^{24}x_5^{13} + x_1^3x_2^2x_3^{12}x_4^{17}x_5^{13}\\
&+ x_1^3x_2^3x_3^4x_4^{13}x_5^{24} + x_1^3x_2^3x_3^8x_4^{12}x_5^{21} + x_1^3x_2^3x_3^{12}x_4^{12}x_5^{17}\\
&+ Sq^1(\sum X) + Sq^2(\sum Y) + Sq^4(\sum Z)\\
&+ Sq^8(\sum T)\ \text{modulo}(\mathscr{P}_5^-(\overline{\omega}_{(1)})),
\end{aligned}
$$

where

$$
\begin{aligned}
\sum X = {}& x_1^3x_2x_3^3x_4^{13}x_5^{26} + x_1^3x_2x_3^3x_4^{14}x_5^{25} + x_1^3x_2x_3^3x_4^{17}x_5^{22} + x_1^3x_2x_3^3x_4^{18}x_5^{21}\\
&+ x_1^3x_2x_3^3x_4^{21}x_5^{18} + x_1^3x_2x_3^3x_4^{22}x_5^{17} + x_1^3x_2x_3^3x_4^{25}x_5^{14} + x_1^3x_2x_3^3x_4^{26}x_5^{13}\\
&+ x_1^3x_2x_3^7x_4^{17}x_5^{18} + x_1^3x_2x_3^7x_4^{18}x_5^{17} + x_1^3x_2x_3^{10}x_4^7x_5^{25} + x_1^3x_2x_3^{10}x_4^{11}x_5^{21}\\
&+ x_1^3x_2x_3^{11}x_4^{14}x_5^{17} + x_1^3x_2x_3^{11}x_4^{17}x_5^{14} + x_1^3x_2x_3^{18}x_4^7x_5^{17} + x_1^3x_2x_3^{18}x_4^{11}x_5^{13}\\
&+ x_1^3x_2^3x_3^3x_4^{13}x_5^{24} + x_1^3x_2^3x_3^3x_4^{17}x_5^{20} + x_1^3x_2^3x_3^3x_4^{20}x_5^{17} + x_1^3x_2^3x_3^3x_4^{24}x_5^{13}\\
&+ x_1^3x_2^3x_3^9x_4^{13}x_5^{18} + x_1^3x_2^3x_3^9x_4^{17}x_5^{14} + x_1^3x_2^3x_3^{12}x_4^{11}x_5^{17} + x_1^3x_2^3x_3^{16}x_4^{11}x_5^{13}\\
&+ x_1^3x_2^4x_3^9x_4^{17}x_5^{13},\\
\sum Y = {}& x_1^2x_2x_3^3x_4^{13}x_5^{26} + x_1^2x_2x_3^3x_4^{14}x_5^{25} + x_1^2x_2x_3^3x_4^{25}x_5^{14} + x_1^2x_2x_3^3x_4^{26}x_5^{13}\\
&+ x_1^2x_2x_3^{10}x_4^7x_5^{25} + x_1^2x_2x_3^{10}x_4^{11}x_5^{21} + x_1^2x_2x_3^{11}x_4^{14}x_5^{17} + x_1^2x_2x_3^{11}x_4^{17}x_5^{14}\\
&+ x_1^2x_2x_3^{18}x_4^{11}x_5^{13} + x_1^2x_2^3x_3^3x_4^{13}x_5^{24} + x_1^2x_2^3x_3^3x_4^{24}x_5^{13} + x_1^2x_2^3x_3^9x_4^{13}x_5^{18}\\
&+ x_1^2x_2^3x_3^9x_4^{17}x_5^{14} + x_1^2x_2^3x_3^{12}x_4^{11}x_5^{17} + x_1^2x_2^3x_3^{16}x_4^{11}x_5^{13} + x_1^5x_2x_3^7x_4^{14}x_5^{18}\\
&+ x_1^5x_2x_3^7x_4^{18}x_5^{14} + x_1^5x_2x_3^{10}x_4^7x_5^{22} + x_1^5x_2x_3^{18}x_4^7x_5^{14} + x_1^5x_2^2x_3^3x_4^{13}x_5^{22}\\
&+ x_1^5x_2^2x_3^3x_4^{14}x_5^{21} + x_1^5x_2^2x_3^3x_4^{21}x_5^{14} + x_1^5x_2^2x_3^3x_4^{22}x_5^{13} + x_1^5x_2^2x_3^7x_4^{14}x_5^{17}\\
&+ x_1^5x_2^2x_3^7x_4^{17}x_5^{14} + x_1^5x_2^2x_3^{10}x_4^7x_5^{21} + x_1^5x_2^2x_3^{18}x_4^7x_5^{13} + x_1^5x_2^3x_3^3x_4^{14}x_5^{20}\\
&+ x_1^5x_2^3x_3^3x_4^{20}x_5^{14} + x_1^5x_2^3x_3^5x_4^{14}x_5^{18} + x_1^5x_2^3x_3^5x_4^{18}x_5^{14} + x_1^5x_2^3x_3^{10}x_4^{11}x_5^{16}\\
&+ x_1^5x_2^3x_3^{10}x_4^{13}x_5^{14} + x_1^5x_2^3x_3^{12}x_4^{11}x_5^{14},\\
\sum Z = {}& x_1^3x_2x_3^7x_4^{14}x_5^{18} + x_1^3x_2x_3^7x_4^{18}x_5^{14} + x_1^3x_2x_3^{18}x_4^7x_5^{14} + x_1^3x_2^2x_3^3x_4^{13}x_5^{22}\\
&+ x_1^3x_2^2x_3^3x_4^{14}x_5^{21} + x_1^3x_2^2x_3^3x_4^{21}x_5^{14} + x_1^3x_2^2x_3^3x_4^{22}x_5^{13} + x_1^3x_2^2x_3^7x_4^{14}x_5^{17}\\
&+ x_1^3x_2^2x_3^7x_4^{17}x_5^{14} + x_1^3x_2^2x_3^{10}x_4^7x_5^{21} + x_1^3x_2^2x_3^{18}x_4^7x_5^{13} + x_1^3x_2^3x_3^3x_4^{14}x_5^{20}\\
&+ x_1^3x_2^3x_3^3x_4^{20}x_5^{14} + x_1^3x_2^3x_3^4x_4^{20}x_5^{13} + x_1^3x_2^3x_3^5x_4^{14}x_5^{18} + x_1^3x_2^3x_3^5x_4^{18}x_5^{14}\\
&+ x_1^3x_2^3x_3^{10}x_4^{11}x_5^{16} + x_1^3x_2^3x_3^{10}x_4^{13}x_5^{14} + x_1^3x_2^3x_3^{12}x_4^{11}x_5^{14} + x_1^3x_2^8x_3^5x_4^{13}x_5^{14}\\
&+ x_1^3x_2^8x_3^5x_4^{14}x_5^{13} + x_1^3x_2^8x_3^{12}x_4^7x_5^{13},\\
\sum T = {}& x_1^3x_2^2x_3^{12}x_4^9x_5^{13} + x_1^3x_2^3x_3^8x_4^{12}x_5^{13} + x_1^3x_2^4x_3^5x_4^{13}x_5^{14} + x_1^3x_2^4x_3^5x_4^{14}x_5^{13}
\end{aligned}
$$

$$+ x_1^3x_2^4x_3^{10}x_4^9x_5^{13} + x_1^3x_2^4x_3^{12}x_4^7x_5^{13}.$$

The above relations show that d_1 is strictly inadmissible. The lemma follows. $\square$

Proof of Proposition 3.3.18. Let a be an admissible monomial in $\mathscr{P}_5^+$ such that $\omega(a) = \overline{\omega}_{(1)}$. Then $a = x_tx_kx_mu^2$ with $1 \leq t < k < m \leq 5$ and $u \in (\mathscr{P}_5^+)_{22}$. Since a is admissible, by Theorem 2.2.1, u is also admissible. Further, $u \in \mathscr{B}_5^+(\omega_{(1)})$.

Let $X \in \mathscr{B}_5^+(\omega_{(1)})$ such that $x_tx_kx_mX^2 \in (\mathscr{P}_5^+)_{47}$. By a direct computation using Proposition 3.3.10, we see that if $x_tx_kx_mX^2 \neq \mathcal{Y}_{47,\,j},\ 1 \leq j \leq 370$, then there is a monomial b which is given in one of Lemmas 3.3.16-3.3.21 such that $x_tx_kx_mX^2 = bY^{2^r}$ with suitable monomia $Y \in \mathscr{P}_5$, and $r = \max\{s \in \mathbb{Z} : \omega_s(b) > 0\}$. By Theorem 2.2.1, $x_tx_kx_mX^2$ is inadmissible. On the other hand, we have $a = x_tx_kx_mu^2$ and a is admissible, hence $a = \mathcal{Y}_j$ for some $j = 1, 2, \ldots, 370$. This completes the proof of the proposition. $\square$

Remark 3.3.22. For $1 \leq j \leq 370$, we have $[\mathcal{Y}_j := \mathcal{Y}_{47,\,j}]_{\overline{\omega}_{(1)}} \neq [0]$. Indeed, suppose that there is a linear relation

$$\mathcal{S} = \sum_{1\leq j\leq 170} \gamma_j\mathcal{Y}_j \equiv_{\overline{\omega}_{(1)}} 0,$$

where $\gamma_j \in \mathbb{Z}/2$, for all j. Based on Theorem 3.1.3 and Proposition 3.2.1, for $(k;\mathscr{K}) \in \mathcal{N}_5$, we explicitly compute $\pi_{(k;\mathscr{K})}(\mathcal{S})$ in terms of a given minimal set of $\mathcal{A}_2$-generators in $\mathscr{P}_4$ (modulo$(\mathcal{A}_2^+\mathscr{P}_4)$). By computing from the relations $\pi_{(k;\mathscr{K})}(\mathcal{S}) \equiv_{\overline{\omega}_{(1)}} 0,\ \ell(\mathscr{K}) = 1$, one gets $\gamma_j = 0, \forall j,\ 1 \leq j \leq 370$. Note that these computations are similar to the proof of Propositions 3.3.1 and 3.3.10. Combining this and Proposition 3.3.18, we have a direct corollary.

Corollary 3.3.23. *The set* $[\overline{\Phi}^+(\mathscr{B}_4(\overline{\omega}_{(1)}))\bigcup\big(\bigcup_{1\leq k\leq 5}\overline{\mathscr{B}}^+(k,\overline{\omega}_{(1)})\big)\bigcup\mathcal{F}]_{\overline{\omega}_{(1)}}$ *is a basis of the* $\mathbb{Z}/2$*-vector space* $Q\mathscr{P}_5^+(\overline{\omega}_{(1)})$*. This implies* $\dim(Q\mathscr{P}_5^+(\overline{\omega}_{(1)})) = 370$.

Remark 3.3.24. Consider the weight vector $\omega^{**} = (3, 2, 4)$ with $\deg\omega^{**} = 23$. By using a result in [62], we see that the following monomials are strictly inadmissible:

$e_1 = x_1^7x_2^2x_3^4x_4^5x_5^5$, $e_2 = x_1^7x_2^2x_3^5x_4^4x_5^5$, $e_3 = x_1^7x_2^2x_3^5x_4^5x_5^4$, $e_4 = x_1^3x_2^7x_3^4x_4^4x_5^5$,
$e_5 = x_1^3x_2^7x_3^4x_4^5x_5^4$, $e_6 = x_1^3x_2^7x_3^5x_4^4x_5^4$, $e_7 = x_1^7x_2^3x_3^4x_4^4x_5^5$, $e_8 = x_1^7x_2^3x_3^4x_4^5x_5^4$,
$e_9 = x_1^7x_2^3x_3^5x_4^4x_5^4$, $e_{10} = x_1^3x_2^4x_3^4x_4^5x_5^7$, $e_{11} = x_1^3x_2^4x_3^4x_4^7x_5^5$, $e_{12} = x_1^3x_2^4x_3^5x_4^4x_5^7$,
$e_{13} = x_1^3x_2^4x_3^7x_4^4x_5^5$, $e_{14} = x_1^3x_2^4x_3^5x_4^7x_5^4$, $e_{15} = x_1^3x_2^4x_3^7x_4^5x_5^4$, $e_{16} = x_1^3x_2^5x_3^4x_4^4x_5^7$,
$e_{17} = x_1^3x_2^5x_3^4x_4^7x_5^4$, $e_{18} = x_1^3x_2^5x_3^7x_4^4x_5^4$, $e_{19} = x_1^3x_2^6x_3^4x_4^5x_5^5$, $e_{20} = x_1^3x_2^6x_3^5x_4^4x_5^5$,
$e_{21} = x_1^3x_2^6x_3^5x_4^5x_5^4$, $e_{22} = x_1^3x_2^4x_3^6x_4^5x_5^5$, $e_{23} = x_1^3x_2^4x_3^5x_4^6x_5^5$, $e_{24} = x_1^3x_2^4x_3^5x_4^5x_5^6$,
$e_{25} = x_1^3x_2^5x_3^4x_4^6x_5^5$, $e_{26} = x_1^3x_2^5x_3^6x_4^4x_5^5$, $e_{27} = x_1^3x_2^5x_3^4x_4^5x_5^6$, $e_{28} = x_1^3x_2^5x_3^6x_4^5x_5^4$,
$e_{29} = x_1^3x_2^5x_3^5x_4^4x_5^6$, $e_{30} = x_1^3x_2^5x_3^5x_4^6x_5^4$.

Note that $\omega(e_k) = \omega^{**},\ k = 1, 2, \ldots, 30$. Let $X \in \mathscr{B}_5^+(47)$ such that either $\omega(X) = \overline{\omega}_{(2)}$ or $\omega(X) = \overline{\omega}_{(3)}$. Then $X = X_{(\{\ell,m\},\,5)}y^2$ with $1 \leq \ell < m \leq 5$, and

$y \in (\mathscr{P}_5^+)_{22}$. By a direct computation using Theorem 2.2.1, and Proposition 3.3.14(I), (II), we see that either $y \in \mathscr{B}_5^+(\omega_{(2)})$ or $y \in \mathscr{B}_5^+(\omega_{(3)})$ and there is a monomial $Z = e_i$ for some i, $1 \leq i \leq 30$ such that $X = X_{(\{\ell,m\},5)}y^2 = Zu^{2^t}$, $1 \leq \ell < m \leq 5$, with suitable monomia $u \in \mathscr{P}_5$, and $t = \max\{a \in \mathbb{Z} : \omega_a(Z) > 0\}$. By Theorem 2.2.1, we get either $[X] = [0]_{\overline{\omega}_{(2)}}$ or $[X] = [0]_{\overline{\omega}_{(3)}}$. As a consequence, we get the following.

Proposition 3.3.25. *The $\mathbb{Z}/2$-vector spaces $Q\mathscr{P}_5^+(\overline{\omega}_{(2)})$ and $Q\mathscr{P}_5^+(\overline{\omega}_{(3)})$ are trivial.*

Lemma 3.3.26. *The set $\{[\mathcal{Y}_{47,\,j}]_{\overline{\omega}_{(4)}} : 371 \leq j \leq 479\}$ is a minimal system of generators for $Q\mathscr{P}_5^+(\overline{\omega}_{(4)}))$, where the monomials $\mathcal{Y}_{47,\,j},\ j = 371, \ldots, 479$, are described in Section 6.7 of the online version* [40].

In order to prove the lemma, we need to use some results. As an immediate consequence of a result in [54], we obtain the following.

Lemma 3.3.27. *The following monomials are strictly inadmissible*:

I) $x_a^7x_b^2x_c^{15}x_d^7,\ x_a^7x_b^{15}x_c^2x_d^7,\ x_a^7x_b^6x_c^3x_d^{15},\ x_a^7x_b^3x_c^6x_d^{15},\ x_a^7x_b^3x_c^{14}x_d^7,$
$x_a^7x_b^{14}x_c^3x_d^7,\ x_a^7x_b^6x_c^{11}x_d^7,\ x_a^7x_b^{11}x_c^6x_d^7,$
$1 \leq b < c \leq 5,\ 1 \leq a,\, d \leq 5,\ a \neq d,\ a, d \neq b, c.$

II) $\rho_{(k,\,5)}(v)$, $1 \leq k \leq 5$, *where v is one of the following monomials*:

$$x_1^7x_2^7x_3^7x_4^{10},\ x_1^7x_2^7x_3^{10}x_4^7,\ x_1^7x_2^{10}x_3^7x_4^7.$$

We now consider the weight vector $\omega^{***} = (3, 4, 3, 1)$ with $\deg\omega^{***} = 31$. The following lemma can be easily proved by a direct computation.

Lemma 3.3.28. *All permutations of the following monomials are strictly inadmissible*:

$$\begin{array}{llll}
x_1x_2^2x_3^6x_4^7x_5^{15}, & x_1x_2^2x_3^14x_4^7x_5^7, & x_1x_2^6x_3^7x_4^7x_5^{10}, & x_1^7x_2^2x_3^2x_4^5x_5^{15},\\
x_1^7x_2^2x_3^2x_4^7x_5^{13}, & x_1^3x_2^2x_3^4x_4^7x_5^{15}, & x_1^3x_2^2x_3^7x_4^7x_5^{12}, & x_1^7x_2^2x_3^4x_4^7x_5^{11},\\
x_1^7x_2^2x_3^5x_4^7x_5^{10}, & x_1^7x_2^2x_3^6x_4^7x_5^9, & x_1^7x_2^2x_3^7x_4^7x_5^8, & x_1^3x_2^3x_3^4x_4^6x_5^{15},\\
x_1^3x_2^3x_3^4x_4^7x_5^{14}, & x_1^3x_2^3x_3^6x_4^7x_5^{12}, & x_1^3x_2^4x_3^6x_4^7x_5^{11}, & x_1^3x_2^4x_3^7x_4^7x_5^{10},\\
x_1^3x_2^6x_3^7x_4^7x_5^8. & & &
\end{array}$$

Lemma 3.3.29. *If (m, n, p, q, r) is a permutation of $(1, 2, 3, 4, 5)$, then the following monomials are strictly inadmissible*:

$$x_m^3x_n^{13}x_p^6x_q^3x_r^6 \neq X \in \{x_1^3x_2^3x_3^{13}x_4^6x_5^6,\ x_1^3x_2^{13}x_3^3x_4^6x_5^6\},$$
$$x_m^7x_n^2x_p^5x_q^6x_r^{11} \neq Y \in \{x_1^7x_2^{11}x_3^5x_4^2x_5^6,\ x_1^7x_2^{11}x_3^5x_4^6x_5^2\},$$
$$x_m^3x_n^5x_p^6x_q^6x_r^{11} \neq Z \in \{x_1^3x_2^5x_3^6x_4^6x_5^{11},\ x_1^3x_2^5x_3^{11}x_4^6x_5^6\},$$
$$x_m^3x_n^7x_p^9x_q^6x_r^6 \neq G \in \{x_1^3x_2^7x_3^9x_4^6x_5^6,\ x_1^7x_2^3x_3^9x_4^6x_5^6,\ x_1^7x_2^9x_3^3x_4^6x_5^6\},$$
$$x_m^3x_n^2x_p^6x_q^7x_r^{13} \neq H \in \{x_1^3x_2^7x_3^{13}x_4^2x_5^6,\ x_1^3x_2^7x_3^{13}x_4^6x_5^2,\ x_1^7x_2^3x_3^{13}x_4^2x_5^6,$$
$$x_1^7x_2^3x_3^{13}x_4^6x_5^2\},$$
$$x_m^3x_n^5x_p^6x_q^7x_r^{10} \neq F \in \{x_1^3x_2^5x_3^6x_4^7x_5^{10},\ x_1^3x_2^5x_3^7x_4^6x_5^{10},\ x_1^3x_2^5x_3^7x_4^{10}x_5^6,$$

$$x_1^3x_2^7x_3^5x_4^6x_5^{10},\ x_1^3x_2^7x_3^5x_4^{10}x_5^6,\ x_1^7x_2^3x_3^5x_4^6x_5^{10},\ x_1^7x_2^3x_3^5x_4^{10}x_5^6\},$$

$$x_m^3x_n^5x_p^{14}x_q^3x_r^6 \neq T \in \{x_1^3x_2^3x_3^5x_4^6x_5^{14},\ x_1^3x_2^3x_3^5x_4^{14}x_5^6,\ x_1^3x_2^5x_3^3x_4^6x_5^{14},\ x_1^3x_2^5x_3^3x_4^{14}x_5^6,\ x_1^3x_2^5x_3^6x_4^3x_5^{14},\ x_1^3x_2^5x_3^6x_4^{14}x_5^3\}.$$

Proof. It is easy to see that the weight vector of these monomials is ω^{***}. Note that the monomials X, Y, Z, G, H, F, and T are admissible. We now prove the lemma for the monomials $f = x_m^3x_n^{13}x_p^6x_q^3x_r^6$ and $g = x_m^3x_n^5x_p^{14}x_q^3x_r^6$. The others can be proved by an argument similar to the proofs of Lemmas 3.3.16 and 3.3.17. Applying the Cartan formula, we get

$$f = x_m^3x_n^{13}x_p^3x_q^6x_r^6 + Sq^1(\sum \overline{A}_1) + Sq^2(\sum \overline{B}_1) + Sq^4(\sum \overline{C}_1) \text{ modulo}(\mathscr{P}_5^-(\omega^{***})),$$

where

$$\begin{aligned}\sum \overline{A}_1 &= x_m^3x_n^7x_p^5x_q^5x_r^{10} + x_m^3x_n^7x_p^5x_q^6x_r^9 + x_m^3x_n^7x_p^6x_q^5x_r^9 + x_m^3x_n^7x_p^6x_q^9x_r^5\\ &+ x_m^3x_n^7x_p^9x_q^6x_r^5 + x_m^3x_n^{11}x_p^5x_q^6x_r^5 + x_m^3x_n^{14}x_p^3x_q^5x_r^5 + x_m^6x_n^{11}x_p^5x_q^3x_r^5,\\ \sum \overline{B}_1 &= x_m^3x_n^{11}x_p^5x_q^5x_r^5 + x_m^3x_n^{11}x_p^6x_q^3x_r^6 + x_m^3x_n^{13}x_p^5x_q^3x_r^5 + x_m^5x_n^7x_p^3x_q^5x_r^9\\ &+ x_m^5x_n^7x_p^3x_q^9x_r^5 + x_m^5x_n^7x_p^5x_q^3x_r^9 + x_m^5x_n^7x_p^9x_q^3x_r^5 + x_m^5x_n^9x_p^3x_q^3x_r^9\\ &+ x_m^5x_n^{11}x_p^3x_q^5x_r^5,\\ \sum \overline{C}_1 &= x_m^3x_n^7x_p^3x_q^5x_r^9 + x_m^3x_n^7x_p^3x_q^9x_r^5 + x_m^3x_n^7x_p^5x_q^3x_r^9 + x_m^3x_n^7x_p^9x_q^3x_r^5\\ &+ x_m^3x_n^9x_p^3x_q^3x_r^9 + x_m^3x_n^9x_p^5x_q^5x_r^5 + x_m^3x_n^{11}x_p^3x_q^5x_r^5 + x_m^3x_n^{11}x_p^5x_q^3x_r^5.\end{aligned}$$

Since $x_m^3x_n^{13}x_p^3x_q^6x_r^6 < f$, the monomial f is strictly inadmissible. Next, by a direct computation, we have

$$g = \sum \mathcal{L} + Sq^1(\sum \overline{A}_2) + Sq^2(\sum \overline{B}_2) + Sq^4(\sum \overline{C}_2) \text{ modulo}(\mathscr{P}_5^-(\omega^{***})),$$

where

$$\begin{aligned}\sum \mathcal{L} &= x_m^3x_n^3x_p^{14}x_q^5x_r^6 + x_m^3x_n^5x_p^7x_q^6x_r^{10} + x_m^3x_n^5x_p^{11}x_q^6x_r^6,\\ \sum \overline{A}_2 &= x_m^3x_n^3x_p^{13}x_q^5x_r^6 + x_m^3x_n^3x_p^{14}x_q^5x_r^5 + x_m^3x_n^5x_p^{11}x_q^5x_r^6 + x_m^3x_n^5x_p^{13}x_q^3x_r^6\\ &+ x_m^3x_n^6x_p^7x_q^9x_r^5 + x_m^3x_n^6x_p^{11}x_q^5x_r^5 + x_m^3x_n^{10}x_p^7x_q^5x_r^5 + x_m^6x_n^5x_p^{11}x_q^3x_r^5,\\ \sum \overline{B}_2 &= x_m^3x_n^5x_p^{11}x_q^5x_r^5 + x_m^3x_n^5x_p^{13}x_q^3x_r^5 + x_m^3x_n^6x_p^{11}x_q^3x_r^6 + x_m^5x_n^3x_p^7x_q^5x_r^9\\ &+ x_m^5x_n^3x_p^7x_q^9x_r^5 + x_m^5x_n^3x_p^9x_q^3x_r^9 + x_m^5x_n^3x_p^{11}x_q^5x_r^5 + x_m^5x_n^5x_p^7x_q^3x_r^9\\ &+ x_m^5x_n^9x_p^7x_q^3x_r^5,\\ \sum \overline{C}_2 &= x_m^3x_n^3x_p^7x_q^5x_r^9 + x_m^3x_n^3x_p^7x_q^9x_r^5 + x_m^3x_n^3x_p^9x_q^3x_r^9 + x_m^3x_n^3x_p^{11}x_q^5x_r^5\\ &+ x_m^3x_n^5x_p^7x_q^3x_r^9 + x_m^3x_n^5x_p^7x_q^6x_r^6 + x_m^3x_n^5x_p^9x_q^5x_r^5 + x_m^3x_n^5x_p^{11}x_q^3x_r^5\\ &+ x_m^3x_n^9x_p^7x_q^3x_r^5.\end{aligned}$$

These relations show that g is strictly inadmissible. The lemmas follows. □

The proof of the following lemmas is analogous to the proofs of Lemmas 3.3.16, 3.3.17, 3.3.28 and 3.3.29.

Lemma 3.3.30. *If* (p, q, r) *is a permutation of* $(3, 4, 5)$, *then the following monomials are strictly inadmissible*:

$u_1 = x_1x_2^6x_p^6x_q^3x_r^{15}$, $u_2 = x_1^3x_2^6x_p^6x_qx_r^{15}$, $u_3 = x_1^{16}x_2^6x_p^6x_qx_r^3$, $u_4 = x_1x_2^6x_p^3x_q^7x_r^{14}$,
$u_5 = x_1x_2^{14}x_p^3x_q^6x_r^7$, $u_6 = x_1^3x_2^6x_px_q^7x_r^{14}$, $u_7 = x_1^3x_2^{14}x_px_q^6x_r^7$, $u_8 = x_1^7x_2^6x_px_q^3x_r^{14}$,
$u_9 = x_1^7x_2^{14}x_px_q^3x_r^6$, $u_{10} = x_1x_2^6x_p^6x_q^7x_r^{11}$, $u_{11} = x_1^7x_2^6x_px_q^6x_r^{11}$, $u_{12} = x_1^3x_2^2x_p^5x_q^6x_r^{15}$,
$u_{13} = x_1^3x_2^6x_p^2x_q^5x_r^{15}$, $u_{14} = x_1^{15}x_2^2x_p^3x_q^5x_r^6$, $u_{15} = x_1^{15}x_2^6x_p^2x_q^3x_r^5$, $u_{16} = x_1^3x_2^2x_p^5x_q^7x_r^{14}$,
$u_{17} = x_1^3x_2^{14}x_p^2x_q^5x_r^7$, $u_{18} = x_1^7x_2^2x_p^{14}x_q^3x_r^5$, $u_{19} = x_1^7x_2^{14}x_p^2x_q^3x_r^5$.

Lemma 3.3.31. *The following monomials are strictly inadmissible*:

I) $u_{20} = x_1x_2^{15}x_3^6x_q^3x_r^6$, $u_{21} = x_1^3x_2^{15}x_3^6x_qx_r^6$, $u_{22} = x_1^{15}x_2x_3^6x_q^3x_r^6$,
$u_{23} = x_1^{15}x_2^3x_3^6x_qx_r^6$, $u_{24} = x_1^7x_2x_3^6x_q^3x_r^{14}$, $u_{25} = x_1^7x_2x_3^{14}x_q^3x_r^6$,
$u_{26} = x_1^7x_2^3x_3^6x_qx_r^{14}$, $u_{27} = x_1^7x_2^3x_3^{14}x_qx_r^6$, $u_{28} = x_1x_2^3x_3^{14}x_q^6x_r^7$,
$u_{29} = x_1x_2^7x_3^6x_q^3x_r^{14}$, $u_{30} = x_1x_2^7x_3^{14}x_q^3x_r^6$, $u_{31} = x_1^3x_2x_3^{14}x_q^6x_r^7$,
$u_{32} = x_1^3x_2^7x_3^6x_qx_r^{14}$, $u_{33} = x_1^3x_2^7x_3^{14}x_qx_r^6$, $u_{34} = x_1x_2^7x_3^6x_q^6x_r^{11}$,
$u_{35} = x_1^7x_2x_3^6x_q^6x_r^{11}$, $u_{36} = x_1^7x_2^{11}x_3^6x_qx_r^6$, $u_{37} = x_1^3x_2^{15}x_3^2x_q^5x_r^6$,
$u_{38} = x_1^3x_2^{15}x_3^6x_q^2x_r^5$, $u_{39} = x_1^{15}x_2^3x_3^2x_q^5x_r^6$, $u_{40} = x_1^{15}x_2^3x_3^6x_q^2x_r^5$,
$u_{41} = x_1^3x_2^5x_3^{14}x_q^2x_r^7$, $u_{42} = x_1^3x_2^5x_3^{14}x_q^2x_r^7$, $u_{43} = x_1^3x_2^7x_3^2x_q^5x_r^{14}$,
$u_{44} = x_1^3x_2^7x_3^{14}x_q^2x_r^5$, $u_{45} = x_1^7x_2^3x_3^2x_q^5x_r^{14}$, $u_{46} = x_1^7x_2^3x_3^{14}x_q^2x_r^5$,
where q, $r = 4$, 5, $q \neq r$.

II) $u_{47} = x_1x_2^3x_3^6x_4^{14}x_5^7$, $u_{48} = x_1^3x_2x_3^6x_4^{14}x_5^7$, $u_{49} = x_1^3x_2^5x_3^2x_4^{14}x_5^7$.

Note that $\omega(u_t) = \omega^{***}$ for $t = 1, 2, \ldots, 49$. Next we consider the following lemma.

Lemma 3.3.32. *The following monomials are strictly inadmissible*:

$A_1 = x_1x_2^3x_3^6x_4^{15}x_5^{22}$, $A_2 = x_1x_2^3x_3^{15}x_4^6x_5^{22}$, $A_3 = x_1x_2^3x_3^{15}x_4^{22}x_5^6$,
$A_4 = x_1x_2^{15}x_3^3x_4^6x_5^{22}$, $A_5 = x_1x_2^{15}x_3^3x_4^{22}x_5^6$, $A_6 = x_1x_2^{15}x_3^{19}x_4^6x_5^6$,
$A_7 = x_1^3x_2x_3^6x_4^{15}x_5^{22}$, $A_8 = x_1^3x_2^5x_3^2x_4^{15}x_5^{22}$, $A_9 = x_1^3x_2^5x_3^6x_4^{15}x_5^{18}$,
$A_{10} = x_1^3x_2x_3^{15}x_4^6x_5^{22}$, $A_{11} = x_1^3x_2x_3^{15}x_4^{22}x_5^6$, $A_{12} = x_1^3x_2^5x_3^{15}x_4^2x_5^{22}$,
$A_{13} = x_1^3x_2^5x_3^{15}x_4^6x_5^{18}$, $A_{14} = x_1^3x_2^5x_3^{15}x_4^{18}x_5^6$, $A_{15} = x_1^3x_2^5x_3^{15}x_4^{22}x_5^2$,
$A_{16} = x_1^3x_2^{13}x_3^3x_4^6x_5^{22}$, $A_{17} = x_1^3x_2^{13}x_3^3x_4^{22}x_5^6$, $A_{18} = x_1^3x_2^{13}x_3^{19}x_4^6x_5^6$,
$A_{19} = x_1^3x_2^{15}x_3x_4^6x_5^{22}$, $A_{20} = x_1^3x_2^{15}x_3x_4^{22}x_5^6$, $A_{21} = x_1^3x_2^{15}x_3^5x_4^2x_5^{22}$,
$A_{22} = x_1^3x_2^{15}x_3^5x_4^6x_5^{18}$, $A_{23} = x_1^3x_2^{15}x_3^5x_4^{18}x_5^6$, $A_{24} = x_1^3x_2^{15}x_3^5x_4^{22}x_5^2$,
$A_{25} = x_1^3x_2^{15}x_3^{17}x_4^6x_5^6$, $A_{26} = x_1^3x_2^{15}x_3^{21}x_4^2x_5^6$, $A_{27} = x_1^3x_2^{15}x_3^{21}x_4^6x_5^2$,
$A_{28} = x_1^7x_2^9x_3^3x_4^6x_5^{22}$, $A_{29} = x_1^7x_2^9x_3^3x_4^{22}x_5^6$, $A_{30} = x_1^7x_2^9x_3^{19}x_4^6x_5^6$,
$A_{31} = x_1^7x_2^{11}x_3^{17}x_4^6x_5^6$, $A_{32} = x_1^7x_2^{25}x_3^3x_4^6x_5^6$, $A_{33} = x_1^{15}x_2x_3^3x_4^6x_5^{22}$,

$$A_{34} = x_1^{15}x_2x_3^3x_4^{22}x_5^6,\ A_{35} = x_1^{15}x_2x_3^{19}x_4^6x_5^6,\ A_{36} = x_1^{15}x_2^3x_3x_4^6x_5^{22},$$
$$A_{37} = x_1^{15}x_2^3x_3x_4^{22}x_5^6,\ A_{38} = x_1^{15}x_2^3x_3^5x_4^2x_5^{22},\ A_{39} = x_1^{15}x_2^3x_3^5x_4^6x_5^{18},$$
$$A_{40} = x_1^{15}x_2^3x_3^5x_4^{18}x_5^6,\ A_{41} = x_1^{15}x_2^3x_3^5x_4^{22}x_5^2,\ A_{42} = x_1^{15}x_2^3x_3^{17}x_4^6x_5^6,$$
$$A_{43} = x_1^{15}x_2^3x_3^{21}x_4^2x_5^6,\ A_{44} = x_1^{15}x_2^3x_3^{21}x_4^6x_5^2,\ A_{45} = x_1^{15}x_2^{17}x_3^3x_4^6x_5^6,$$
$$A_{46} = x_1^{15}x_2^{19}x_3x_4^6x_5^6,\ A_{47} = x_1^{15}x_2^{19}x_3^5x_4^2x_5^6,\ A_{48} = x_1^{15}x_2^{19}x_3^5x_4^6x_5^2,$$
$$A_{49} = x_1^3x_2^5x_3^6x_4^{14}x_5^{19},\ A_{50} = x_1^3x_2^5x_3^6x_4^6x_5^{27},\ A_{51} = x_1^3x_2^5x_3^{27}x_4^6x_5^6.$$

Proof. We have $\omega(A_i) = \overline{\omega}_{(4)},\ 1 \le i \le 51$. We prove the lemma for the monomials $A_{23} = x_1^3x_2^{15}x_3^5x_4^{18}x_5^6$ and $A_{29} = x_1^7x_2^9x_3^3x_4^{22}x_5^6$. The others can be proved by using a similar technique as in Lemmas 3.3.20 and 3.3.21. Direct computing from Cartan's formula, we get

$$\begin{aligned} A_{23} = &\sum_{1\le i\le 3} b_i + Sq^1(\sum f) + Sq^2(\sum g) \\ &+ Sq^4(\sum h) + Sq^8(\sum p) \text{ modulo}(\mathscr{P}_5^-(\overline{\omega}_{(4)}), \end{aligned}$$

where

$$b_1 = x_1^3x_2^{11}x_3^5x_4^{22}x_5^6,\ b_2 = x_1^3x_2^{13}x_3^3x_4^{22}x_5^6,\ b_3 = x_1^3x_2^{13}x_3^6x_4^{19}x_5^6,$$

$$\begin{aligned} \sum f = &\, x_1^3x_2^7x_3^6x_4^{21}x_5^9 + x_1^3x_2^7x_3^6x_4^{25}x_5^5 + x_1^3x_2^{11}x_3^6x_4^{21}x_5^5 + x_1^3x_2^{14}x_3^3x_4^{21}x_5^5 \\ &+ x_1^3x_2^{15}x_3^5x_4^{14}x_5^9 + x_1^3x_2^{15}x_3^5x_4^{17}x_5^6 + x_1^3x_2^{15}x_3^9x_4^{14}x_5^5 + x_1^3x_2^{19}x_3^5x_4^{14}x_5^5 \\ &+ x_1^3x_2^{22}x_3^5x_4^{11}x_5^5, \end{aligned}$$

$$\begin{aligned} \sum g = &\, x_1^5x_2^7x_3^3x_4^{21}x_5^9 + x_1^5x_2^7x_3^3x_4^{25}x_5^5 + x_1^5x_2^9x_3^9x_4^{19}x_5^3 + x_1^5x_2^{11}x_3^3x_4^{17}x_5^9 \\ &+ x_1^5x_2^{15}x_3^5x_4^{11}x_5^9 + x_1^5x_2^{15}x_3^9x_4^{11}x_5^5 + x_1^5x_2^{17}x_3^3x_4^{17}x_5^3 + x_1^5x_2^{17}x_3^9x_4^{11}x_5^3 \\ &+ x_1^5x_2^{19}x_3^5x_4^{11}x_5^5, \end{aligned}$$

$$\begin{aligned} \sum h = &\, x_1^3x_2^7x_3^3x_4^{21}x_5^9 + x_1^3x_2^7x_3^3x_4^{25}x_5^5 + x_1^3x_2^9x_3^5x_4^{21}x_5^5 + x_1^3x_2^9x_3^9x_4^{19}x_5^3 \\ &+ x_1^3x_2^{11}x_3^3x_4^{17}x_5^9 + x_1^3x_2^{11}x_3^3x_4^{21}x_5^5 + x_1^3x_2^{13}x_3^9x_4^{13}x_5^5 + x_1^3x_2^{15}x_3^5x_4^{11}x_5^9 \\ &+ x_1^3x_2^{15}x_3^6x_4^{13}x_5^6 + x_1^3x_2^{15}x_3^9x_4^{11}x_5^5 + x_1^3x_2^{17}x_3^3x_4^{17}x_5^3 + x_1^3x_2^{17}x_3^9x_4^{11}x_5^3 \\ &+ x_1^3x_2^{19}x_3^5x_4^{11}x_5^5, \end{aligned}$$

$$\sum p = x_1^3x_2^{13}x_3^5x_4^{13}x_5^5 + x_1^3x_2^{13}x_3^6x_4^{11}x_5^6.$$

Since $b_i < A_{23},\ 1 \le i \le 3$, A_{23} is strictly inadmissible. By a similar computation, we obtain

$$\begin{aligned} A_{29} = &\sum Z + Sq^1(\sum \overline{f}) + Sq^2(\sum \overline{g}) \\ &+ Sq^4(\sum \overline{h}) + Sq^8(x_1^7x_2^5x_3^6x_4^{15}x_5^6) \text{ modulo}(\mathscr{P}_5^-(\overline{\omega}_{(4)}), \end{aligned}$$

where

$$\sum Z = x_1^5x_2^3x_3^6x_4^{23}x_5^{10} + x_1^5x_2^3x_3^6x_4^{27}x_5^6 + x_1^5x_2^3x_3^{10}x_4^{23}x_5^6 + x_1^5x_2^7x_3^3x_4^{22}x_5^{10}$$

$$+ x_1^5x_2^7x_3^3x_4^{26}x_5^6 + x_1^5x_2^7x_3^6x_4^{19}x_5^{10} + x_1^5x_2^7x_3^{10}x_4^{19}x_5^6 + x_1^5x_2^{11}x_3^3x_4^{22}x_5^6$$
$$+ x_1^5x_2^{11}x_3^6x_4^{19}x_5^6 + x_1^7x_2^3x_3^5x_4^{22}x_5^{10} + x_1^7x_2^3x_3^6x_4^{23}x_5^8 + x_1^7x_2^3x_3^6x_4^{25}x_5^6$$
$$+ x_1^7x_2^3x_3^8x_4^{23}x_5^6 + x_1^7x_2^3x_3^9x_4^{22}x_5^6 + x_1^7x_2^7x_3^3x_4^{22}x_5^8 + x_1^7x_2^7x_3^3x_4^{24}x_5^6,$$
$$\sum \overline{f} = x_1^7x_2^3x_3^5x_4^{19}x_5^{12} + x_1^7x_2^3x_3^5x_4^{21}x_5^{10} + x_1^7x_2^3x_3^9x_4^{21}x_5^6 + x_1^7x_2^3x_3^{12}x_4^{19}x_5^5$$
$$+ x_1^7x_2^6x_3^9x_4^{15}x_5^9 + x_1^7x_2^7x_3^5x_4^{19}x_5^8 + x_1^7x_2^7x_3^5x_4^{21}x_5^6 + x_1^7x_2^7x_3^8x_4^{19}x_5^5$$
$$+ x_1^7x_2^{10}x_3^5x_4^{15}x_5^9 + x_1^7x_2^{10}x_3^9x_4^{15}x_5^5,$$
$$\sum \overline{g} = x_1^7x_2^3x_3^6x_4^{19}x_5^{10} + x_1^7x_2^3x_3^6x_4^{23}x_5^6 + x_1^7x_2^3x_3^9x_4^{17}x_5^9 + x_1^7x_2^3x_3^{10}x_4^{19}x_5^6$$
$$+ x_1^7x_2^5x_3^9x_4^{15}x_5^9 + x_1^7x_2^7x_3^3x_4^{22}x_5^6 + x_1^7x_2^7x_3^6x_4^{19}x_5^6 + x_1^7x_2^9x_3^3x_4^{17}x_5^9$$
$$+ x_1^7x_2^9x_3^5x_4^{15}x_5^9 + x_1^7x_2^9x_3^9x_4^{15}x_5^5 + x_1^7x_2^9x_3^9x_4^{17}x_5^3 + x_1^9x_2^3x_3^9x_4^{15}x_5^9$$
$$+ x_1^9x_2^9x_3^3x_4^{15}x_5^9 + x_1^9x_2^9x_3^9x_4^{15}x_5^3,$$
$$\sum \overline{h} = x_1^5x_2^3x_3^6x_4^{23}x_5^6 + x_1^5x_2^7x_3^3x_4^{22}x_5^6 + x_1^5x_2^7x_3^6x_4^{19}x_5^6.$$

The above equalities imply that A_{29} is strictly inadmissible. The lemma is proved. $\square$

Proof of Lemma 3.3.26. Let b be an admissible monomial in $(\mathscr{P}_5^+)_{47}$ such that $\omega(b) = \overline{\omega}_{(4)}$. Then $\omega_1(b) = 3$ and $b = X_{(\{t,\,k\},\,5)}Y^2$ with $1 \leq t < k \leq 5$ and Y a monomial of degree 22 in $\mathscr{P}_5$. Since b is admissible, according to Theorem 2.2.1, $Y \in \mathscr{B}_5(\omega_{(4)})$.

Using Proposition 3.3.14(III) and a simple computation shows that if $Z \in \mathscr{B}_5(\omega_{(4)})$, $1 \leq t < k \leq 5$, and $X_{(\{t,\,k\},\,5)}Z^2 \neq \mathcal{Y}_{47,\,j}$, $\forall j$, $371 \leq j \leq 479$, then there is a monomial u which is given in one of Lemmas 3.3.27-3.3.32 such that $X_{(\{t,\,k\},\,5)}Z^2 = ug^{2^s}$ with a monomial $g \in \mathscr{P}_5$, and $s = \max\{\ell \in \mathbb{Z} : \omega_\ell(u) > 0\}$. Then, By Theorem 2.2.1, $X_{(\{t,\,k\},\,5)}Z^2$ is inadmissible. Finally, we see that $b = X_{(\{t,\,k\},\,5)}Y^2$ is admissible with $Y \in \mathscr{B}_5(\omega_{(4)})$; hence $b = \mathcal{Y}_{47,\,j}$ for some $j \in \{371, \ldots, 479\}$. This implies $\mathscr{B}_5^+(\overline{\omega}_{(4)}) \subseteq \{\mathcal{Y}_{47,\,j} : 371 \leq j \leq 479\}$. $\square$

Proposition 3.3.33. *$Q\mathscr{P}_5^+(\overline{\omega}_{(4)})$ is the $\mathbb{Z}/2$-vector space of dimension 109 with a basis consisting of all the classes represented by the monomials $Y_{47,\,j}$, $371 \leq j \leq 479$.*

Proof. First, we show that the set $[V := \{Y_{47,\,j} : 371 \leq j \leq 479\}]_{\overline{\omega}_{(4)}}$ is linearly independent in the space $Q\mathscr{P}_5^+(\overline{\omega}_{(4)})$. Indeed, suppose there is a linear relation $\mathcal{S} = \sum_{371 \leq j \leq 479} \gamma_j \mathcal{Y}_j \equiv_{\overline{\omega}_{(4)}} 0$ with $\gamma_j \in \mathbb{Z}/2$ and $\mathcal{Y}_{47,\,j} \in V$. By using Theorem 3.1.3 and Proposition 3.2.1, we determine explicitly $\pi_{(k;\mathscr{K})}(\mathcal{S})$ in terms of the admissible monomials in $(\mathscr{P}_4^+)_{47}$. From the relations $\pi_{(k;\mathscr{K})}(\mathcal{S}) \equiv_{\overline{\omega}_{(4)}} 0$ with $\ell(\mathscr{K}) > 0$, one gets $\gamma_j = 0$ for $j = 371, 372, \ldots, 479$.

Now, by Lemma 3.3.26, to prove $[V]_{\overline{\omega}_{(4)}}$ is a basis of $Q\mathscr{P}_5^+(\overline{\omega}_{(4)})$ we need to show that $[\mathcal{Y}_{47,\,j}]_{\overline{\omega}_{(4)}} \neq [0]$ for all $\mathcal{Y}_{47,\,j} \in V$. By a similar argument as given in the proof of Propositions 3.3.1 and 3.3.10, we can prove that the set $[\mathscr{B}_5^+(\overline{\omega}_{(1)}) \cup V]$ is linearly independent in $(Q\mathscr{P}_5^+)_{47}$. This fact shows that $[\mathcal{Y}_{47,\,j}]_{\overline{\omega}_{(4)}} \neq [0]$ for all $\mathcal{Y}_{47,\,j}$. The proposition is proved. □

Proposition 3.3.34. *There exist exactly* 15 *admissible monomials in* $(\mathscr{P}_5^+)_{47}$ *such that their weight vectors are* $\overline{\omega}_{(5)}$. *Consequently* $\dim(QP_5^+(\overline{\omega}_{(5)})) = 15$.

We prove the proposition by showing that

$$\mathscr{B}_5^+(\overline{\omega}_{(5)}) = \{\mathcal{Y}_{47,\,m} : 480 \leq m \leq 494\},$$

where the monomials $\mathcal{Y}_{47,\,m} : 480 \leq m \leq 494$, are listed in Section 6.7 of the online version [40]. We need some lemmas for the proof of this proposition. The Lemmas 3.3.35 and 3.3.36 below are proved by using a result in [62].

Lemma 3.3.35. *The following monomials are strictly inadmissible*:

a) $x_ux_v^2x_m^6x_n^7x_p^7,\ x_u^7x_v^2x_m^2x_n^5x_p^7,\ x_u^3x_v^2x_m^4x_n^7x_p^7,\ x_u^3x_v^4x_m^6x_n^3x_p^7,$
$x_u^3x_v^6x_m^6x_n^3x_p^5$, *where* (u, v, m, n, p) *is a permutation of* $(1, 2, 3, 4, 5)$;

b) $x_1x_2^6x_q^6x_r^3x_t^7,\ x_1^3x_2^6x_q^6x_rx_t^7,\ x_1^7x_2^6x_q^6x_rx_t^3,\ x_1^3x_2^2x_q^6x_r^5x_t^7,\ x_1^3x_2^6x_q^2x_r^5x_t^7,$
$x_1^7x_2^2x_q^6x_r^3x_t^5, x_1^7x_2^6x_q^2x_r^3x_t^5$, *where* (q, r, t) *is a permutation of* $(3, 4, 5)$;

c) $x_1x_2^7x_3^6x_4^6x_5^3,\ x_1x_2^7x_3^6x_4^3x_5^6,\ x_1^3x_2^7x_3^6x_4^6x_5,\ x_1^3x_2^7x_3^6x_4x_5^6,$
$x_1^7x_2x_3^6x_4^6x_5^3,\ x_1^7x_2^3x_3^6x_4^6x_5,\ x_1^7x_2x_3^6x_4^3x_5^6,\ x_1^7x_2^3x_3^6x_4x_5^6,$
$x_1^3x_2^7x_3^2x_4^6x_5^5,\ x_1^3x_2^7x_3^6x_4^2x_5^5,\ x_1^3x_2^7x_3^2x_4^5x_5^6,\ x_1^3x_2^7x_3^6x_4^5x_5^2,$
$x_1^7x_2^3x_3^2x_4^6x_5^5,\ x_1^7x_2^3x_3^6x_4^2x_5^5,\ x_1^7x_2^3x_3^2x_4^5x_5^6,\ x_1^7x_2^3x_3^6x_4^5x_5^2.$

Lemma 3.3.36. *Let Z be the set of the following monomials*:

$x_1^2x_2^7x_3^7x_4^7,\ x_1^7x_2^2x_3^7x_4^7,\ x_1^7x_2^7x_3^2x_4^7,\ x_1^7x_2^7x_3^7x_4^2,\ x_1^6x_2^3x_3^7x_4^7,\ x_1^6x_2^7x_3^3x_4^7,$
$x_1^6x_2^7x_3^7x_4^3,\ x_1^3x_2^6x_3^7x_4^7,\ x_1^7x_2^6x_3^3x_4^7,\ x_1^7x_2^6x_3^7x_4^3,\ x_1^3x_2^7x_3^6x_4^7,\ x_1^7x_2^3x_3^6x_4^7,$
$x_1^7x_2^7x_3^6x_4^3,\ x_1^3x_2^7x_3^7x_4^6,\ x_1^7x_2^3x_3^7x_4^6,\ x_1^7x_2^7x_3^3x_4^6.$

Then, the monomials $\rho_{(k,\,5)}(Z),\ 1 \leq k \leq 5$, *are strictly inadmissible.*

Lemma 3.3.37. *The following monomials are strictly inadmissible*:

$T_1 = x_1x_2^3x_3^{14}x_4^{14}x_5^{15},\ T_2 = x_1x_2^3x_3^{14}x_4^{15}x_5^{14},\ T_3 = x_1x_2^3x_3^{15}x_4^{14}x_5^{14},$

$T_4 = x_1x_2^{15}x_3^3x_4^{14}x_5^{14},\ T_5 = x_1^3x_2x_3^{14}x_4^{14}x_5^{15},\ T_6 = x_1^3x_2x_3^{14}x_4^{15}x_5^{14},$

$T_7 = x_1^3x_2x_3^{15}x_4^{14}x_5^{14},\ T_8 = x_1^3x_2^{15}x_3x_4^{14}x_5^{14},\ T_9 = x_1^{15}x_2x_3^3x_4^{14}x_5^{14},$

$T_{10} = x_1^{15}x_2^3x_3x_4^{14}x_5^{14},\ T_{11} = x_1^3x_2^{13}x_3^2x_4^{14}x_5^{15},\ T_{12} = x_1^3x_2^{13}x_3^2x_4^{15}x_5^{14},$

$T_{13} = x_1^3x_2^{13}x_3^{14}x_4^2x_5^{15},\ T_{14} = x_1^3x_2^{13}x_3^{14}x_4^{15}x_5^2,\ T_{15} = x_1^3x_2^{13}x_3^{15}x_4^2x_5^{14},$

$T_{16} = x_1^3x_2^{13}x_3^{15}x_4^{14}x_5^2,\ T_{17} = x_1^3x_2^{15}x_3^{13}x_4^2x_5^{14},\ T_{18} = x_1^3x_2^{15}x_3^{13}x_4^{14}x_5^2,$

$T_{19} = x_1^{15}x_2^3x_3^{13}x_4^2x_5^{14},\ T_{20} = x_1^{15}x_2^3x_3^{13}x_4^{14}x_5^2,\ T_{21} = x_1^3x_2^3x_3^{13}x_4^{14}x_5^{14},$

$T_{22} = x_1^3x_2^{13}x_3^3x_4^{14}x_5^{14},\ T_{23} = x_1^3x_2^{13}x_3^{14}x_4^3x_5^{14},\ T_{24} = x_1^3x_2^{13}x_3^{14}x_4^{14}x_5^3,$

$T_{25} = x_1^3x_2^5x_3^{10}x_4^{14}x_5^{15},\ T_{26} = x_1^3x_2^5x_3^{10}x_4^{15}x_5^{14},\ T_{27} = x_1^3x_2^5x_3^{14}x_4^{10}x_5^{15},$

$T_{28} = x_1^3x_2^5x_3^{14}x_4^{15}x_5^{10},\ T_{29} = x_1^3x_2^5x_3^{15}x_4^{10}x_5^{14},\ T_{30} = x_1^3x_2^5x_3^{15}x_4^{14}x_5^{10},$

$T_{31} = x_1^3x_2^{15}x_3^5x_4^{10}x_5^{14},\ T_{32} = x_1^3x_2^{15}x_3^5x_4^{14}x_5^{10},\ T_{33} = x_1^{15}x_2^3x_3^5x_4^{10}x_5^{14},$

$$T_{34} = x_1^{15}x_2^{3}x_3^{5}x_4^{14}x_5^{10},\ T_{35} = x_1^{3}x_2^{5}x_3^{14}x_4^{11}x_5^{14},\ T_{36} = x_1^{3}x_2^{5}x_3^{14}x_4^{14}x_5^{11},$$
$$T_{37} = x_1^{3}x_2^{13}x_3^{6}x_4^{10}x_5^{15},\ T_{38} = x_1^{3}x_2^{13}x_3^{6}x_4^{15}x_5^{10},\ T_{39} = x_1^{3}x_2^{13}x_3^{15}x_4^{6}x_5^{10},$$
$$T_{40} = x_1^{3}x_2^{15}x_3^{13}x_4^{6}x_5^{10},\ T_{41} = x_1^{15}x_2^{3}x_3^{13}x_4^{6}x_5^{10},\ T_{42} = x_1^{3}x_2^{13}x_3^{6}x_4^{11}x_5^{14},$$
$$T_{43} = x_1^{3}x_2^{13}x_3^{6}x_4^{14}x_5^{11},\ T_{44} = x_1^{3}x_2^{13}x_3^{14}x_4^{6}x_5^{11},\ T_{45} = x_1^{7}x_2^{9}x_3^{3}x_4^{14}x_5^{14},$$
$$T_{46} = x_1^{3}x_2^{13}x_3^{7}x_4^{10}x_5^{14},\ T_{47} = x_1^{3}x_2^{13}x_3^{7}x_4^{14}x_5^{10},\ T_{48} = x_1^{3}x_2^{13}x_3^{14}x_4^{7}x_5^{10}.$$

Proof. It is easy to see that $\omega(T_s) = \overline{\omega}_{(5)}$ for $s = 1, 2, \ldots, 48$. We prove the lemma for the monomials T_{35} and T_{42}. The others can be proved by using a similar technique as in Lemmas 3.3.20, 3.3.21 and 3.3.32. Computing from the Cartan formula, we get

$$\begin{aligned} T_{35} = {} & Sq^1\big(x_1^{3}x_2^{6}x_3^{11}x_4^{13}x_5^{13} + x_1^{3}x_2^{6}x_3^{13}x_4^{11}x_5^{13}\big)\\ & + Sq^2\big(x_1^{5}x_2^{3}x_3^{7}x_4^{13}x_5^{17} + x_1^{5}x_2^{3}x_3^{7}x_4^{17}x_5^{13} + x_1^{5}x_2^{3}x_3^{9}x_4^{11}x_5^{17} + x_1^{5}x_2^{3}x_3^{9}x_4^{17}x_5^{11}\\ & + x_1^{5}x_2^{3}x_3^{11}x_4^{9}x_5^{17} + x_1^{5}x_2^{3}x_3^{11}x_4^{13}x_5^{13} + x_1^{5}x_2^{3}x_3^{13}x_4^{7}x_5^{17} + x_1^{5}x_2^{3}x_3^{13}x_4^{11}x_5^{13}\\ & + x_1^{5}x_2^{3}x_3^{17}x_4^{7}x_5^{13} + x_1^{5}x_2^{3}x_3^{17}x_4^{9}x_5^{11}\big)\\ & + Sq^4(f) + x_1^{3}x_2^{5}x_3^{11}x_4^{14}x_5^{14} \text{ modulo}(\mathscr{P}_5^-(\overline{\omega}_{(5)})), \end{aligned}$$

where the polynomial f is determined as follows:

$$\begin{aligned} f = {} & x_1^{3}x_2^{3}x_3^{7}x_4^{13}x_5^{17} + x_1^{3}x_2^{3}x_3^{7}x_4^{17}x_5^{13} + x_1^{3}x_2^{3}x_3^{9}x_4^{11}x_5^{17} + x_1^{3}x_2^{3}x_3^{9}x_4^{17}x_5^{11}\\ & + x_1^{3}x_2^{3}x_3^{11}x_4^{9}x_5^{17} + x_1^{3}x_2^{3}x_3^{11}x_4^{13}x_5^{13} + x_1^{3}x_2^{3}x_3^{13}x_4^{7}x_5^{17} + x_1^{3}x_2^{3}x_3^{13}x_4^{11}x_5^{13}\\ & + x_1^{3}x_2^{3}x_3^{17}x_4^{7}x_5^{13} + x_1^{3}x_2^{3}x_3^{17}x_4^{9}x_5^{11}. \end{aligned}$$

Obviously, $T_{35} > x_1^{3}x_2^{5}x_3^{11}x_4^{14}x_5^{14}$, hence T_{35} is strictly inadmissible. By a similar computation, we claim that $T_{42} = (p_1 + p_2)$ modulo$(\mathcal{A}_2^+\mathscr{P}_5 + \mathscr{P}_5^-(\overline{\omega}_{(5)}))$, where $p_1 = x_1^{3}x_2^{7}x_3^{10}x_4^{13}x_5^{14}$, and $p_2 = x_1^{3}x_2^{13}x_3^{3}x_4^{14}x_5^{14}$. This completes the proof. $\square$

Proof of Proposition 3.3.34. Suppose that X is an admissible monomial in $\mathscr{P}_5^+$ and $\omega(X) = \overline{\omega}_{(5)}$. Then $\omega_1(X) = 3$ and $X = x_ax_bx_cu^2$ with $1 \le a < b < c \le 5$ and $u \in (\mathscr{P}_5^+)_{22}$. Then, by Theorem 2.2.1, u is admissible and $u \in \mathscr{B}_5^+(\omega_{(5)})$.

Let $Z \in B_5^+(\omega_{(5)})$ such that $x_ax_bx_cZ^2 \in (\mathscr{P}_5^+)_{47}$ with $1 \le a < b < c \le 5$. Denote by $\overline{V}$ the set of all the monomials as given in Proposition 3.3.34. By a direct computation using Proposition 3.3.14(IV), we see that if $x_ax_bx_cZ^2 \notin \overline{V}$, then there is a monomial X_1 which is given in Lemmas 3.3.35, 3.3.36 and 3.3.37 such that $x_ax_bx_cZ^2 = X_1Z_1^{2^t}$ with suitable monomial $Z_1 \in \mathscr{P}_5$ and $t = \max\{q \in \mathbb{Z} : \omega_q(X_1) > 0\}$. Based on Theorem 2.2.1, we deduce that $x_ax_bx_cZ^2$ is inadmissible. Combining this with the above data, one gets $X \in \overline{V}$. This means $\mathscr{B}_5^+(\overline{\omega}_{(5)}) \subseteq \overline{V}$.

Next, we show that the set $[\overline{V}]_{\overline{\omega}_{(5)}}$ is linearly independent in the space $Q\mathscr{P}_5^+(\overline{\omega}_{(5)})$. Indeed, suppose there is a linear relation $\mathcal{S} = \sum_{480 \le m \le 494} \gamma_m \mathcal{Y}_{47,\,m} \equiv_{\overline{\omega}_{(5)}} 0$, where with $\gamma_m \in \mathbb{Z}/2,\ m = 480, \ldots, 494$ and $\mathcal{Y}_{47,\,m} \in \overline{V}$. By combining Theorem 3.1.3 and Proposition 3.2.1, we explicitly calculate $\pi_{(k;\mathscr{K})}(\mathcal{S})$ in terms of a given minimal set of $\mathcal{A}_2$-generators in $\mathscr{P}_4$ (modulo($\mathcal{A}_2^+\mathscr{P}_4$)). From the relations $\pi_{(k;\mathscr{K})}(\mathcal{S}) \equiv_{\overline{\omega}_{(5)}} 0$ with $\ell(\mathscr{K}) \leq 2$, we get $\gamma_m = 0$ for all m.

To prove $[\overline{V}]_{\overline{\omega}_{(5)}}$ is a basis of $Q\mathscr{P}_5^+(\overline{\omega}_{(5)})$ we need to show that $[\mathcal{Y}_{47,\,m}]_{\overline{\omega}_{(5)}} \neq [0]$ for all $\mathcal{Y}_{47,m} \in \overline{V}$. Denote by $\overline{V}$ the set of all the monomials as given in Proposition 3.3.34. By a similar argument as given in the proof of Propositions 3.3.1 and 3.3.10, we can prove that the set $[\mathscr{B}_5^+(\overline{\omega}_{(1)}) \cup V \cup \overline{V}]$ is linearly independent in $(Q\mathscr{P}_5^+)_{47}$, where V the set of all the admissible monomials as given in Proposition 3.3.33. This implies $[\mathcal{Y}_{47,\,m}]_{\overline{\omega}_{(5)}} \neq [0]$ for all $\mathcal{Y}_{47,\,m}$. The proposition is proved. □

Now, since $\dim(Q\mathscr{P}_5^0)_{47} = 560$ and $\dim(Q\mathscr{P}_5^0)_{21} = 460$, by Corollaries 3.3.15, 3.3.23 and Propositions 3.3.25, 3.3.33, 3.3.34, we conclude that $Q\mathscr{P}_5$ has dimension 1894 in degree 47. The proof of Theorem 1.1 is completed.

Final remarks. Recall that Kameko's map $(\widetilde{Sq^0_*})_{(5,13.2^t-5)}$ is an epimorphism of $\mathbb{Z}/2(GL_5)$-modules. This implies that

$$(Q\mathscr{P}_5)_{13.2^t-5} \cong \mathrm{Ker}(\widetilde{Sq^0_*})_{(5,13.2^t-5)} \bigoplus (Q\mathscr{P}_5)_{13.2^{t-1}-5}.$$

From a result in [60], we deduce that $\mathscr{B}_5(13.2^0-5) = \bigcup_{1\leq j\leq 3}\mathscr{B}_5(\widetilde{\omega}_{(j)})$, where $\widetilde{\omega}_{(1)} = (2,1,1)$, $\widetilde{\omega}_{(2)} = (2,3)$, and $\widetilde{\omega}_{(3)} = (4,2)$. For $m,\ k \in \mathbb{N}$ and $1 \leq k \leq 5$, we denote

$$\overline{\mathscr{B}}(k,8) := \big\{x_k^{2^m-1}\rho_{(k,\,5)}(x) \in (\mathscr{P}_5)_8 : x \in \mathscr{B}_4(9-2^m),\ \alpha(13-2^m) \leq 4\big\}.$$

As well known (see [26]), $\overline{\mathscr{B}}(k,8) \subseteq \mathscr{B}_5(8)$ for all $k,\ 1 \leq k \leq 5$. We set

$$\overline{\mathscr{B}}(k,\widetilde{\omega}_{(j)}) := \overline{\mathscr{B}}(k,8) \cap \mathscr{P}_5(\widetilde{\omega}_{(j)}) \text{ for } 1 \leq j \leq 3,\ 1 \leq k \leq 5.$$

Then, by a simple computation, we get

$$\big|\overline{\Phi}(\mathscr{B}_4(\widetilde{\omega}_{(1)}))\bigcup\big(\bigcup_{1\leq k\leq 5}\overline{\mathscr{B}}(k,\widetilde{\omega}_{(1)})\big)\big| = 105,$$
$$\big|\overline{\Phi}(\mathscr{B}_4(\widetilde{\omega}_{(2)}))\bigcup\big(\bigcup_{1\leq k\leq 5}\overline{\mathscr{B}}(k,\widetilde{\omega}_{(2)})\big)\big| = 24,$$
$$\big|\overline{\Phi}(\mathscr{B}_4(\widetilde{\omega}_{(3)}))\bigcup\big(\bigcup_{1\leq k\leq 5}\overline{\mathscr{B}}(k,\widetilde{\omega}_{(3)})\big)\big| = 45.$$

Furthermore,

$$\mathscr{B}_5(\widetilde{\omega}_{(j)}) = \overline{\Phi}(\mathscr{B}_4(\widetilde{\omega}_{(j)}))\bigcup\big(\bigcup_{1\leq k\leq 5}\overline{\mathscr{B}}(k,\widetilde{\omega}_{(j)})\big),\ j = 1,2,3.$$

Note that $\sum_{1\leq j\leq 3}|\mathscr{B}_5(\widetilde{\omega}_{(j)})| = \dim(Q\mathscr{P}_5)_{13.2^0-5} = 105+24+45 = 174.$

Now, according to Lemma 3.3.2, $\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,13.2^1-5)} \cong Q\mathscr{P}_5(\omega)\bigoplus Q\mathscr{P}_5(\omega')$, where $\omega = (3,3,1,1)$, and $\omega' = (3,3,3)$. A routine computation shows that

$$\mathscr{B}_5^0(13.2^1-5) = \overline{\Phi}(\mathscr{B}_4(13.2^1-5)) = \mathscr{B}_5^0(\omega) \cup \mathscr{B}_5^0(\omega').$$

Combining this with Propositions 3.3.6 and 3.3.7 gives

$$\overline{\Phi}(\mathscr{B}_4(\omega)) \subset \mathscr{B}_5(\omega), \quad \overline{\Phi}(\mathscr{B}_4(\omega')) \subset \mathscr{B}_5(\omega').$$

Next, we have $(Q\mathscr{P}_5)_{13.2^t-5} \cong (Q\mathscr{P}_5)_{13.2^2-5}$ for all $t \geq 2$ and

$$\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,13.2^2-5)} \cong \bigoplus_{1\leq k\leq 5} Q\mathscr{P}_5(\overline{\omega}_{(k)}).$$

From the above computations,

$$\mathscr{B}^0_5(\overline{\omega}_{(1)}) = \mathscr{B}^0_5(13.2^2-5) = \overline{\Phi}(\mathscr{B}_4(13.2^2-5)) = \overline{\Phi}(\mathscr{B}_4(\overline{\omega}_{(1)})).$$

Then, by Corollary 3.3.23, and Propositions 3.3.25, 3.3.33, 3.3.34, we conclude

$$\overline{\Phi}(\mathscr{B}_4(\overline{\omega}_{(1)})) \subset \mathscr{B}_5(\overline{\omega}_{(1)}).$$

If $\overline{\omega}$ is a weight vector of degree 13.2^2-5 and $\overline{\omega} \neq \overline{\omega}_{(1)}$, then $\mathscr{B}_4(\overline{\omega}) = \emptyset$. Furthermore, if $t > 2$, then $\mathscr{B}_4(13.2^t-5) = \emptyset$.

From the above remarks, Conjecture 3.2.2 also satisfies in case of five variables and generic degree 13.2^t-5 for t an arbitrary non-negative integer.

4. An application of Theorem 1.1

The goal of this section is to prove Theorem 1.2. More precisely, by using our results in Section 3 and a result in [60], we describe the $\mathbb{Z}/2(GL_d)$-modules structure of $Q\mathscr{P}_5$ in degree 13.2^t-5 for $t \in \{0,1\}$. Then, we explicitly determine all GL_5-invariants of these spaces.

Before coming to the proof of the theorem, we introduce some notations and homomorphisms. We note that $(\mathbb{Z}/2)^{\times d}$ regarded as a $\mathbb{Z}/2$-vector space of dimension d and

$$(\mathbb{Z}/2)^{\times d} \cong \langle x_1, \ldots, x_d\rangle \subset \mathscr{P}_d.$$

For $1 \leq t \leq d$, define the $\mathbb{Z}/2$-linear map $\tau_t : (\mathbb{Z}/2)^{\times d} \to (\mathbb{Z}/2)^{\times d}$, which is determined by

$$\tau_t(x_t) = x_{t+1},\ \tau_t(x_{t+1}) = x_t,\ \tau_t(x_m) = x_m \quad (m \neq t, t+1,\ 1 \leq t \leq d-1),$$

and

$$\tau_d(x_1) = x_1 + x_2,\ \tau_d(x_m) = x_m \quad (m > 1).$$

Denote by S_d the symmetric group of degree d. Then, S_d is generated by $\tau_t,\ 1 \leq t \leq d-1$. For each permutation in S_d, consider corresponding permutation matrix; these form a group of matrices isomorphic to S_d. So, $GL_d = GL(d,\mathbb{Z}/2) \cong GL((\mathbb{Z}/2)^{\times d})$ is generated by S_d and τ_d. Let $X = x_1^{a_1}x_2^{a_2}\cdots x_d^{a_d}$ be an monomial in $\mathscr{P}_d$. Then, the weight vector $\omega(X)$ is invariant under the permutation of the generators x_j, $j = 1, 2, \ldots, d$; hence $Q\mathscr{P}_d(\omega)$ also has a S_d-module structure. We have a homomorphism $\tau_t : \mathscr{P}_d \to \mathscr{P}_d$ of algebras, which is induced by τ_t. Hence, a class $[u]_\omega \in Q\mathscr{P}_d(\omega)$ is a GL_d-invariant if and only if $\tau_t(u) + u \equiv_\omega 0$ for $1 \leq t \leq d$. If $\tau_t(u) + u \equiv_\omega 0$ for $1 \leq t \leq d-1$, then $[u]_\omega$ is an S_d-invariant. Note that $\dim((QP_d)_n^{GL_d}) \leq \sum_{\deg(\omega)=n} \dim(QP_d(\omega)^{GL_d})$ (see Section 2).

Let ω be a weight vector of degree n and let $\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s$ be the monomials in $\mathscr{P}_d(\omega)$ for $s \geq 1$. We consider a subgroup $L \subseteq GL_d$ and denote by

$$L(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s) = \{\sigma(\mathcal{Y}_j) : \sigma \in L,\, 1 \leq j \leq s\} \subset \mathscr{P}_d(\omega),$$
$$[\mathscr{B}(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s)]_\omega = [\mathscr{B}_d(\omega)]_\omega \cap \langle [L(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s)]_\omega \rangle,$$
$$\theta(\mathcal{Y}_j) = \sum_{x \in \mathscr{B}_d(n) \cap L(\mathcal{Y}_j)} x.$$

Note that $\langle [L(\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_s)]_\omega \rangle$ is the L-submodule of $Q\mathscr{P}_d(\omega)$ generated by the set

$$\{[\mathcal{Y}_1]_\omega, [\mathcal{Y}_2]_\omega, \ldots, [\mathcal{Y}_s]_\omega\}.$$

Now, we have $13.2^1 - 5 = 8$, and $13.2^1 - 5 = 21$. Recall that Kameko's map $(\widetilde{Sq^0_*})_{(5,21)} : (Q\mathscr{P}_5)_{21} \to (Q\mathscr{P}_5)_8$ is an epimorphism of GL_5-modules. So, to prove Theorem 1.2, we need to compute GL_5-invariants of $(Q\mathscr{P}_5)_8$.

4.1. Computation of $(Q\mathscr{P}_5)_8^{GL_5}$

According to Tín [60], the $\mathbb{Z}/2$-vector space $(Q\mathscr{P}_5)_8$ has the basis $[\{\mathcal{Y}_{8,\,i} : 1 \leq i \leq 174\}]$, where the monomials $\mathcal{Y}_i := \mathcal{Y}_{8,\,i}, 1 \leq i \leq 174$, are given in Section 6.1 of the online version [40].

Theorem 4.1.1. *The space $(Q\mathscr{P}_5)^{GL_5}$ is trivial in degree* 8.

We prepare some lemmas for the proof of the theorem. We have

$$(Q\mathscr{P}_5)_8 \cong Q\mathscr{P}_5^0(\widetilde{\omega}_{(1)}) \bigoplus Q\mathscr{P}_5(\widetilde{\omega}_{(2)}) \bigoplus Q\mathscr{P}_5(\widetilde{\omega}_{(3)}),$$

where $\widetilde{\omega}_{(1)} = (2,1,1)$, $\widetilde{\omega}_{(2)} = (2,3)$, $\widetilde{\omega}_{(3)} = (4,2)$. We see that $\dim Q\mathscr{P}_5^0(\widetilde{\omega}_{(1)}) = 105$ with the basis $\bigcup_{1 \leq i \leq 6} [\mathscr{B}(\mathcal{Y}_i)]_{\widetilde{\omega}_{(1)}}$, where

$$\mathcal{Y}_1 = x_4x_5^7, \quad \mathcal{Y}_2 = x_4^3x_5^5, \quad \mathcal{Y}_3 = x_3x_4x_5^6,$$
$$\mathcal{Y}_4 = x_3x_4^2x_5^5, \quad \mathcal{Y}_5 = x_3x_4^3x_5^4, \quad \mathcal{Y}_6 = x_2x_3x_4^2x_5^4.$$

Observe that $\widetilde{\omega}_{(1)}$ is the weight vector of the mimimal spike $x_1^7x_2$, so $[X]_{\widetilde{\omega}_{(1)}} = [X]$ for all $X \in (\mathscr{P}_5)_8$. A direct computation shows that

$$[S_5(\mathcal{Y}_1)] = \langle [\mathcal{Y}_i] : 1 \leq i \leq 20 \rangle,$$
$$[S_5(\mathcal{Y}_2)] = \langle [\mathcal{Y}_i] : 21 \leq i \leq 30 \rangle,$$
$$[S_5(\mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5)] = \langle [\mathcal{Y}_i] : 31 \leq i \leq 90 \rangle,$$
$$[S_5(\mathcal{Y}_6)] = \langle [\mathcal{Y}_i] : 91 \leq i \leq 105 \rangle$$

are S_5-submodules of $Q\mathscr{P}_5^0(\widetilde{\omega}_{(1)})$. Hence, we have a direct summand decomposition of S_5-modules:

$$QP_5^0(\widetilde{\omega}_{(1)}) = [S_5(\mathcal{Y}_1)] \bigoplus [S_5(\mathcal{Y}_2)] \bigoplus [S_5(\mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5)] \bigoplus [S_5(\mathcal{Y}_6)].$$

Lemma 4.1.2. $Q\mathscr{P}_5^0(\widetilde{\omega}_{(1)}))^{S_5}$ *has dimension* 4.

Proof. We prove the following:

$$[S_5(\mathcal{Y}_j)]^{S_5} = \langle[\theta(\mathcal{Y}_j)]\rangle,\ j = 1, 2, 6,$$
$$[S_5(\mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5)]^{S_5} = \langle[q := \mathcal{Y}_{31} + \mathcal{Y}_{32} + \cdots + \mathcal{Y}_{70}]\rangle.$$

Indeed, we compute $[S_5(\mathcal{Y}_j)]^{S_5}$ for $j = 2$ and $[S_5(\mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5)]^{S_5}$. The others can be proved by a similar technique. Note that $\dim[S_5(\mathcal{Y}_2)] = 10$ with a basis consisting of all the classes represented by the monomials $\mathcal{Y}_i : 21 \leq i \leq 30$. Suppose that $a = \sum_{21\leq i\leq 30} \gamma_i\mathcal{Y}_i$ with $\gamma_i \in \mathbb{Z}/2$ and $[a] \in [S_5(\mathcal{Y}_2)]^{S_5}$. By a direct computation using Theorem 3.1.3, we have

$$\tau_1(a) + a = \sum X \ \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5) = 0 \ \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5),$$
$$\tau_2(a) + a = \sum Y \ \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5) = 0 \ \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5),$$
$$\tau_3(a) + a = \sum Z \ \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5) = 0 \ \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5),$$
$$\tau_4(a) + a = \sum W \ \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5) = 0 \ \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5),$$

where

$$\sum X = (\gamma_{24}+\gamma_{27})(\mathcal{Y}_{24}+\mathcal{Y}_{27}) + (\gamma_{25}+\gamma_{28})(\mathcal{Y}_{25}+\mathcal{Y}_{28}) + (\gamma_{26}+\gamma_{29})(\mathcal{Y}_{26}+\mathcal{Y}_{29}),$$
$$\sum Y = (\gamma_{22}+\gamma_{24})(\mathcal{Y}_{22}+\mathcal{Y}_{24}) + (\gamma_{23}+\gamma_{25})(\mathcal{Y}_{23}+\mathcal{Y}_{25}) + (\gamma_{29}+\gamma_{30})(\mathcal{Y}_{29}+\mathcal{Y}_{30}),$$
$$\sum Z = (\gamma_{21}+\gamma_{22})(\mathcal{Y}_{21}+\mathcal{Y}_{22}) + (\gamma_{25}+\gamma_{26})(\mathcal{Y}_{25}+\mathcal{Y}_{26}) + (\gamma_{28}+\gamma_{29})(\mathcal{Y}_{28}+\mathcal{Y}_{29}),$$
$$\sum W = (\gamma_{22}+\gamma_{23})(\mathcal{Y}_{22}+\mathcal{Y}_{23}) + (\gamma_{24}+\gamma_{25})(\mathcal{Y}_{24}+\mathcal{Y}_{25}) + (\gamma_{27}+\gamma_{28})(\mathcal{Y}_{27}+\mathcal{Y}_{28}).$$

These relations imply that $\gamma_i = \gamma_{21}$ for $i = 22, \ldots, 30$. Hence, we get $a = \theta(\mathcal{Y}_2)$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$ with $\theta(Y_2) = \sum_{21\leq j\leq 30} \mathcal{Y}_j$.

Now, we have the set $\{[\mathcal{Y}_i] : 31 \leq i \leq 90]\}$ is a basis of $[S_5(\mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5)]$. Suppose that $b = \sum_{31\leq i\leq 90} \gamma_i\mathcal{Y}_i$ with $\gamma_i \in \mathbb{Z}/2$ and $[b] \in [S_5(\mathcal{Y}_3, \mathcal{Y}_4, \mathcal{Y}_5)]^{S_5}$. By a similar computation from the relations $\tau_t(b) + b = 0$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$, $t = 1, 2, 3, 4$, one gets $\gamma_i = 0$ for $71 \leq i \leq 90$ and $\gamma_i = \gamma_{31},\ i = 32, 33, \ldots, 70$. This means $b = q$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$ with $q = \sum_{31\leq i\leq 70}$. The lemma is proved. □

Lemma 4.1.3. *The subspace $(Q\mathscr{P}_5(\widetilde{\omega}_{(3)}))^{GL_5}$ is trivial.*

Proof. Using a result in [60], we see that $Q\mathscr{P}_5(\widetilde{\omega}_{(3)})$ is the $\mathbb{Z}/2$-vector space of dimension 45 with the basis

$$[\mathscr{B}(\mathcal{Y}_{130} = x_2x_3x_4^3x_5^3)]_{\widetilde{\omega}_{(3)}} \bigcup [\mathscr{B}(\mathcal{Y}_{160} = x_1x_2x_3x_4^2x_5^3)]_{\widetilde{\omega}_{(3)}}.$$

Furthermore, $[S_5(\mathcal{Y}_{130})]$ and $[S_5(\mathcal{Y}_{160})]$ are S_5-submodules of $Q\mathscr{P}_5^0(\widetilde{\omega}_{(3)})$, where $[S_5(\mathcal{Y}_{130})]_{\widetilde{\omega}_{(3)}} = \langle[\mathcal{Y}_i : 130 \leq i \leq 159]_{(\widetilde{\omega}_{(3)})}\rangle$ and $[S_5(\mathcal{Y}_{160})]_{\widetilde{\omega}_{(3)}} = \langle[\mathcal{Y}_i : 160 \leq i \leq 174]_{(\widetilde{\omega}_{(3)})}\rangle$. So, we have a direct summand decomposition of S_5-modules: $Q\mathscr{P}_5(\widetilde{\omega}_{(3)}) = [S_5(\mathcal{Y}_{130})]_{\widetilde{\omega}_{(3)}} \bigoplus [S_5(\mathcal{Y}_{160})]_{\widetilde{\omega}_{(3)}}$. The set $[\mathscr{B}(\mathcal{Y}_{130})]_{\widetilde{\omega}_{(3)}}$ is a basis of $[S_5(\mathcal{Y}_{130})]_{\widetilde{\omega}_{(3)}}$. The action of S_5 on $Q\mathscr{P}_5$ induces the one of it on $[\mathscr{B}(\mathcal{Y}_{130})]_{\widetilde{\omega}_{(3)}}$. On the other hand, this action is transitive, hence if $a = \sum_{130\leq i\leq 159} \gamma_i\mathcal{Y}_i$ with $\gamma_i \in \mathbb{Z}/2$ and $[a] \in [S_5(\mathcal{Y}_{130})]^{S_5}$, then from the relations $\tau_t(a) + a \equiv_{\widetilde{\omega}_{(3)}} 0,\ 1 \leq$

$t \leq 4$, we get $\gamma_i = \gamma_{130}$, $\forall i$, $131 \leq i \leq 159$. In other words, $[S_5(\mathcal{Y}_{130})]^{S_5}_{\widetilde{\omega}_{(3)}} = \langle[\theta(\mathcal{Y}_{130})]_{\widetilde{\omega}_{(3)}}\rangle$ with $\theta(\mathcal{Y}_{130}) = \sum_{130\leq i\leq 159} \mathcal{Y}_i$.

Next, we have $\dim[S_5(\mathcal{Y}_{160})]_{\widetilde{\omega}_{(3)}} = 15$ with the basis $[\mathscr{B}(\mathcal{Y}_{160})]_{\widetilde{\omega}_{(3)}}$. Suppose $b = \sum_{160\leq i\leq 174} \gamma_j \mathcal{Y}_j$ with $\gamma_j \in \mathbb{Z}/2$ and $[b] \in [S_5(\mathcal{Y}_{160})]^{S_5}$. A direct computation shows:

$$\begin{aligned}
\tau_1(b) + b \equiv_{\widetilde{\omega}_{(3)}} &\gamma_{169}(\mathcal{Y}_{160} + \mathcal{Y}_{162}) + \gamma_{170}(\mathcal{Y}_{161} + \mathcal{Y}_{163}) + \gamma_{171}(\mathcal{Y}_{164} + \mathcal{Y}_{165})\\
&+ (\gamma_{166} + \gamma_{172})(\mathcal{Y}_{166} + \mathcal{Y}_{172}) + (\gamma_{167} + \gamma_{173})(\mathcal{Y}_{167} + \mathcal{Y}_{173})\\
&+ (\gamma_{168} + \gamma_{174})(\mathcal{Y}_{168} + \mathcal{Y}_{174}),\\
\tau_2(b) + b \equiv_{\widetilde{\omega}_{(3)}} &(\gamma_{162} + \gamma_{169})(\mathcal{Y}_{162} + \mathcal{Y}_{169}) + (\gamma_{163} + \gamma_{170})(\mathcal{Y}_{163} + \mathcal{Y}_{170})\\
&+ (\gamma_{164} + \gamma_{166})(\mathcal{Y}_{164} + \mathcal{Y}_{166}) + (\gamma_{165} + \gamma_{167})(\mathcal{Y}_{165} + \mathcal{Y}_{167})\\
&+ (\gamma_{168} + \gamma_{171})(\mathcal{Y}_{168} + \mathcal{Y}_{171}) + (\gamma_{165} + \gamma_{167})(\mathcal{Y}_{165} + \mathcal{Y}_{167})\\
&+ \gamma_{174}(\mathcal{Y}_{172} + \mathcal{Y}_{173}),\\
\tau_3(b) + b \equiv_{\widetilde{\omega}_{(3)}} &(\gamma_{160} + \gamma_{162})(\mathcal{Y}_{160} + \mathcal{Y}_{162}) + (\gamma_{161} + \gamma_{164})(\mathcal{Y}_{161} + \mathcal{Y}_{164})\\
&+ (\gamma_{163} + \gamma_{165})(\mathcal{Y}_{163} + \mathcal{Y}_{165}) + (\gamma_{167} + \gamma_{168})(\mathcal{Y}_{167} + \mathcal{Y}_{168})\\
&+ (\gamma_{170} + \gamma_{171})(\mathcal{Y}_{170} + \mathcal{Y}_{171}) + (\gamma_{173} + \gamma_{174})(\mathcal{Y}_{173} + \mathcal{Y}_{174}),\\
\tau_4(b) + b \equiv_{\widetilde{\omega}_{(3)}} &(\gamma_{160} + \gamma_{161})(\mathcal{Y}_{160} + \mathcal{Y}_{161}) + (\gamma_{162} + \gamma_{163})(\mathcal{Y}_{162} + \mathcal{Y}_{163})\\
&+ (\gamma_{164} + \gamma_{165})(\mathcal{Y}_{164} + \mathcal{Y}_{165}) + (\gamma_{166} + \gamma_{167})(\mathcal{Y}_{166} + \mathcal{Y}_{167})\\
&+ (\gamma_{169} + \gamma_{170})(\mathcal{Y}_{169} + \mathcal{Y}_{170}) + (\gamma_{172} + \gamma_{173})(\mathcal{Y}_{172} + \mathcal{Y}_{173}).
\end{aligned}$$

Then, from the relations $\tau_t(b) + b \equiv_{\widetilde{\omega}_{(3)}} 0$, we obtain $\gamma_j = 0$, $\forall j$.

Now, let $[X]_{\widetilde{\omega}_{(3)}} \in (Q\mathscr{P}_5(\widetilde{\omega}_{(3)}))^{GL_5}$ with $X \in \mathscr{P}_5(\widetilde{\omega}_{(3)})$, then $[X]_{\widetilde{\omega}_{(3)}} \in (Q\mathscr{P}_5(\widetilde{\omega}_{(3)}))^{S_5}$. So, we have $X \equiv_{\widetilde{\omega}_{(3)}} \gamma\theta(\mathcal{Y}_{130})$ with $\gamma \in \mathbb{Z}/2$. By a direct computation, we have

$$\tau_5(X) + X \equiv_{\widetilde{\omega}_{(3)}} \gamma\mathcal{Y}_{130} + \text{other terms} \equiv_{\widetilde{\omega}_{(3)}} 0.$$

This implies $\gamma = 0$. The proposition follows. □

By a simple computation using the techniques as given in the proof of Lemmas 4.1.2 and 4.1.3, we claim that:

Lemma 4.1.4. *The following results are true*:

i) *We have a direct summand decomposition of the S_5-modules*:

$$Q\mathscr{P}_5(\widetilde{\omega}_{(2)}) = [S_5(\mathcal{Y}_{106})]_{\widetilde{\omega}_{(2)}} \bigoplus [S_5(\mathcal{Y}_{126})]_{\widetilde{\omega}_{(2)}}.$$

ii) *The subspace $(Q\mathscr{P}_5(\widetilde{\omega}_{(2)}))^{GL_5}$ is trivial.*

Proof of Theorem 4.1.1. Let $[X] \in (Q\mathscr{P}_5)^{GL_5}_8$. Then, from Lemmas 4.1.2-4.1.4, we have

$$X = \beta_1\theta(\mathcal{Y}_1) + \beta_2\theta(\mathcal{Y}_2) + \beta_3 q + \beta_4\theta(\mathcal{Y}_3) \ \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5),$$

with $\beta_t \in \mathbb{Z}/2,\ 1 \leq t \leq 4$. By using Theorem 3.1.3 and computing $\tau_5(X)+X$ in terms of the admissible monomials (modulo($\mathcal{A}_2^+\mathscr{P}_5$)), we conclude

$$\tau_5(X)+X \equiv \beta_1\mathcal{Y}_7+(\beta_1+\beta_2)\mathcal{Y}_{16}+\beta_3\mathcal{Y}_{33}+\beta_4(\mathcal{Y}_{91}+\mathcal{Y}_{92}+\mathcal{Y}_{93}) + \text{other terms } \equiv 0.$$

This relation shows $\beta_1=\beta_2=\beta_3=\beta_4=0$. The proposition is proved. □

As an immediate consequence of Theorem 4.1.1, we get the following.

Corollary 4.1.5. *The fifth algebraic transfer*

$$Tr_5 : \mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2} H_8(B(\mathbb{Z}/2)^{\times 5}) \to \mathrm{Ext}_{\mathcal{A}_2}^{5,5+8}(\mathbb{Z}/2,\mathbb{Z}/2)$$

is a trivial isomorphism.

4.2. Computation of $(\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)})^{S_5}$

From the results in Section 3.3.1, we see that $\dim \mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)} = 666$ with the basis $\{[\mathcal{Y}_{21,\,t}] : 1 \leq t \leq 666\}$. Here, the admissible monomials $\mathcal{Y}_t := \mathcal{Y}_{21,\,t},\ 1 \leq t \leq 666$, is described in Sections 6.6 and 6.7 of the online version [40]. Recall that $(\widetilde{Sq^0_*})_{(5,21)}$ is an epimorphism of GL_5-modules. Combining this and the results in Section 4.1, we get $(Q\mathscr{P}_5)_{21}^{GL_5} \subseteq (\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)})^{GL_5}$. By Lemma 3.3.2, we have a direct summand decomposition of the S_5-modules:

$$\mathrm{Ker}(\widetilde{Sq^0_*})_{(5,21)} = Q\mathscr{P}_5(3,3,1,1) \bigoplus Q\mathscr{P}_5(3,3,3).$$

For $\omega = (3,3,1,1)$, according to the results in Section 3.3.1, we get $Q\mathscr{P}_5(\omega) = Q\mathscr{P}_5^0(\omega) \bigoplus Q\mathscr{P}_5^+(\omega)$ with $\dim Q\mathscr{P}_5^0(\omega) = 340$ and $\dim Q\mathscr{P}_5^+(\omega) = 196$. Note that $\mathscr{Z} = x_1^{15}x_2^3x_3^3$ is the minimal spike monomial in $(\mathscr{P}_5)_{21}$ and $\omega(\mathscr{Z}) = \omega$. So, $[x]_\omega = [x]$ for any $x \in (\mathscr{P}_5)_{21}$. By using the results in Section 3.3.1, we see that there is a direct summand decomposition of the S_5-modules:

$$Q\mathscr{P}_5^0(\omega) = \langle[S_5(\mathcal{Y}_1)]\rangle \bigoplus \langle[S_5(\mathcal{Y}_{31})]\rangle \bigoplus \langle[S_5(\mathcal{Y}_{61})]\rangle \bigoplus \langle[S_5(\mathcal{Y}_{121})]\rangle \bigoplus \langle\mathbb{V}_1\rangle,$$

where

$$\begin{aligned}
\mathscr{B}(\mathcal{Y}_1) &= \{\mathcal{Y}_t : 1 \leq t \leq 30\}, \quad \mathscr{B}(\mathcal{Y}_{31}) = \{\mathcal{Y}_t : 31 \leq t \leq 60\},\\
\mathscr{B}(\mathcal{Y}_{61}) &= \{\mathcal{Y}_t : 61 \leq t \leq 120\}, \quad \mathscr{B}(\mathcal{Y}_{121}) = \{\mathcal{Y}_t : 121 \leq t \leq 150\},\\
\mathbb{V}_1 &= \mathscr{B}(\mathcal{Y}_{151},\mathcal{Y}_{181},\mathcal{Y}_{201},\mathcal{Y}_{241},\mathcal{Y}_{256},\mathcal{Y}_{266},\mathcal{Y}_{286},\mathcal{Y}_{296},\mathcal{Y}_{301},\mathcal{Y}_{316})\\
&= \{\mathcal{Y}_t : 151 \leq t \leq 340\}.
\end{aligned}$$

Lemma 4.2.1. *The following results are true*:

a) $\langle[S_5(\mathcal{Y}_j)]\rangle^{S_5} = \langle[\theta(\mathcal{Y}_j)]\rangle$ *for* $j = 1, 31, 61, 121$.

b) *The subspace* $\langle[S_5(\mathbb{V}_1)]\rangle^{S_5}$ *is trivial.*

Outline of the proof. For $j = 1$, let $[f_1] \in \langle[S_5(\mathcal{Y}_1)]\rangle^{S_5}$. Then, we have

$$\tau_m(f_1) = \sum_{X \in \mathscr{B}(\mathcal{Y}_1)} \beta_X.X \ \ \text{modulo}(\mathcal{A}_2^+\mathscr{P}_5),\ 0 < m < 5$$

with $f_1 = \sum_{X\in\mathscr{B}(\mathcal{Y}_1)}\beta_X.X$ and $\beta_X \in \mathbb{Z}/2$. By a direct computation, we can see that the action of the symmetric group S_5 on $Q\mathscr{P}_5$ induces the one of it on the set $[\mathscr{B}(\mathcal{Y}_j)]$ and this action is transitive. So, we get $\beta_X = \beta_{X'} = \beta \in \mathbb{Z}/2$ for all $X, X' \in \mathscr{B}(\mathcal{Y}_j)$. This means $f_1 = \theta(\mathcal{Y}_1)$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$. For $j = 31, 61, 121$, we determine $\tau_m(f_j) + f_j$ in terms of $\mathcal{Y}_j$. Then, by a simple computation using the relations $\tau_m(f_j) = f_j$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$, we conclude $f_j = \theta(\mathcal{Y}_j)$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$.

Next, we have $\dim\langle\mathbb{V}_1\rangle = 190$ with the basis $\{[\mathcal{Y}_t] : 151 \le t \le 340\}$. Assume that $g = \sum_{u\in\mathbb{V}_1}\gamma_u.u$ with $\gamma_u \in \mathbb{Z}/2$ and $[g] \in \langle[\mathbb{V}_1]\rangle^{S_5}$. By using Theorem 3.1.3 and a similar computation as given in the proof of Lemmas 4.1.2 and 4.1.3, we obtain $\gamma_u = 0$ for all $u \in \mathbb{V}_1$. This implies that g is $\mathcal{A}_2$-decomposable. The lemma follows. □

Lemma 4.2.2. $Q\mathscr{P}_5^+(\omega)^{S_5} = \langle[p := \mathcal{Y}_{411} + \mathcal{Y}_{412} + \cdots + \mathcal{Y}_{419}]\rangle$.

Proof. From Proposition 3.3.6, we see that the sets $[S_5(\mathcal{Y}_{401})] = \langle[\mathcal{Y}_t] : 401 \le t \le 410\rangle$ and $\mathbb{V}_2 = \langle[\mathcal{Y}_t] : 411 \le t \le 596\rangle$, are S_5-submodules of $Q\mathscr{P}_5^+(\omega)$. Hence, we have a direct summand decomposition of the S_5-modules:

$$Q\mathscr{P}_5^+(\omega) = [S_5(\mathcal{Y}_{401})] \bigoplus \mathbb{V}_2.$$

The set $[\mathscr{B}(\mathcal{Y}_{401})]$ is a basis of $[S_5(\mathcal{Y}_{401})]$. Assume that Z is a polynomial such that $[Z] \in [S_5(\mathcal{Y}_{401})]^{S_5}$ and $Z = \sum_{401\le t\le 410}\ell_t\mathcal{Y}_t$ with $\ell_t \in \mathbb{Z}/2$. For $1 \le j \le 4$, we explicitly compute $\tau_j(Z) + Z$ in the terms of the admissible monomials $\mathcal{Y}_t,\ 401 \le t \le 410$. By a direct computation using Theorem 3.1.3, we get

$$\begin{aligned}
\tau_1(Z) + Z \equiv\ & \ell_{401}\mathcal{Y}_{404} + \ell_{402}\mathcal{Y}_{405} + \ell_{403}\mathcal{Y}_{407} + (\ell_{406} + \ell_{409})(\mathcal{Y}_{406} + \mathcal{Y}_{409})\\
&+ (\ell_{408} + \ell_{410})(\mathcal{Y}_{408} + \mathcal{Y}_{410}),\\
\tau_2(Z) + Z \equiv\ & (\ell_{401} + \ell_{404})(\mathcal{Y}_{401} + \mathcal{Y}_{404}) + (\ell_{402} + \ell_{405})(\mathcal{Y}_{402} + \mathcal{Y}_{405})\\
&+ (\ell_{403} + \ell_{406})(\mathcal{Y}_{403} + \mathcal{Y}_{406}) + (\ell_{407} + \ell_{408})(\mathcal{Y}_{407} + \mathcal{Y}_{408}) + \ell_{409}\mathcal{Y}_{410},\\
\tau_3(Z) + Z \equiv\ & \ell_{401}\mathcal{Y}_{404} + (\ell_{402} + \ell_{403})(\mathcal{Y}_{402} + \mathcal{Y}_{403}) + (\ell_{405} + \ell_{407})(\mathcal{Y}_{405} + \mathcal{Y}_{407}),\\
&+ (\ell_{406} + \ell_{408})(\mathcal{Y}_{406} + \mathcal{Y}_{408}) + (\ell_{409} + \ell_{410})(\mathcal{Y}_{409} + \mathcal{Y}_{410}),\\
\tau_4(Z) + Z \equiv\ & (\ell_{401} + \ell_{402})(\mathcal{Y}_{401} + \mathcal{Y}_{402}) + \ell_{403}\mathcal{Y}_{407} + (\ell_{404} + \ell_{405})(\mathcal{Y}_{404} + \mathcal{Y}_{405})\\
&+ \ell_{406}\mathcal{Y}_{408} + \ell_{409}\mathcal{Y}_{410}.
\end{aligned}$$

Then, by the relations $\tau_j(Z) = Z$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$, $j = 1, 2, 3, 4$, one gets $\ell_t = 0,\ \forall t$. This implies that Z is $\mathcal{A}_2$-decomposable.

Note that $\dim\mathbb{V}_2 = 186$ with the basis $\{[\mathcal{Y}_j] : 411 \le j \le 596\}$. Then if the polynomial $g = \sum_{u\in\mathbb{V}_2}\sigma_u.u,\ \sigma_u \in \mathbb{Z}/2$ such that $[g] \in \mathbb{V}_2^{S_5}$, then by a similar argument as given above, we obtain $g = p$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$. This completes the proof of the lemma. □

Proposition 4.2.3. *For $\overline{\omega} = (3,3,3)$, we have $Q\mathscr{P}_5(\overline{\omega})^{GL_5} = \langle [q_2]_{\overline{\omega}}\rangle$, where*

$$\begin{aligned} q_2 &= \mathcal{Y}_{652} + \mathcal{Y}_{653} + \mathcal{Y}_{654} + \mathcal{Y}_{656} + \mathcal{Y}_{657} + \mathcal{Y}_{658} + \sum_{661\leq t\leq 666}\mathcal{Y}_t \\ &= x_1x_2^6x_3^3x_4^5x_5^6 + x_1x_2^3x_3^6x_4^5x_5^6 + x_1x_2^3x_3^5x_4^6x_5^6 + x_1^3x_2x_3^5x_4^6x_5^6 \\ &\quad + x_1^3x_2^5x_3x_4^6x_5^6 + x_1^3x_2^5x_3^6x_4x_5^6 + x_1^3x_2^3x_3^5x_4^4x_5^6 + x_1^3x_2^3x_3^4x_4^5x_5^6 \\ &\quad + x_1^3x_2^3x_3^5x_4^6x_5^4 + x_1^3x_2^4x_3^3x_4^5x_5^6 + x_1^3x_2^5x_3^3x_4^6x_5^4 + x_1^3x_2^5x_3^6x_4^3x_5^4. \end{aligned}$$

Based on the results in Section 3.3, we have

$$\dim Q\mathscr{P}_5(\overline{\omega}) = \dim Q\mathscr{P}_5^0(\overline{\omega}) + \dim Q\mathscr{P}_5^+(\overline{\omega}) = 60 + 70 = 130.$$

Consider the following monomials:

$$\begin{aligned} &\mathcal{Y}_{341} = x_3^7x_4^7x_5^7, && \mathcal{Y}_{351} = x_2x_3^6x_4^7x_5^7, && \mathcal{Y}_{381} = x_2^3x_3^5x_4^6x_5^7,\\ &\mathcal{Y}_{597} = x_1x_2^2x_3^4x_4^7x_5^7, && \mathcal{Y}_{607} = x_1x_2^6x_3x_4^6x_5^7, && \mathcal{Y}_{617} = x_1x_2^2x_3^5x_4^6x_5^7,\\ &\mathcal{Y}_{622} = x_1x_2^3x_3^4x_4^6x_5^7, && \mathcal{Y}_{642} = x_1^3x_2^5x_3^2x_4^4x_5^7, && \mathcal{Y}_{647} = x_1^3x_2^3x_3^4x_4^4x_5^7,\\ &\mathcal{Y}_{652} = x_1x_2^6x_3^3x_4^5x_5^6, && \mathcal{Y}_{659} = x_1^3x_2^5x_3^2x_4^5x_5^6, && \mathcal{Y}_{660} = x_1^3x_2^5x_3^3x_4^4x_5^6. \end{aligned}$$

The following lemma can be easily proved by a direct computation.

Lemma 4.2.4.

i) *The following subspaces are S_5-submodules of $Q\mathscr{P}_5(\overline{\omega})$:*

$$\langle [S_5(\mathcal{Y}_a)]_{\overline{\omega}}\rangle,\ a = 341, 351, 381, 597, \mathbb{V}_3 := \langle [S_5(\mathcal{Y}_{607}, \mathcal{Y}_{617}, \mathcal{Y}_{622}, \mathcal{Y}_{642}, \mathcal{Y}_{647})]_{\overline{\omega}}\rangle,$$

$$\mathbb{V}_4 := \langle [S_5(\mathcal{Y}_{652}, \mathcal{Y}_{659}, \mathcal{Y}_{660})]_{\overline{\omega}}\rangle.$$

ii) *We have a direct summand decomposition of the S_5-modules*:

$$\begin{aligned} Q\mathscr{P}_5(\overline{\omega}) &= \langle [S_5(\mathcal{Y}_{341})]_{\overline{\omega}}\rangle \bigoplus \langle [S_5(\mathcal{Y}_{351})]_{\overline{\omega}}\rangle \bigoplus \langle [S_5(\mathcal{Y}_{381})]_{\overline{\omega}}\rangle \\ &\quad \bigoplus \langle [S_5(\mathcal{Y}_{597})]_{\overline{\omega}}\rangle \bigoplus \mathbb{V}_3 \bigoplus \mathbb{V}_4. \end{aligned}$$

Lemma 4.2.5. *We have the following results*:

i) $\langle [S_5(\mathcal{Y}_a)]_{\overline{\omega}}\rangle^{S_5} = \langle [\theta(\mathcal{Y}_a)]_{\overline{\omega}}\rangle$ *for* $a = 341, 351, 381, 597$.

ii) $\langle [S_5(\mathbb{V}_3)]_{\overline{\omega}}\rangle^{S_5} = \langle [q_1 := \mathcal{Y}_{610} + \mathcal{Y}_{611} + \mathcal{Y}_{613} + \mathcal{Y}_{614} + \mathcal{Y}_{616} + \sum_{647\leq t\leq 651}\mathcal{Y}_t]_{\overline{\omega}}\rangle$.

iii) $\langle [S_5(\mathbb{V}_4)]_{\overline{\omega}}\rangle^{S_5} = \langle [q_2 := \mathcal{Y}_{652} + \mathcal{Y}_{653} + \mathcal{Y}_{654} + \mathcal{Y}_{656} + \mathcal{Y}_{657} + \mathcal{Y}_{658} + \sum_{661\leq t\leq 666}\mathcal{Y}_t]_{\overline{\omega}}\rangle$,

The proof of the lemma is straightforward.

Proof of Proposition 4.2.3. By Lemmas 4.2.4 and 4.2.5, we get

$$Q\mathscr{P}_5(\overline{\omega})^{S_5} = \langle [\theta(\mathcal{Y}_{341})]_{\overline{\omega}},\ [\theta(\mathcal{Y}_{351})]_{\overline{\omega}},\ [\theta(\mathcal{Y}_{381})]_{\overline{\omega}},\ [\theta(\mathcal{Y}_{597})]_{\overline{\omega}},\ [q_1]_{\overline{\omega}},\ [q_2]_{\overline{\omega}}\rangle.$$

Let X be a polynomial in $\mathscr{P}_5(\overline{\omega})$ such that $[X]_{\overline{\omega}} \in Q\mathscr{P}_5(\overline{\omega})^{GL_5}$. Then, we have

$$X \equiv_{\overline{\omega}} v_1\theta(\mathcal{Y}_{341}) + v_2\theta(\mathcal{Y}_{351}) + v_3\theta(\mathcal{Y}_{381}) + v_4\theta(\mathcal{Y}_{597}) + v_5q_1 + v_6q_2,$$

with $v_j \in \mathbb{Z}/2$ for $1 \le j \le 6$. We explicitly compute $\tau_5(X)$ in terms of the admissible monomials $\mathcal{Y}_t$ with $t = 341, 342, \ldots, 400, 597, 598, \ldots, 666$. By a direct computation, one gets

$$\tau_5(X) + X \equiv_{\overline{\omega}} (v_1 + v_2)\mathcal{Y}_{342} + (v_2 + v_4)\mathcal{Y}_{351} + v_2\mathcal{Y}_{354} + v_3\mathcal{Y}_{383} + v_5\mathcal{Y}_{650} + \text{other terms}.$$

Since $[X]_{\overline{\omega}} \in Q\mathscr{P}_5(\overline{\omega})^{GL_5}$, $v_j = 0,\ 1 \le j \le 5$. The proposition is proved. □

Combining Lemmas 4.2.1, 4.2.2 and 4.2.5 gives:

Corollary 4.2.6. *There exist exactly* 11 *non-zero classes in the kernel of* $(\widetilde{Sq^0_*})_{(5,21)}$ *invariant under the action of* S_5.

4.3. Proof of Theorem 1.2

Suppose that T is a polynomial in $(\mathscr{P}_5)_{21}$ such that $[T] \in (Q\mathscr{P}_5)_{21}^{GL_5}$. According to Proposition 4.2.3, we have $T = T^* + \zeta_6 q_2$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$ with $T^* \in \mathscr{P}_5^-(\overline{\omega})$ and $\zeta_6 \in \mathbb{Z}/2$. By a simple computation, we see that $[q_2] \in (Q\mathscr{P}_5)_{21}^{S_5}$. This implies that $[T^*]$ is S_5-invariant. On the other hand, $[\mathscr{P}_5^-(\overline{\omega})] = Q\mathscr{P}_5(\omega)$. Hence, by Lemmas 4.2.1 and 4.2.2, we obtain

$$T^* = \zeta_1\theta(\mathcal{Y}_1) + \zeta_2\theta(\mathcal{Y}_{31}) + \zeta_3\theta(\mathcal{Y}_{61}) + \zeta_4\theta(\mathcal{Y}_{121}) + \zeta_5 p \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5),$$

where $\zeta_i \in \mathbb{Z}/2$. Using Theorem 1.1 and computing $\tau_5(T) + T$ in terms of the admissible monomials $\mathcal{Y}_t,\ 1 \le t \le 666$, we conclude

$$\tau_5(T) + T = (\zeta_1 + \zeta_3)\mathcal{Y}_4 + (\zeta_2 + \zeta_4)\mathcal{Y}_{35} + \zeta_3\mathcal{Y}_{61} + (\zeta_3 + \zeta_4)\mathcal{Y}_{107} + \zeta_5\mathcal{Y}_{153} + \text{other terms modulo}(\mathcal{A}_2^+\mathscr{P}_5).$$

By the relation $\tau_5(T) = T$ modulo$(\mathcal{A}_2^+\mathscr{P}_5)$, one gets $\zeta_i = 0,\ 1 \le i \le 5$. This shows that

$$T = \zeta_6 q_2 \text{ modulo}(\mathcal{A}_2^+\mathscr{P}_5).$$

The proof of the theorem is completed.

5. Proof of Theorem 1.3

Obviously, $\lambda_3 \in \Lambda^{1,3}$ and $\overline{f}_0 = \lambda_3\lambda_5\lambda_6\lambda_4 + \lambda_3^2\lambda_7\lambda_5 + \lambda_7\lambda_5\lambda_2\lambda_3^2 + \lambda_7\lambda_5\lambda_4\lambda_2 \in \Lambda^{4,18}$ are the cycles in the lambda algebra Λ. By Lin [20], we have

$$\mathrm{Ext}_{\mathcal{A}_2}^{5,5+21}(\mathbb{Z}/2, \mathbb{Z}/2) = \langle h_2 f_0 \rangle,$$

where

$$h_2 = [\lambda_3] \in \mathrm{Ext}_{\mathcal{A}_2}^{1,4}(\mathbb{Z}/2, \mathbb{Z}/2), \text{ and } f_0 = [\overline{f}_0] \in \mathrm{Ext}_{\mathcal{A}_2}^{4,22}(\mathbb{Z}/2, \mathbb{Z}/2).$$

We note that $h_2 f_0 = h_1 g_1$ with $h_1 \in \mathrm{Ext}_{\mathcal{A}_2}^{1,2}(\mathbb{Z}/2, \mathbb{Z}/2)$ and $g_1 \in \mathrm{Ext}_{\mathcal{A}_2}^{4,24}(\mathbb{Z}/2, \mathbb{Z}/2)$. By direct computations, we find that the following element is $\mathcal{A}_2^+$-annihilated in $H_{21}(B(\mathbb{Z}/2)^{\times 5})$:

$$Z = \left\{\begin{array}{l}
a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(1)}a_5^{(9)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(2)}a_5^{(8)} + a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(1)}a_5^{(8)} \\
+ a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(2)}a_5^{(7)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(4)}a_5^{(6)} + a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(3)}a_5^{(6)} \\
+ a_1^{(3)}a_2^{(5)}a_3^{(6)}a_4^{(1)}a_5^{(6)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(5)}a_5^{(5)} + a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(4)}a_5^{(5)} \\
+ a_1^{(3)}a_2^{(5)}a_3^{(6)}a_4^{(2)}a_5^{(5)} + a_1^{(3)}a_2^{(3)}a_3^{(9)}a_4^{(1)}a_5^{(5)} + a_1^{(3)}a_2^{(5)}a_3^{(7)}a_4^{(1)}a_5^{(5)} \\
+ a_1^{(3)}a_2^{(3)}a_3^{(9)}a_4^{(2)}a_5^{(4)} + a_1^{(3)}a_2^{(5)}a_3^{(7)}a_4^{(2)}a_5^{(4)} + a_1^{(3)}a_2^{(3)}a_3^{(10)}a_4^{(1)}a_5^{(4)} \\
+ a_1^{(3)}a_2^{(6)}a_3^{(7)}a_4^{(1)}a_5^{(4)} + a_1^{(3)}a_2^{(5)}a_3^{(6)}a_4^{(4)}a_5^{(3)} + a_1^{(3)}a_2^{(6)}a_3^{(7)}a_4^{(2)}a_5^{(3)} \\
+ a_1^{(3)}a_2^{(3)}a_3^{(10)}a_4^{(2)}a_5^{(3)} + a_1^{(3)}a_2^{(3)}a_3^{(11)}a_4^{(2)}a_5^{(2)} + a_1^{(3)}a_2^{(5)}a_3^{(9)}a_4^{(2)}a_5^{(2)} \\
+ a_1^{(3)}a_2^{(6)}a_3^{(10)}a_4^{(1)}a_5^{(1)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(9)}a_5^{(1)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(8)}a_5^{(2)} \\
+ a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(8)}a_5^{(1)} + a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(7)}a_5^{(2)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(6)}a_5^{(4)} \\
+ a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(6)}a_5^{(3)} + a_1^{(3)}a_2^{(5)}a_3^{(6)}a_4^{(6)}a_5^{(1)} + a_1^{(3)}a_2^{(3)}a_3^{(5)}a_4^{(5)}a_5^{(5)} \\
+ a_1^{(3)}a_2^{(3)}a_3^{(6)}a_4^{(5)}a_5^{(4)} + a_1^{(3)}a_2^{(5)}a_3^{(6)}a_4^{(5)}a_5^{(2)} + a_1^{(3)}a_2^{(3)}a_3^{(9)}a_4^{(5)}a_5^{(1)} \\
+ a_1^{(3)}a_2^{(5)}a_3^{(7)}a_4^{(5)}a_5^{(1)} + a_1^{(3)}a_2^{(3)}a_3^{(9)}a_4^{(4)}a_5^{(2)} + a_1^{(3)}a_2^{(5)}a_3^{(7)}a_4^{(4)}a_5^{(2)} \\
+ a_1^{(3)}a_2^{(3)}a_3^{(10)}a_4^{(4)}a_5^{(1)} + a_1^{(3)}a_2^{(6)}a_3^{(7)}a_4^{(4)}a_5^{(1)} + a_1^{(3)}a_2^{(5)}a_3^{(6)}a_4^{(3)}a_5^{(4)} \\
+ a_1^{(3)}a_2^{(6)}a_3^{(7)}a_4^{(3)}a_5^{(2)} + a_1^{(3)}a_2^{(3)}a_3^{(10)}a_4^{(3)}a_5^{(2)} + a_1^{(3)}a_2^{(3)}a_3^{(11)}a_4^{(2)}a_5^{(2)} \\
+ a_1^{(3)}a_2^{(5)}a_3^{(9)}a_4^{(2)}a_5^{(2)} + a_1^{(3)}a_2^{(6)}a_3^{(10)}a_4^{(1)}a_5^{(1)} + a_1^{(3)}a_2^{(3)}a_3^{(12)}a_4^{(1)}a_5^{(2)} \\
+ a_1^{(3)}a_2^{(7)}a_3^{(8)}a_4^{(1)}a_5^{(2)} + a_1^{(3)}a_2^{(11)}a_3^{(4)}a_4^{(1)}a_5^{(2)} + a_1^{(3)}a_2^{(13)}a_3^{(2)}a_4^{(1)}a_5^{(2)} \\
+ a_1^{(3)}a_2^{(14)}a_3^{(1)}a_4^{(1)}a_5^{(2)} + a_1^{(3)}a_2^{(12)}a_3^{(3)}a_4^{(1)}a_5^{(2)} + a_1^{(3)}a_2^{(8)}a_3^{(7)}a_4^{(1)}a_5^{(2)} \\
+ a_1^{(3)}a_2^{(4)}a_3^{(11)}a_4^{(1)}a_5^{(2)} + a_1^{(3)}a_2^{(2)}a_3^{(13)}a_4^{(1)}a_5^{(2)} + a_1^{(3)}a_2^{(1)}a_3^{(14)}a_4^{(1)}a_5^{(2)} \\
+ a_1^{(3)}a_2^{(6)}a_3^{(6)}a_4^{(3)}a_5^{(3)} + a_1^{(3)}a_2^{(5)}a_3^{(5)}a_4^{(5)}a_5^{(3)} + a_1^{(3)}a_2^{(3)}a_3^{(3)}a_4^{(9)}a_5^{(3)} \\
+ a_1^{(3)}a_2^{(5)}a_3^{(3)}a_4^{(7)}a_5^{(3)} + a_1^{(3)}a_2^{(7)}a_3^{(7)}a_4^{(2)}a_5^{(2)} + a_1^{(3)}a_2^{(6)}a_3^{(9)}a_4^{(1)}a_5^{(2)} \\
+ a_1^{(3)}a_2^{(9)}a_3^{(6)}a_4^{(1)}a_5^{(2)} + a_1^{(3)}a_2^{(10)}a_3^{(5)}a_4^{(1)}a_5^{(2)} + a_1^{(3)}a_2^{(5)}a_3^{(10)}a_4^{(2)}a_5^{(1)} \\
+ a_1^{(3)}a_2^{(13)}a_3^{(3)}a_4^{(1)}a_5^{(1)} + a_1^{(3)}a_2^{(5)}a_3^{(11)}a_4^{(1)}a_5^{(1)} + a_1^{(3)}a_2^{(9)}a_3^{(7)}a_4^{(1)}a_5^{(1)}
\end{array}\right\}.$$

According to the proof of Theorem 1.2, $\{[q_2]\}$ is a basis of $(Q\mathscr{P}_5)^{GL_5}$ in degree 13.2^1-5. Obviously, $\langle Z, q_2\rangle = 1$. Since $Z \in P_{\mathcal{A}_2}H_{13.2^1-5}(B(\mathbb{Z}/2)^{\times 5})$, $[Z]$ is dual to $[q_2]$. Using the presentation of Tr_5 over the algebra Λ and the differential (1.3) in Section 1, we obtain

$$\begin{aligned}
\psi_5(Z) &= \lambda_3^2\lambda_5\lambda_6\lambda_4 + \lambda_3^3\lambda_7\lambda_5 + \lambda_3\lambda_7\lambda_5\lambda_2\lambda_3^2 + \lambda_3\lambda_7\lambda_5\lambda_4\lambda_2 + \lambda_3^2\lambda_5\lambda_3\lambda_7 \\
&= \lambda_3\overline{f}_0 + \partial(\lambda_3^2\lambda_5\lambda_{11}).
\end{aligned}$$

Since $Z \in P_{\mathcal{A}_2} H_{13.2^1-5}(B(\mathbb{Z}/2)^{\times 5})$, $\psi_5(Z)$ is a cycle in $\Lambda^{5,21}$. This implies that $h_2 f_0$ is in the image of Tr_5. Further, by Theorem 1.2,

$$\mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2} H_{13.2^1-5}(B(\mathbb{Z}/2)^{\times 5}) \text{ and } \mathrm{Ext}_{\mathcal{A}_2}^{5,5+(13.2^1-5)}(\mathbb{Z}/2, \mathbb{Z}/2)$$

that have the same dimensions are 1. Hence, Tr_5 is an isomorphism when acting on the space $\mathbb{Z}/2 \otimes_{GL_5} P_{\mathcal{A}_2} H_{13.2^1-5}(B(\mathbb{Z}/2)^{\times 5})$. Theorem 1.3 is proved.

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