J. Korean Math. Soc. **58** (2021), No. 3, pp. 525–552
https://doi.org/10.4134/JKMS.j180808
pISSN: 0304-9914 / eISSN: 2234-3008

GENERAL ITERATIVE ALGORITHMS FOR MONOTONE INCLUSION, VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce two general iterative algorithms (one implicit algorithm and one explicit algorithm) for finding a common element of the solution set of the variational inequality problems for a continuous monotone mapping, the zero point set of a set-valued maximal monotone operator, and the fixed point set of a continuous pseudocontractive mapping in a Hilbert space. Then we establish strong convergence of the proposed iterative algorithms to a common point of three sets, which is a solution of a certain variational inequality. Further, we find the minimum-norm element in common set of three sets.

1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and let C be a nonempty closed convex subset of H. For the mapping $T : C \to C$, we denote the fixed point set of T by $Fix(T)$, that is, $Fix(T) = \{x \in C : Tx = x\}$.

The monotone inclusion problem plays an essential role in the theory of nonlinear analysis and optimization. Let $B : H \to 2^H$ be a maximal monotone operator. The monotone inclusion problem consists of finding a zero element of B, that is, a solution of the inclusion problem:

$$0 \in Bx. \tag{1.1}$$

The solution set of the problem (1.1) is denoted by $B^{-1}0$. A classical method for solving the problem is proximal point algorithm, proposed by Martinet [9] and generalized by Rockafellar [11]. In some concrete cases including variational inequalities, the monotone inclusion problem requires to find a zero of the sum

Received November 28, 2018; Accepted March 10, 2021.
2010 *Mathematics Subject Classification.* 47H05, 47H09, 47H10, 47J05, 47J20, 47J25.
Key words and phrases. Maximal monotone operator, variational inequality, zeros, fixed points, continuous monotone mapping, continuous pseudocontractive mapping, minimum-norm point.
This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2018R1D1A1B07045718).

of two monotone operator. That is, in the case of $F = A + B$, where A and B are monotone operators, the problem is reduced to as follows:

$$\text{find } z \in C \text{ such that } 0 \in Fz.$$

The solution set of this problem is denoted by $F^{-1}0$.

Let $A : C \to H$ be a nonlinear mapping. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

This problem is called Hartmann-Stampacchia variational inequality ([13]). We denote the set of solutions of the variational inequality problem (1.2) by $VI(C, A)$. As we also know, variational inequality theory has emerged as an important tool in studying a wide class of numerous problem in physics, optimization, variational inequalities, minimax problem, Nash equilibrium problem in noncooperative games and others.

A fixed point problem is to find a fixed point z of a nonlinear mapping T with property:

$$z \in C,\ Tz = z. \tag{1.3}$$

In order to study the variational inequality problem (1.2) coupled with the fixed point problem (1.3), many researchers have invented some iterative algorithms for finding an element of $VI(C, A) \cap Fix(T)$, where A and T are nonlinear mappings. For instance, in case that $A : C \to H$ is an inverse-strongly monotone mapping and $T : C \to C$ is a nonexpansive mapping, see [5, 6] and the references therein, and in case that $A : C \to H$ is a continuous monotone mapping and $T : C \to C$ is a continuous pseudocontractive mapping, see [4, 16, 20].

In 2016, Jung [8] proposed an iterative algorithm for finding an element of $Fix(T) \cap VI(C, A) \cap B^{-1}0$, where $T : C \to C$ is a continuous pseudocontractive mapping and $A : C \to H$ is a continuous monotone mapping.

Some iterative algorithms for finding an element of $Fix(T) \cap (A + B)^{-1}0$ have been provided by several authors. For instance, in case that $T : C \to C$ is a nonexpansive mapping and $A : C \to H$ is an inverse-strongly monotone mapping, see [15]. In [1], Afassinou et al. considered a ceratin iterative algorithm for split monotone variational inclusion problem combined with variational inequality and fixed point problems in case that $T : C \to C$ is a demicontractive mapping and $A : C \to H$ is an inverse-strongly monotone mapping.

In this paper, as a continuation of study in this direction, we introduce implicit and explicit iterative algorithms for finding a common element of the set $\Omega := Fix(T) \cap VI(C, A) \cap B^{-1}0$, where $T : C \to C$ is a continuous pseudocontractive mapping, $A : C \to H$ is a continuous monotone mapping and $B : H \to 2^H$ is a maximal monotone operator. Then we establish strong convergence of the sequences generated by the proposed iterative algorithms to a common point of three sets, which is a solution of a certain variational inequality. As a direct consequence, we find the unique minimum-norm element of Ω.

2. Preliminaries and lemmas

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. A mapping A of C into H is called *monotone* if

$$\langle x - y, Ax - Ay \rangle \geq 0, \quad \forall x, y \in C.$$

A mapping A of C into H is called *α-inverse-strongly monotone* (see [5]) if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Clearly, the class of monotone mappings includes the class of α-inverse-strongly monotone mappings.

A mapping T of C into H is said to be *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

and T is said to be *k-strictly pseudocontractive* (see [3]) if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

where I is the identity mapping. Note that the class of k-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a subclass. That is, T is *nonexpansive* (i.e., $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$) if and only if T is 0-strictly pseudocontractive. Clearly, the class of pseudocontractive mappings includes the class of strictly pseudocontractive mappings and the class of nonexpansive mappings as a subclass. Moreover, this inclusion is strict (see Example 5.7.1 and Example 5.7.2 in [2]).

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $dom(B)$, that is, $dom(B) = \{x \in H : Bx \neq \emptyset\}$. A set-valued mapping B is said to be *monotone* on H if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in dom(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator $J_\lambda^B = (I + \lambda B)^{-1} : H \to dom(B)$, which is called the *resolvent* of B. It is well known that $B^{-1}0 = \{x \in H : 0 \in Bx\} = Fix(J_\lambda^B)$ for all $\lambda > 0$ is closed and convex and the resolvent J_λ^B is firmly nonexpansive, that is,

$$\|J_\lambda^B x - J_\lambda^B y\|^2 \leq \langle x - y, J_\lambda^B x - J_\lambda^B y \rangle, \quad \forall x, y \in H, \tag{2.1}$$

and that the resolvent identity

$$J_\lambda^B x = J_\mu^B \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda} \right) J_\lambda^B x \right) \tag{2.2}$$

holds for all $\lambda,\ \mu > 0$ and $x \in H$.

In a real Hilbert space H, the following hold:

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle \tag{2.3}$$

and

$$\|\alpha x + \beta y\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 - \alpha\beta\|x - y\|^2 \le \alpha\|x\|^2 + \beta\|y\|^2 \tag{2.4}$$

for all $x, y \in H$ and $\alpha,\ \beta \in (0, 1)$ with $\alpha + \beta = 1$.

We recall that

(i) a mapping $V : C \to H$ is said to be *l-Lipschitzian* if there exists a constant $l \ge 0$ such that

$$\|Vx - Vy\| \le l\|x - y\|, \quad \forall x, y \in C;$$

(ii) a mapping $G : C \to H$ is said to be *η-strongly monotone* if there exists a constant $\eta > 0$ such that

$$\langle Gx - Gy, x - y\rangle \ge \eta\|x - y\|^2, \quad \forall x, y \in C.$$

We need the following lemmas for the proof of our main results.

Lemma 2.1 ([2]). *In a real Hilbert space H, the following inequality holds*:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.2 ([14]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a real Banach space E, and let $\{\gamma_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition*:

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n)z_n$ for all $n \ge 1$ and

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.3 ([17]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \le (1 - \xi_n)s_n + \xi_n\delta_n, \quad \forall n \ge 1,$$

where $\{\xi\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) *$\{\xi_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \xi_n = \infty$;*

(ii) *$\limsup_{n\to\infty} \delta_n \le 0$ or $\sum_{n=1}^{\infty} \xi_n|\delta_n| < \infty$.*

Then $\lim_{n\to\infty} s_n = 0$.

The following lemmas are Lemma 2.3 and Lemma 2.4 of Zegeye [19], respectively.

Lemma 2.4 ([19])**.** *Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping. Then, for $\nu > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Az\rangle + \frac{1}{\nu}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

For $\nu > 0$ and $x \in H$, define $A_\nu : H \to C$ by

$$A_\nu x = \left\{ z \in C : \langle y - z, Az\rangle + \frac{1}{\nu}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}.$$

Then the following hold:

(i) *A_ν is single-valued*;
(ii) *A_ν is firmly nonexpansive, that is,*

$$\|A_\nu x - A_\nu y\|^2 \leq \langle x - y, A_\nu x - A_\nu y\rangle, \quad \forall x, y \in H;$$

(iii) *$Fix(A_\nu) = VI(C, A)$*;
(iv) *$VI(C, A)$ is a closed convex subset of C.*

Lemma 2.5 ([19])**.** *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudocontractive mapping. Then, for $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0, \quad \forall y \in C.$$

For $r > 0$ and $x \in H$, define $T_r : H \to C$ by

$$T_r x = \left\{ z \in C : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0,\ \forall y \in C \right\}.$$

Then the following hold:

(i) *T_r is single-valued*;
(ii) *T_r is firmly nonexpansive, that is,*

$$\|T_r x - T_r y\|^2 \leq \langle x - y, T_r x - T_r y\rangle, \quad \forall x, y \in H;$$

(iii) *$Fix(T_r) = Fix(T)$*;
(iv) *$Fix(T)$ is a closed convex subset of C.*

The following lemma is a variant of a Minty lemma (see [10]).

Lemma 2.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $G : C \to H$ is monotone and weakly continuous along segments, that is, $G(x+ty) \to G(x)$ weakly as $t \to 0$. Then the variational inequality*

$$\widetilde{x} \in C, \quad \langle G\widetilde{x}, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C,$$

is equivalent to the dual variational inequality

$$\widetilde{x} \in C, \quad \langle Gp, p - \widetilde{x}\rangle \geq 0, \quad \forall p \in C.$$

The following lemmas can be easily proven (see [18]), and therefore, we omit their proof.

Lemma 2.7. *Let H be a real Hilbert space. Let $V : H \to H$ be an l-Lipschitzian mapping with a constant $l \geq 0$, and let $G : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa, \eta > 0$. Then for $0 \leq \gamma l < \mu\eta$,*

$$\langle(\mu G - \gamma V)x - (\mu G - \gamma V)y, x - y\rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x, y \in H.$$

That is, $\mu G - \gamma V$ is strongly monotone with constant $\mu\eta - \gamma l$.

Lemma 2.8. *Let H be a real Hilbert space. Let $G : H \to H$ be a κ-Lipschitzian and η-strongly monotone mapping with constants $\kappa > 0$ and $\eta > 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 < t < 1$. Then $I - t\mu G : H \to H$ is a contractive mapping with a constant $1 - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.*

In the following, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

3. Iterative algorithms

Throughout the rest of this paper, we always assume the following:

- H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$;
- C is a nonempty closed convex subset of H;
- $B : H \to 2^H$ is a maximal monotone operator with $dom(B) \subset C$;
- $B^{-1}0$ is the set of zero points of B, that is, $B^{-1}0 = \{z \in H : 0 \in Bz\}$;
- $J^B_{\lambda_t} : H \to \mathrm{dom}(B)$ is the resolvent of B for $\lambda_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t\to 0} \lambda_t > 0$;
- $J^B_{\lambda_n} : H \to \mathrm{dom}(B)$ is the resolvent of B for $\lambda_n \in (0, \infty)$ and $\liminf_{n\to\infty} \lambda_n > 0$;
- $G : C \to C$ is a κ-Lipschitzian and η-strongly monotone mapping with constants κ, $\eta > 0$;
- $V : C \to C$ is an l-Lipschitzian mapping with constant $l \in [0, \infty)$;
- Constants $\mu > 0$ and $\gamma \geq 0$ satisfy $0 < \mu < \frac{2\eta}{\kappa^2}$ and $0 \leq \gamma l < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $A : C \to H$ is a continuous monotone mapping;
- $VI(C, A)$ is the solution set of the variational inequality problem (1.2) for A;
- $T : C \to C$ is a continuous pseudocontractive mapping with $Fix(T) \neq \emptyset$;
- $A_{\nu_t} : H \to C$ is a mapping defined by
$$A_{\nu_t}x = \left\{z \in C : \langle y - z, Az\rangle + \frac{1}{\nu_t}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}$$
for $x \in H$ and $\nu_t \in (0, \infty), t \in (0, 1)$, $\liminf_{t\to 0} \nu_t > 0$;
- $A_{\nu_n} : H \to C$ is a mapping defined by
$$A_{\nu_n}x = \left\{z \in C : \langle y - z, Az\rangle + \frac{1}{\nu_n}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}$$

for $x \in H$ and $\nu_n \in (0, \infty)$, $\liminf_{n\to\infty} \nu_n > 0$;

- $T_{r_t} : H \to C$ is a mapping defined by

$$T_{r_t}x = \left\{ z \in C : \langle Tz, y - z\rangle - \frac{1}{r_t}\langle y - z, (1 + r_t)z - x\rangle \le 0,\ \forall y \in C \right\}$$

for $x \in H$ and $r_t \in (0, \infty)$, $t \in (0, 1)$, and $\liminf_{t\to 0} r_t > 0$;

- $T_{r_n} : H \to C$ is a mapping defined by

$$T_{r_n}x = \left\{ z \in C : \langle Tz, y - z\rangle - \frac{1}{r_n}\langle y - z, (1 + r_n)z - x\rangle \le 0,\ \forall y \in C \right\}$$

for $x \in H$ and $r_n \in (0, \infty)$, and $\liminf_{n\to\infty} r_n > 0$;

- $\Omega := Fix(T) \cap VI(C, A) \cap B^{-1}0 \neq \emptyset$.

By Lemma 2.4 and Lemma 2.5, we note that A_{ν_t}, A_{ν_n}, T_{r_t} and T_{r_n} are nonexpansive, $VI(C, A) = Fix(A_{\nu_t}) = Fix(A_{\nu_n})$ and $Fix(T_{r_t}) = Fix(T_{r_n}) = Fix(T)$.

Now, we introduce the following iterative algorithm that generates a net $\{x_t\}$ in an implicit way:

$$x_t = \theta_t x_t + (1 - \theta_t)T_{r_t}(t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t),\ t \in (0, 1), \tag{3.1}$$

where $0 < \theta_t < 1$ for $t \in (0, 1)$. For $t \in (0, 1)$, consider the following mapping Q_t on C defined by

$$Q_t x = \theta_t x + (1 - \theta_t)T_{r_t}(t\gamma V x + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x),\ \forall x \in C.$$

Then Q_t is contractive. In fact, since T_{r_t}, $J^B_{\lambda_t}$ and A_{ν_t} are nonexpansive, for $x, y \in C$, we have

$$\begin{aligned}
&\|Q_t x - Q_t y\| \\
&= \|\theta_t x + (1 - \theta_t)T_{r_t}(t\gamma V x + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x) \\
&\qquad - (\theta_t y + (1 - \theta_t)T_{r_t}(t\gamma V y + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} y))\| \\
&\le \theta_t\|x - y\| \\
&\quad + (1 - \theta_t)\|T_{r_t}(t\gamma V x + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x) - T_{r_t}(t\gamma V y + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} y)\| \\
&\le \theta_t\|x - y\| + (1 - \theta_t)\|t\gamma V x + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x - (t\gamma V y + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} y)\| \\
&\le \theta_t\|x - y\| \\
&\quad + (1 - \theta_t)(t\|\gamma V x - \gamma V y\| + \|(I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x - (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} y\|) \\
&\le \theta_t\|x - y\| + (1 - \theta_t)(t\gamma l\|x - y\| + (1 - t\tau)\|x - y\|) \\
&= (1 - (1 - \theta_t)(\tau - \gamma l)t)\|x - y\|.
\end{aligned}$$

Since $0 < 1 - (1 - \theta_t)(\tau - \gamma l)t < 1$, Q_t is a contractive mapping. By Banach contraction principle, Q_t has a unique fixed point $x_t \in C$, which uniquely solves the fixed point equation

$$x_t = \theta_t x_t + (1 - \theta_t)T_{r_t}(t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t),\ t \in (0, 1).$$

We summarize the basic property of $\{x_t\}$ and $\{y_t\}$, where $y_t = t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t$.

Proposition 3.1. *Let the net* $\{x_t\}$ *be defined by* (3.1) *and let the net* $\{y_t\}$ *be defined by* $y_t = t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t$ *for* $t \in (0,1)$. *Let* $w_t = A_{\nu_t} x_t$ *for* $t \in (0,1)$. *Then*

(1) $\{x_t\}$ *and* $\{y_t\}$ *are bounded for* $t \in (0,1)$;
(2) x_t *defines a continuous path from* $(0,1)$ *into* C *and so does* y_t *provided* $\theta_t : (0,1) \to (0,1)$ *is continuous and* $r_t, \lambda_t, \nu_t : (0,1) \to (0,\infty)$ *are continuous and* $0 < a \le \min\{r_t, \lambda_t, \nu_t\}$ *for* $t \in (0,1)$;
(3) $\lim_{t\to 0} \|A_{\nu_t} x_t - J^B_{\lambda_t} A_{\nu_t} x_t\| = \lim_{t\to 0} \|w_t - J^B_{\lambda_t} w_t\| = 0$;
(4) $\lim_{t\to 0} \|x_t - w_t\| = 0$;
(5) $\lim_{t\to 0} \|x_t - y_t\| = 0$;
(6) $\lim_{t\to 0} \|x_t - J^B_{\lambda_t} A_{\nu_t} x_t\| = \lim_{t\to 0} \|x_t - J^B_{\lambda_t} w_t\| = 0$;
(7) $\lim_{t\to 0} \|x_t - T_{r_t} x_t\| = 0$;
(8) $\lim_{t\to 0} \|y_t - T_{r_t} y_t\| = 0$.

Proof. (1) Let $p \in \Omega$. Observing $p = T_{r_t} p$, $p = A_{\nu_t} p$, $p = J^B_{\lambda_t} p$ and $p = J^B_{\lambda_t} A_{\nu_t} p$, we derive

$$\begin{aligned}\|x_t - p\| &= \|\theta_t x_t + (1-\theta_t) T_{r_t} y_t - p\| \\ &\le \theta_t \|x_t - p\| + (1-\theta_t)\|T_{r_t} y_t - p\| \\ &\le \theta_t \|x_t - p\| + (1-\theta_t)\|y_t - p\|,\end{aligned}$$

and so

$$\begin{aligned}\|x_t - p\| &\le \|y_t - p\| \\ &= \|t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t - p\| \\ &\le t\|\gamma V x_t - \mu G p\| + \|(I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t - (I - t\mu G)p\| \\ &\le t(\|\gamma V x_t - \gamma V p\| + \|\gamma V p - \mu G p\|) + (1 - t\tau)\|x_t - p\| \\ &\le t(\gamma l \|x_t - p\| + \|\gamma V p - \mu G p\|) + (1 - t\tau)\|x_t - p\| \\ &= (1 - (\tau - \gamma l)t)\|x_t - p\| + t\|\gamma V p - \mu G p\|.\end{aligned}$$

Thus, it follows that

$$\|x_t - p\| \le \frac{\|\gamma V p - \mu G p\|}{\tau - \gamma l} \quad \text{and} \quad \|y_t - p\| \le \frac{\|\gamma V p - \mu G p\|}{\tau - \gamma l}.$$

Hence $\{x_t\}$ and $\{y_t\}$ are bounded and so are $\{V x_t\}$, $\{T_{r_t} x_t\}$, $\{J^B_{\lambda_t} A_{\nu_t} x_t\} = \{J^B_{\lambda_t} w_t\}$, $\{T_{r_t} y_t\}$, $\{G x_t\}$, $\{G J^B_{\lambda_t} A_{\nu_t} x_t\}$ and $\{J^B_{\lambda_t} A_{\nu_t} y_t\}$.

(2) Let $t,\ t_0 \in (0,1)$ and calculate

$$\begin{aligned}&\|x_t - x_{t_0}\| \\ &= \|\theta_t x_t + (1-\theta_t) T_{r_t} y_t - (\theta_{t_0} x_{t_0} + (1-\theta_{t_0}) T_{r_{t_0}} y_{t_0})\| \\ &= \|(\theta_t - \theta_{t_0}) x_t + \theta_{t_0}(x_t - x_{t_0}) + (\theta_{t_0} - \theta_t) T_{r_t} y_t + (1-\theta_{t_0})(T_{r_t} y_t - T_{r_{t_0}} y_{t_0})\| \\ &\le |\theta_t - \theta_{t_0}|\|x_t\| + \theta_{t_0}\|x_t - x_{t_0}\| + (1-\theta_{t_0})\|y_t - y_{t_0}\| \\ &\quad + (1-\theta_{t_0})\|T_{r_t} y_{t_0} - T_{r_{t_0}} y_{t_0}\| + |\theta_t - \theta_{t_0}|\|T_{r_t} y_t\|,\end{aligned}$$

which implies

$$(1-\theta_{t_0})\|x_t - x_{t_0}\| \le |\theta_t - \theta_{t_0}|(\|x_t\| + \|T_{r_t} y_t\|) \\ + (1-\theta_{t_0})(\|y_t - y_{t_0}\| + \|T_{r_t} y_{t_0} - T_{r_{t_0}} y_{t_0}\|).$$

Thus, we have

$$\|x_t - x_{t_0}\| \le \frac{|\theta_t - \theta_{t_0}|}{1-\theta_{t_0}}(\|x_t\| + \|T_{r_t} y_t\|) + \|y_t - y_{t_0}\| + \|T_{r_t} y_{t_0} - T_{r_{t_0}} y_{t_0}\|. \tag{3.2}$$

And, from definition of y_t, we derive

$$\begin{aligned}
&\|y_t - y_{t_0}\| \\
&\le \|(t-t_0)\gamma V x_t + t_0(\gamma V x_t - \gamma V x_{t_0})\| \\
&\quad + \|(I - t\mu G)J^B_{\lambda_t} A_{\nu_t} x_t - (I - t_0\mu G)J^B_{\lambda_t} A_{\nu_t} x_t\| \\
&\quad + \|(I - t_0\mu G)J^B_{\lambda_t} A_{\nu_t} x_t - (I - t_0\mu G)J^B_{\lambda_{t_0}} A_{\nu_{t_0}} x_{t_0}\| \\
&\le |t-t_0|\|\gamma V x_t\| + t_0\gamma l\|x_t - x_{t_0}\| + |t-t_0|\|\mu G J^B_{\lambda_t} A_{\nu_t} x_t\| \\
&\quad + (1-t_0\tau)(\|J^B_{\lambda_t} A_{\nu_t} x_t - J^B_{\lambda_t} A_{\nu_t} x_{t_0}\| + \|J^B_{\lambda_t} A_{\nu_t} x_{t_0} - J^B_{\lambda_{t_0}} A_{\nu_{t_0}} x_{t_0}\|) \\
&\le |t-t_0|\|\gamma V x_t\| + t_0\gamma l\|x_t - x_{t_0}\| + |t-t_0|\|\mu G J^B_{\lambda_t} A_{\nu_t} x_t\| \\
&\quad + (1-t_0\tau)(\|x_t - x_{t_0}\| + \|J^B_{\lambda_t} A_{\nu_t} x_{t_0} - J^B_{\lambda_{t_0}} A_{\nu_{t_0}} x_{t_0}\|) \\
&= |t-t_0|(\|\gamma V x_t\| + \|\mu G J^B_{\lambda_t} A_{\nu_t} x_t\|) + (1-(\tau-\gamma l)t_0)\|x_t - x_{t_0}\| \\
&\quad + (1-t_0\tau)\|J^B_{r_t} A_{r_t} x_{t_0} - J^B_{\lambda_{t_0}} A_{\nu_{t_0}} x_{t_0}\|.
\end{aligned} \tag{3.3}$$

Combining (3.2) and (3,3), we obtain

$$\begin{aligned}
&\|x_t - x_{t_0}\| \\
&\le \frac{\|x_t\| + \|T_{r_t} y_t\|}{(1-\theta_{t_0})(\tau-\gamma l)t_0}|\theta_t - \theta_{t_0}| + \frac{\|\gamma V x_t\| + \|\mu G J^B_{\lambda_t} A_{\nu_t} x_t\|}{(\tau-\gamma l)t_0}|t-t_0| \\
&\quad + \frac{1}{(\tau-\gamma l)t_0}\|T_{r_t} y_{t_0} - T_{r_{t_0}} y_{t_0}\| + \frac{1-t_0\tau}{(\tau-\gamma l)t_0}\|J^B_{\lambda_t} A_{\nu_t} x_{t_0} - J^B_{\lambda_{t_0}} A_{\nu_{t_0}} x_{t_0}\|.
\end{aligned} \tag{3.4}$$

Now, let $A_{\nu_t} x_{t_0} = w_t$ and $A_{\nu_{t_0}} x_{t_0} = w_{t_0}$. Then, from Lemma 2.4, we get

$$\langle y - w_t, A w_t\rangle + \frac{1}{\nu_t}\langle y - w_t, w_t - x_{t_0}\rangle \ge 0,\ \forall y \in C \tag{3.5}$$

and

$$\langle y - w_{t_0}, A w_{t_0}\rangle + \frac{1}{\nu_{t_0}}\langle y - w_{t_0}, w_{t_0} - x_{t_0}\rangle \ge 0, \quad \forall y \in C. \tag{3.6}$$

Putting $y := w_{t_0}$ in (3.5) and $y := w_t$ in (3.6), we obtain

$$\langle w_{t_0} - w_t, A w_t\rangle + \frac{1}{\nu_t}\langle w_{t_0} - w_t, w_t - x_{t_0}\rangle \ge 0 \tag{3.7}$$

and

$$\langle w_t - w_{t_0}, A w_{t_0}\rangle + \frac{1}{\nu_{t_0}}\langle w_t - w_{t_0}, w_{t_0} - x_{t_0}\rangle \ge 0. \tag{3.8}$$

Adding up (3.7) and (3.8), we deduce

$$-\langle w_t - w_{t_0}, Aw_t - Aw_{t_0}\rangle + \langle w_{t_0} - w_t, \frac{w_t - x_{t_0}}{\nu_t} - \frac{w_{t_0} - x_{t_0}}{\nu_{t_0}}\rangle \geq 0.$$

Since A is monotone, we get

$$\langle w_{t_0} - w_t, \frac{w_t - x_{t_0}}{\nu_t} - \frac{w_{t_0} - x_{t_0}}{\nu_{t_0}}\rangle \geq 0. \tag{3.9}$$

From (3.9), we derive

$$\begin{aligned}\|w_t - w_{t_0}\|^2 &\leq \langle w_t - w_{t_0}, (1 - \frac{\nu_t}{\nu_{t_0}})(w_{t_0} - x_{t_0})\rangle \\ &\leq \|w_t - w_{t_0}\||\nu_t - \nu_{t_0}|\frac{\|w_{t_0} - x_{t_0}\|}{a},\end{aligned}$$

and hence

$$\|w_t - w_{t_0}\| \leq |\nu_t - \nu_{t_0}|\frac{M_1}{a}, \tag{3.10}$$

where $M_1 > 0$ is an appropriate constant. Moreover, from the resolvent identity (2.2) and (3.10), we deduce

$$\begin{aligned}\|J^B_{\lambda_t}A_{\nu_t}x_{t_0} - J^B_{\lambda_{t_0}}A_{\nu_{t_0}}x_{t_0}\| &= \|J^B_{\lambda_t}w_t - J^B_{\lambda_{t_0}}w_{t_0}\| \\ &= \left\|J^B_{\lambda_{t_0}}\left(\frac{\lambda_{t_0}}{\lambda_t}w_t + \left(1 - \frac{\lambda_{t_0}}{\lambda_t}\right)J^B_{\lambda_t}w_t\right) - J^B_{\lambda_{t_0}}w_{t_0}\right\| \\ &\leq \left\|\frac{\lambda_{t_0}}{\lambda_t}(w_t - w_{t_0}) + \left(1 - \frac{\lambda_{t_0}}{\lambda_t}\right)(J^B_{\lambda_t}w_t - w_{t_0})\right\| \\ &\leq \|w_t - w_{t_0}\| + |\lambda_t - \lambda_{t_0}|\frac{\|J^B_{\lambda_t}w_t - w_t\|}{a} \\ &\leq |\nu_t - \nu_{t_0}|\frac{M_1}{a} + |\lambda_t - \lambda_{t_0}|\frac{M_2}{a},\end{aligned} \tag{3.11}$$

where $M_2 > 0$ is an appropriate constant.

Again, let $T_{r_t}y_{t_0} = z_t$ and $T_{r_{t_0}}y_{t_0} = z_{t_0}$. Then, by Lemma 2.5, we have

$$\langle y - z_t, Tz_t\rangle - \frac{1}{r_t}\langle y - z_t, (1 + r_t)z_t - y_{t_0}\rangle \leq 0, \quad \forall y \in C \tag{3.12}$$

and

$$\langle y - z_{t_0}, Tz_{t_0}\rangle - \frac{1}{r_{t_0}}\langle y - z_{t_0}, (1 + r_{t_0})z_{t_0} - y_{t_0}\rangle \leq 0, \quad \forall y \in C. \tag{3.13}$$

Putting $y := z_{t_0}$ in (3.12) and $y := z_t$ in (3.13), we get

$$\langle z_{t_0} - z_t, Tz_t\rangle - \frac{1}{r_t}\langle z_{t_0} - z_t, (1 + r_t)z_t - y_{t_0}\rangle \leq 0 \tag{3.14}$$

and

$$\langle z_t - z_{t_0}, Tz_{t_0}\rangle - \frac{1}{r_{t_0}}\langle z_t - z_{t_0}, (1 + r_{t_0})z_{t_0} - y_{t_0}\rangle \leq 0. \tag{3.15}$$

Adding up (3.14) and (3.15), we obtain

$$\begin{aligned} &\langle z_{t_0} - z_t, Tz_t - Tz_{t_0}\rangle \\ &- \langle z_{t_0} - z_t, \frac{(1+r_t)z_t - y_{t_0}}{r_t} - \frac{(1+r_{t_0})z_{t_0} - y_{t_0}}{r_{t_0}}\rangle \le 0. \end{aligned} \tag{3.16}$$

Since T is pseudocontractive, by (3.16), we have

$$\langle z_{t_0} - z_t, \frac{z_t - y_{t_0}}{r_t} - \frac{z_{t_0} - y_{t_0}}{r_{t_0}}\rangle \ge 0,$$

and hence

$$\langle z_{t_0} - z_t, z_t - z_{t_0} + z_{t_0} - y_{t_0} - \frac{r_t}{r_{t_0}}(z_{t_0} - y_{t_0})\rangle \ge 0. \tag{3.17}$$

From (3.17), we derive

$$\begin{aligned} \|z_t - z_{t_0}\|^2 &\le \langle z_{t_0} - z_t, (1 - \frac{r_t}{r_{t_0}})(z_{t_0} - y_{t_0})\rangle \\ &\le \|z_{t_0} - z_t\| |r_t - r_{t_0}| \frac{\|z_{t_0} - y_{t_0}\|}{a}, \end{aligned}$$

and hence

$$\|T_{r_t} y_{t_0} - T_{r_{t_0}} y_{t_0}\| = \|z_t - z_{t_0}\| \le |r_t - r_{t_0}| \frac{M_3}{a}, \tag{3.18}$$

where $M_3 > 0$ is an appropriate constant. From (3.4), (3.11) and (3.18), we have

$$\begin{aligned} &\|x_t - x_{t_0}\| \\ \le\ & \frac{\|x_t\| + \|T_{r_t} y_t\|}{(1-\theta_{t_0})(\tau - \gamma l)t_0} |\theta_t - \theta_{t_0}| + \frac{\|\gamma V x_t\| + \|\mu G J^B_{\lambda_t} A_{\nu_t} x_t\|}{(\tau - \gamma l)t_0} |t - t_0| \\ &+ \frac{1}{(\tau - \gamma l)t_0} \|T_{r_t} y_{t_0} - T_{r_{t_0}} y_{t_0}\| + \frac{1 - t_0\tau}{(\tau - \gamma l)t_0} \|J^B_{\lambda_t} A_{\nu_t} x_{t_0} - J^B_{\lambda_{t_0}} A_{\nu_{t_0}} x_{t_0}\| \\ \le\ & \frac{\|x_t\| + \|T_{r_t} y_t\|}{(1-\theta_{t_0})(\tau - \gamma l)t_0} |\theta_t - \theta_{t_0}| + \frac{\|\gamma V x_t\| + \|\mu G J^B_{\lambda_t} A_{\nu_t} x_t\|}{(\tau - \gamma l)t_0} |t - t_0| \\ &+ \frac{1}{(\tau - \gamma l)t_0} |r_t - r_{t_0}| \frac{M_3}{a} + \frac{1 - t_0}{(\tau - \gamma l)t_0} (|\nu_t - \nu_{t_0}| \frac{M_1}{a} + |\lambda_t - \lambda_{t_0}| \frac{M_2}{a}). \end{aligned}$$

Since $\theta_t : (0, 1) \to (0, 1)$ and $r_t, \nu_t, \lambda_t : (0, 1) \to (0, \infty)$ are continuous, we deduce that x_t is continuous. Also, it follows from (3.3) and (3.11) that y_t is continuous.

(3) Let $p \in \Omega$. Then, it follows from the resolvent identity (2.2) that

$$J^B_{\lambda_t} A_{\nu_t} x_t = J^B_{\lambda_t} w_t = J^B_{\frac{\lambda_t}{2}} (\frac{1}{2} w_t + \frac{1}{2} J^B_{\lambda_t} w_t).$$

Then we get

$$\begin{aligned}\|J^B_{\lambda_t}w_t - p\| &= \|J^B_{\frac{\lambda_t}{2}}(\frac{1}{2}w_t + \frac{1}{2}J^B_{\lambda_t}w_t) - p\| \\ &\le \|(\frac{1}{2}w_t + \frac{1}{2}J^B_{\lambda_t}w_t) - A_{\nu_t}p\| \\ &= \|\frac{1}{2}(w_t - A_{\nu_t}p) + \frac{1}{2}(J^B_{\lambda_t}w_t - A_{\nu_t}p)\|.\end{aligned}$$

Thus, by (2.4), we obtain

$$\begin{aligned}\|J^B_{\lambda_t}w_t - p\|^2 &\le \|\frac{1}{2}(w_t - A_{\nu_t}p) + \frac{1}{2}(J^B_{\lambda_t}w_t - A_{\nu_t}p)\|^2 \\ &= \frac{1}{2}\|w_t - A_{\nu_t}p\|^2 + \frac{1}{2}\|J^B_{\lambda_t}w_t - A_{\nu_t}p\|^2 - \frac{1}{4}\|w_t - J^B_{\lambda_t}w_t\|^2 \\ &= \frac{1}{2}\|A_{\nu_t}x_t - A_{r_t}p\|^2 + \frac{1}{2}\|J^B_{\lambda_t}A_{\nu_t}x_t - J^B_{\lambda_t}A_{\nu_t}p\|^2 \\ &\quad - \frac{1}{4}\|w_t - J^B_{\lambda_t}w_t\|^2 \\ &\le \frac{1}{2}\|x_t - p\|^2 + \frac{1}{2}\|x_t - p\|^2 - \frac{1}{4}\|w_t - J^B_{\lambda_t}w_t\|^2 \\ &= \|x_t - p\|^2 - \frac{1}{4}\|w_t - J^B_{\lambda_t}w_t\|^2.\end{aligned} \tag{3.19}$$

Therefore, from (3.1), definition of y_t and (3.19), we derive

$$\begin{aligned}\|x_t - p\|^2 &\le \|y_t - p\|^2 \\ &\le (t\|\gamma Vx_t - \mu Gp\| + \|(I - t\mu G)J^B_{\lambda_t}A_{\nu_t}x_t - (I - t\mu G)p\|)^2 \\ &\le (t\|\gamma Vx_t - \mu Gp\| + (1 - t\tau)\|J^B_{\lambda_t}w_t - p\|)^2 \\ &= t^2\|\gamma Vx_t - \mu Gp\|^2 + 2t(1 - t\tau)\|\gamma Vx_t - \mu Gp\|\|J^B_{\lambda_t}w_t - p\| \\ &\quad + (1 - t\tau)^2\|J^B_{\lambda_t}w_t - p\|^2 \\ &\le tM_4 + (1 - t\tau)\|J^B_{\lambda_t}w_t - p\|^2 \\ &\le tM_4 + (1 - t\tau)[\|x_t - p\|^2 - \frac{1}{4}\|w_t - J^B_{\lambda_t}w_t\|^2],\end{aligned} \tag{3.20}$$

where $M_4 > 0$ is an appropriate constant. This implies that

$$\|w_t - J^B_{\lambda_t}w_t\|^2 \le \frac{4t}{1 - t\tau}(M_4 - \tau\|x_t - p\|^2).$$

By boundedness of $\{x_t\}$, letting $t \to 0$ in above inequality yields

$$\lim_{t\to 0}\|w_t - J^B_{\lambda_t}w_t\| = 0.$$

(4) Let $p \in \Omega$. Then, since $p = A_{\nu_t} p$ and A_{ν_t} is firmly nonexpansive, we deduce from (2.3)

$$\begin{aligned}\|w_t - p\|^2 &= \|A_{\nu_t} x_t - A_{\nu_t} p\|^2 \\ &\leq \langle w_t - p, x_t - p\rangle \\ &= \frac{1}{2}(\|x_t - p\|^2 + \|w_t - p\|^2 - \|x_t - w_t\|^2),\end{aligned}$$

and hence

$$\|w_t - p\|^2 \leq \|x_t - p\|^2 - \|x_t - w_t\|^2.$$

Thus, we have

$$\|J^B_{\lambda_t} w_t - p\|^2 \leq \|w_t - p\|^2 \leq \|x_t - p\|^2 - \|x_t - w_t\|^2. \tag{3.21}$$

From (3.20) and (3.21), we derive

$$\begin{aligned}\|x_t - p\|^2 &\leq \|y_t - p\|^2 \\ &\leq tM_4 + (1 - t\tau)\|J^B_{\lambda_t} w_t - p\|^2 \\ &\leq tM_4 + \|J^B_{\lambda_t} w_t - p\|^2 \\ &\leq tM_4 + (\|x_t - p\|^2 - \|x_t - w_t\|^2),\end{aligned}$$

and hence

$$\|x_t - w_t\|^2 \leq tM_4.$$

This implies that

$$\lim_{t\to 0} \|x_t - w_t\| = 0.$$

(5) Since

$$\begin{aligned}\|x_t - y_t\| &= \|x_t - (t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t} w_t)\| \\ &= \|t(\mu G x_t - \gamma V x_t) + (I - t\mu G)x_t - (I - t\mu G)J^B_{\lambda_t} w_t\| \\ &\leq t\|\mu G x_t - \gamma V x_t\| + (1 - t\tau)\|x_t - J^B_{\lambda_t} w_t\| \\ &\leq t\|\mu G x_t - \gamma V x_t\| + \|x_t - J^B_{\lambda_t} w_t\| \\ &\leq t\|\mu G x_t - \gamma V x_t\| + \|x_t - w_t\| + \|w_t - J^B_{\lambda_t} w_t\|\end{aligned}$$

by boundedness of $\{Gx_t\}$ and $\{Vx_t\}$, (3) and (4), we obtain

$$\lim_{t\to 0} \|x_t - y_t\| = 0.$$

(6) Since

$$\|x_t - J^B_{\lambda_t} A_{\nu_t} x_t\| = \|x_t - J^B_{\lambda_t} w_t\| \leq \|x_t - w_t\| + \|w_t - J^B_{\lambda_t} w_t\|$$

by (3) and (4), we have

$$\lim_{t\to 0} \|x_t - J^B_{\lambda_t} A_{\nu_t} x_t\| = \lim_{t\to 0} \|x_t - J^B_{\lambda_t} w_t\| = 0.$$

(7) In fact, since

$$\begin{aligned}\|x_t - T_{r_t}x_t\| &= \|\theta_t x_t + (1-\theta_t)T_{r_t}y_t - T_{r_t}x_t\| \\ &= \|\theta_t(x_t - T_{r_t}x_t) + (1-\theta_t)(T_{r_t}y_t - T_{r_t}x_t)\| \\ &\le \theta_t\|x_t - T_{r_t}x_t\| + (1-\theta_t)\|T_{r_t}y_t - T_{r_t}x_t\|\end{aligned}$$

by (5), we obtain

$$\|x_t - T_{r_t}x_t\| \le \|T_{r_t}y_t - T_{r_t}x_t\| \le \|y_t - x_t\| \to 0 \quad (t \to 0).$$

(8) From (5) and (7), it follows that

$$\begin{aligned}\|y_t - T_{r_t}y_t\| &\le \|y_t - x_t\| + \|x_t - T_{r_t}x_t\| + \|T_{r_t}x_t - T_{r_t}y_t\| \\ &\le \|y_t - x_t\| + \|x_t - T_{r_t}x_t\| + \|x_t - y_t\| \\ &= 2\|y_t - x_t\| + \|x_t - T_{r_t}x_t\| \to 0 \quad (t \to 0).\end{aligned}$$ □

By using Proposition 3.1, we establish strong convergence of the path x_t, which guarantees the existence of solutions of the variational inequality (3.22) below.

Theorem 3.2. *Let the net $\{x_t\}$ be defined by* (3.1). *Let $\theta_t : (0,1) \to (0,1)$ be continuous and let $r_t, \lambda_t, \nu_t : (0,1) \to (0,\infty)$ be continuous and $0 < a \le \min\{r_t, \lambda_t, \nu_t\}$ for $t \in (0,1)$. Then $\{x_t\}$ converges strongly, as $t \to 0$, to a point $q \in \Omega$, which is the unique solution of the variational inequality*:

$$\langle (\mu G - \gamma V)q, p - q\rangle \ge 0, \quad \forall p \in \Omega. \tag{3.22}$$

Proof. We first note that the uniqueness of a solution of the variational inequality (3.22) is a consequence of the strong monotonicity of $\mu G - \gamma V$ (see Lemma 2.7). In fact, suppose that both $q_1 \in \Omega$ and $q_2 \in \Omega$ are solutions to (3.22). Then we have

$$\langle (\mu G - \gamma V)q_1, q_1 - q_2\rangle \le 0 \tag{3.23}$$

and

$$\langle (\mu G - \gamma V)q_2, q_2 - q_1\rangle \le 0. \tag{3.24}$$

Adding up (3.23) and (3.24) yields

$$\langle (\mu G - \gamma V)q_1 - (\mu G - \gamma V)q_2, q_1 - q_2\rangle \le 0.$$

The strong monotonicity of $\mu G - \gamma V$ implies that $q_1 = q_2$ and the uniqueness is proved.

Let the net $\{y_t\}$ be defined by $y_t = t\gamma V x_t + (I - t\mu G)J^B_{\lambda_t}A_{\nu_t}x_t$ for $t \in (0,1)$. Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$, $y_n := y_{t_n}$, $r_n := r_{t_n}$, $\lambda_n := \lambda_{t_n}$ and $\nu_n := \nu_{t_n}$. Since $\{x_n\}$ is bounded by (1) in Proposition 3.1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$, which converges weakly to q. First of all, we show $q \in \Omega$. To this end, we divide its proof into three steps.

Step 1. We prove that $q \in Fix(T)$. To show this, put $z_n = T_{r_n} y_n$. Then, by Lemma 2.5, we have

$$\langle y - z_n, Tz_n\rangle - \frac{1}{r_n}\langle y - z_n, (1 + r_n)z_n - y_n\rangle \leq 0, \quad \forall y \in C. \tag{3.25}$$

Put $l_\epsilon = \epsilon v + (1 - \epsilon)q$ for $\epsilon \in (0, 1]$ and $v \in C$. Then $l_\epsilon \in C$, and from (3.25) and pseudocontractivity of T, it follows that

$$\begin{aligned} \langle z_n - l_\epsilon, Tl_\epsilon\rangle &\geq \langle z_n - l_\epsilon, Tl_\epsilon\rangle + \langle l_\epsilon - z_n, Tz_n\rangle \\ &\quad - \frac{1}{r_n}\langle l_\epsilon - z_n, (1 + r_n)z_n - y_n\rangle \\ &= -\langle l_\epsilon - z_n, Tl_\epsilon - Tz_n\rangle - \frac{1}{r_n}\langle l_\epsilon - z_n, z_n - y_n\rangle \\ &\quad - \langle l_\epsilon - z_n, z_n\rangle \\ &\geq -\|l_\epsilon - z_n\|^2 - \frac{1}{r_n}\langle l_\epsilon - z_n, z_n - y_n\rangle - \langle l_\epsilon - z_n, z_n\rangle \\ &= -\langle l_\epsilon - z_n, l_\epsilon\rangle - \langle l_\epsilon - z_n, \frac{z_n - y_n}{r_n}\rangle. \end{aligned} \tag{3.26}$$

Since

$$\begin{aligned} \|x_n - z_n\| &\leq \|x_n - T_{r_n} x_n\| + \|T_{r_n} x_n - T_{r_n} y_n\| \\ &\leq \|x_n - T_{r_n} x_n\| + \|x_n - y_n\| \to 0 \ \ (n \to \infty) \end{aligned}$$

by (5) and (7) in Proposition 3.1, it follows that $z_{n_i} \rightharpoonup q$ as $i \to \infty$. Also, by (8) in Proposition 3.1, we have $\frac{\|z_n - y_n\|}{r_n} \leq \frac{\|z_n - y_n\|}{a} \to 0$. So, replacing n by n_i and letting $i \to \infty$, we derive from (3.26)

$$\langle q - l_\epsilon, Tl_\epsilon\rangle \geq \langle q - l_\epsilon, l_\epsilon\rangle$$

and

$$-\langle v - q, Tl_\epsilon\rangle \geq -\langle v - q, l_\epsilon\rangle, \quad \forall v \in C.$$

Letting $\epsilon \to 0$ and using the fact that T is continuous, we obtain

$$-\langle v - q, Tq\rangle \geq -\langle v - q, q\rangle, \quad \forall v \in C. \tag{3.27}$$

Let $v = Tq$ in (3.27). Then we have $q = Tq$, that is, $q \in Fix(T)$.

Step 2. We prove that $q \in VI(C, A)$. To this end, let $w_n = A_{\nu_n} x_n$. Then, by Lemma 2.1, we induce

$$\langle y - w_n, Aw_n\rangle + \langle y - w_n, \frac{w_n - x_n}{\nu_n}\rangle \geq 0, \quad \forall y \in C. \tag{3.28}$$

Set $l_\epsilon = \epsilon v + (1 - \epsilon)q$ for $\epsilon \in (0, 1]$ and $v \in C$. Then $l_\epsilon \in C$, and it follows from (3.28) that

$$\begin{aligned} \langle l_\epsilon - w_n, Al_\epsilon\rangle &\geq \langle l_\epsilon - w_n, Al_\epsilon\rangle - \langle l_\epsilon - w_n, Aw_n\rangle \\ &\quad - \langle l_\epsilon - w_n, \frac{w_n - x_n}{\nu_n}\rangle. \end{aligned} \tag{3.29}$$

By (4) in Proposition 3.1, we have $\frac{\|w_n - x_n\|}{\nu_n} \le \frac{\|w_n - x_n\|}{a} \to 0$ as $n \to \infty$, and so $w_{n_i} \rightharpoonup q$ as $i \to \infty$. From monotonicity of A, it follows that

$$\langle l_\epsilon - w_n, Al_\epsilon - Aw_n \rangle \ge 0.$$

Thus, replacing n by n_i and letting $i \to \infty$, from (3.29), we obtain

$$0 \le \langle l_\epsilon - q, Al_\epsilon \rangle,$$

and hence

$$\langle v - q, Al_\epsilon \rangle \ge 0, \quad \forall v \in C.$$

If $\epsilon \to 0$, then the continuity of A yields that

$$\langle v - q, Aq \rangle \ge 0, \quad \forall v \in C.$$

This means that $q \in VI(C, A)$.

Step 3. We prove that $q \in B^{-1}0$. To this end, let $u_n = J^B_{\lambda_n} w_n$. Then it follows that

$$w_n \in (I + \lambda_n B)u_n, \ \text{ that is, } \ \frac{w_n - u_n}{\lambda_n} \in Bu_n.$$

Since B is monotone, we know that for any $v \in Bu$,

$$\langle u_n - u, \frac{w_n - u_n}{\lambda_n} - v \rangle \ge 0. \tag{3.30}$$

Since $\frac{\|w_n - u_n\|}{\lambda_n} \le \frac{\|w_n - J^B_{\lambda_n} w_n\|}{a} \to 0$ as $n \to \infty$ by (3) in Proposition 3.1 and $\|x_n - w_n\| \to 0$ as $n \to \infty$ by (4) in Proposition 3.1, we have $u_{n_i} \rightharpoonup q$ as $i \to \infty$. By replacing n by n_i in (3.30) and letting $i \to \infty$, we have

$$\langle q - u, -v \rangle \ge 0.$$

Since B is maximal monotone, we get $0 \in Bq$, that is, $q \in B^{-1}0$. This along with Steps 1 and 2 obtains $q \in \Omega$.

Next, we show that q is a solution of the variational inequality (3.22). In fact, from (3.1), we write for $p \in \Omega$,

$$x_t - p = \theta_t(x_t - p) + (1 - \theta_t)(T_{r_t} y_t - p) \Longleftrightarrow x_t - p = T_{r_t} y_t - p,$$

and $\|x_t - p\| = \|T_{r_t} y_t - p\| \le \|y_t - p\|$. Observing

$$y_t - p = t(\gamma V x_t - \gamma V p) + t(\gamma V p - \mu G p) + (I - t\mu G) J^B_{\lambda_t} A_{\nu_t} x_t - (I - t\mu G)p,$$

we deduce

$$\begin{aligned}
\|y_t - p\|^2 &= \langle y_t - p, y_t - p \rangle \\
&= \langle t(\gamma V x_t - \gamma V p), y_t - p \rangle + t\langle \gamma V p - \mu G p, y_t - p \rangle \\
&\quad + \langle (I - t\mu G) J^B_{\lambda_t} A_{\nu_t} x_t - (I - t\mu G)p, y_t - p \rangle \\
&\le t\gamma l \|x_t - p\| \|y_t - p\| + t\langle \gamma V p - \mu G p, y_t - p \rangle \\
&\quad + (1 - t\tau)\|x_t - p\| \|y_t - p\| \\
&\le t\gamma l \|y_t - p\|^2 + (1 - t\tau)\|y_t - p\|^2 + t\langle \gamma V p - \mu G p, y_t - p \rangle \\
&= (1 - (\tau - \gamma l)t)\|y_t - p\|^2 + t\langle \gamma V p - \mu G p, y_t - p \rangle,
\end{aligned}$$

and hence

$$\|y_t - p\|^2 \leq \frac{1}{\tau - \gamma l}\langle \gamma Vp - \mu Gp, y_t - p\rangle.$$

In particular,

$$\|y_{n_i} - p\|^2 \leq \frac{1}{\tau - \gamma l}\langle \gamma Vp - \mu Gp, y_{n_i} - p\rangle. \tag{3.31}$$

Since $q \in \Omega$, by (3.31), we obtain

$$\|y_{n_i} - q\|^2 \leq \frac{1}{\tau - \gamma l}\langle \gamma Vq - \mu Gq, y_{n_i} - q\rangle. \tag{3.32}$$

Since $x_{n_i} \rightharpoonup q$ and $\|x_n - y_n\| \to 0$ $(n \to \infty)$ by (5) in Proposition 3.1, it follows that $y_{n_i} \rightharpoonup q$ as $i \to \infty$. Thus, from (3.32), we derive $y_{n_i} \to q$ as $i \to \infty$. Moreover, by taking the limit as $i \to \infty$ in (3.31), we get

$$\|q - p\|^2 \leq \frac{1}{\tau - \gamma l}\langle (\mu G - \gamma V)p, p - q\rangle.$$

In particular, q solves the following variational inequality

$$q \in \Omega, \quad \langle (\mu G - \gamma V)p, p - q\rangle \geq 0, \quad p \in \Omega,$$

or the equivalent dual variational inequality (see Lemma 2.6)

$$q \in \Omega, \quad \langle (\mu G - \gamma V)q, p - q\rangle \geq 0, \quad p \in \Omega.$$

Finally, we show that the net $\{x_t\}$ converges strongly, as $t \to 0$, to q. For this purpose, let $\{s_k\} \subset (0,1)$ be another sequence such that $s_k \to 0$ as $k \to \infty$. Put $x_k := x_{s_k}$ and $y_k := y_{s_k}$. Let $\{x_{k_j}\}$ be a subsequence of $\{x_k\}$ and assume that $x_{k_j} \rightharpoonup \overline{q}$. Then, by the same proof as the one above, we have $\overline{q} \in \Omega$. Moreover, from strong monotonicity of $\mu G - \gamma V$, it follows that $q = \overline{q}$. Therefore, we conclude that $x_t \to q \in \Omega$ as $t \to 0$, which is the unique solution to the variational inequality (3.22). This completes the proof. □

By taking $V \equiv 0$, $G \equiv I$, $\mu = 1$ in Theorem 3.2, we obtain the following result.

Corollary 3.3. *Let the net $\{x_t\}$ be defined by*

$$x_t = \theta_t x_t + (1 - \theta_t)T_{r_t}((1 - t)J^B_{\lambda_t}A_{\nu_t}x_t), \ t \in (0,1).$$

Let $\theta_t : (0,1) \to (0,1)$ be continuous and let $r_t, \lambda_t, \nu_t : (0,1) \to (0,\infty)$ be continuous and $0 < a \leq \min\{r_t, \lambda_t, \nu_t\}$ for $t \in (0,1)$. Then $\{x_t\}$ converges strongly, as $t \to 0$, to q, which solves the following minimum-norm problem: find $q \in \Omega$ such that

$$\|q\| = \min_{x \in \Omega}\|x\|.$$

Proof. From (3.22) with $V \equiv 0$, $G \equiv I$ and $\mu = 1$, we derive

$$0 \leq \langle q, p - q\rangle, \quad \forall p \in \Omega.$$

This obviously implies that

$$\|q\|^2 \le \langle p, q\rangle \le \|p\|\|q\|, \quad \forall p \in \Omega.$$

It turns out that $\|q\| \le \|p\|$ for all $p \in \Omega$. Therefore, q is the minimum-norm point of Ω. □

Now, we propose a new iterative algorithm which generates a sequence $\{x_n\}$ in an explicit way: for an arbitrarily chosen $x_0 \in C$,

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T_{r_n}(\alpha_n\gamma V x_n + (I-\alpha_n\mu G)J^B_{\lambda_n}A_{\nu_n}x_n), \ n \ge 0, \tag{3.33}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$, and $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$, and establish strong convergence of this sequence to an element of Ω.

Theorem 3.4. *Let the sequence $\{x_n\}$ be generated iteratively by the explicit algorithm* (3.33). *Let $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ and $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ satisfy the following conditions*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(C4) $0 < a \le r_n < \infty$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(C5) $0 < a \le \lambda_n < \infty$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$;
(C6) $0 < a \le \nu_n < \infty$ *and* $\lim_{n\to\infty} |\nu_{n+1} - \mu_n| = 0$.

Then $\{x_n\}$ converges strongly to a point $q \in \Omega$, which is the unique solution of the variational inequality (3.22).

Proof. Let $q \in \Omega$ be the unique solution of the variational inequality (3.22). (The existence of q follows from Theorem 3.2.)

From now, we put $y_n = \alpha_n\gamma V x_n + (I-\alpha_n\mu G)J^B_{\lambda_n}A_{\nu_n}x_n$ and $w_n = A_{\nu_n}x_n$ for $n \ge 0$.

We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $p \in \Omega$. It is obvious that $p = J^B_{\lambda_n}A_{\nu_n}p$, $p = A_{\nu_n}p$, $p = J^B_{\lambda_n}p$ and $T_{r_n}p = p$. And we obtain

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n\gamma V x_n + (I-\alpha_n\mu G)J^B_{r_n}w_n - p\| \\ &\le \|\alpha_n(\gamma V x_n - \gamma V p) + \alpha_n(\gamma V p - \mu G p)\| \\ &\quad + \|(I-\alpha_n\mu G)J^B_{\lambda_n}w_n - (I-\alpha_n\mu G)p\| \\ &\le \alpha_n\|\gamma V x_n - \gamma V p\| + \alpha_n\|\gamma V p - \mu G p\| + (1-\alpha_n\tau)\|w_n - p\| \\ &\le \alpha_n\gamma l\|x_n - p\| + \alpha_n\|\gamma V p - \mu G p\| + (1-\alpha_n\tau)\|x_n - p\| \\ &= (1-(\tau-\gamma l)\alpha_n)\|x_n - p\| + \alpha_n\|\gamma V p - \mu G p\|. \end{aligned} \tag{3.34}$$

Thus, since T_{r_n} is nonexpansive (by Lemma 2.5), from (3.33) and (3.34), we deduce

$$\begin{aligned} \|x_{n+1} - p\| &\le \beta_n\|x_n - p\| + (1-\beta_n)\|T_{r_n}y_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \end{aligned}$$

$$\begin{aligned}
&\le \beta_n\|x_n - p\| \\
&\quad + (1-\beta_n)[(1-(\tau-\gamma l)\alpha_n)\|x_n - p\| + \alpha_n\|\gamma Vp - \mu Gp\|] \\
&= (1-(1-\beta_n)(\tau-\gamma l)\alpha_n)\|x_n - p\| \\
&\quad + (1-\beta_n)(\tau-\gamma l)\alpha_n \frac{\|\gamma Vp - \mu Gp\|}{\tau-\gamma l} \\
&\le \max\left\{\|x_n - p\|, \frac{\|\gamma Vp - \mu Gp\|}{\tau-\gamma l}\right\}.
\end{aligned}$$

Using an induction, we have

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|\gamma Vp - \mu Gp\|}{\tau-\gamma l}\right\}.$$

Hence, $\{x_n\}$ is bounded. Also, $\{y_n\}$, $\{w_n\} = \{A_{\nu_n}x_n\}$, $\{u_n\} = \{J^B_{\lambda_n}w_n\}$, $\{Gx_n\}$, $\{GJ^B_{\lambda_n}w_n\}$, $\{z_n\} = \{T_{r_n}y_n\}$ and $\{Vx_n\}$ are bounded.

Step 2. We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. For this purpose, first, we derive

$$\begin{aligned}
(3.35)\quad &\|y_n - y_{n-1}\| \\
&= \|\alpha_n\gamma Vx_n + (I-\alpha_n\mu G)J^B_{\lambda_n}A_{\nu_n}x_n \\
&\qquad - (\alpha_{n-1}\gamma Vx_{n-1} + (I-\alpha_{n-1}\mu G)J^B_{\lambda_{n-1}}A_{\nu_{n-1}}x_{n-1})\| \\
&\le \|(\alpha_n - \alpha_{n-1})\gamma Vx_{n-1} + \alpha_n(\gamma Vx_n - \gamma Vx_{n-1})\| \\
&\quad + \|(I-\alpha_n\mu G)J^B_{\lambda_n}w_n - (I-\alpha_n\mu G)J^B_{\lambda_{n-1}}w_{n-1}\| \\
&\quad + \|(I-\alpha_n\mu G)J^B_{\lambda_{n-1}}w_{n-1} - (I-\alpha_{n-1}\mu G)J^B_{\lambda_{n-1}}w_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|\|\gamma Vx_{n-1}\| + \alpha_n\gamma l\|x_n - x_{n-1}\| \\
&\quad + (1-\alpha_n\tau)\|J^B_{\lambda_n}w_n - J^B_{\lambda_{n-1}}w_{n-1}\| + |\alpha_n - \alpha_{n+1}|\|\mu GJ^B_{\lambda_{n-1}}w_{n-1}\| \\
&= |\alpha_n - \alpha_{n-1}|(\|\gamma Vx_{n-1}\| + \|\mu GJ^B_{\lambda_{n-1}}w_{n-1}\|) \\
&\quad + \alpha_n\gamma l\|x_n - x_{n-1}\| + (1-\alpha_n\tau)\|J^B_{\lambda_n}w_n - J^B_{\lambda_{n-1}}w_{n-1}\| \\
&\le |\alpha_n - \alpha_{n-1}|M_5 + \alpha_n\gamma l\|x_n - x_{n-1}\| + (1-\alpha_n\tau)\|J^B_{\lambda_n}w_n - J^B_{\lambda_{n-1}}w_{n-1}\|,
\end{aligned}$$

where $M_5 > 0$ is an appropriate constant. Let $w_n = A_{\nu_n}x_n$ and $w_{n-1} = A_{\nu_{n-1}}x_{n-1}$ again. Then we get

$$(3.36)\qquad \langle y - w_n, Aw_n\rangle + \frac{1}{\nu_n}\langle y - w_n, w_n - x_n\rangle \ge 0, \quad \forall y \in C$$

and

$$(3.37)\quad \langle y - w_{n-1}, Aw_{n-1}\rangle + \frac{1}{\nu_{n-1}}\langle y - w_{n-1}, w_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C.$$

Putting $y := w_{n-1}$ in (3.36) and $y := w_n$ in (3.37), we obtain

$$(3.38)\qquad \langle w_{n-1} - w_n, Aw_n\rangle + \frac{1}{\nu_n}\langle w_{n-1} - w_n, w_n - x_n\rangle \ge 0$$

and

$$\langle w_n - w_{n-1}, Aw_{n-1}\rangle + \frac{1}{\nu_{n-1}}\langle w_n - w_{n-1}, w_{n-1} - x_{n-1}\rangle \geq 0. \tag{3.39}$$

Adding up (3.38) and (3.39), we deduce

$$-\langle w_n - w_{n-1}, Aw_n - Aw_{n-1}\rangle + \langle w_{n-1} - w_n, \frac{w_n - x_n}{\nu_n} - \frac{w_{n-1} - x_{n-1}}{\nu_{n-1}}\rangle \geq 0.$$

Since A is monotone, we get

$$\langle w_{n-1} - w_n, \frac{w_n - x_n}{\nu_n} - \frac{w_{n-1} - x_{n-1}}{\nu_{n-1}}\rangle \geq 0,$$

and hence

$$\langle w_n - w_{n-1}, w_{n-1} - w_n + w_n - x_{n-1} - \frac{\nu_{n-1}}{\nu_n}(w_n - x_n)\rangle \geq 0. \tag{3.40}$$

From (3.40), we derive

$$\begin{aligned}\|w_n - w_{n-1}\|^2 &\leq \langle w_n - w_{n-1}, w_n - x_n + x_n - x_{n-1} - \frac{\nu_{n-1}}{\nu_n}(w_n - x_n)\rangle \\ &= \langle w_n - w_{n-1}, x_n - x_{n-1} + \left(1 - \frac{\nu_{n-1}}{\nu_n}\right)(w_n - x_n)\rangle \\ &\leq \|w_n - w_{n-1}\|\left[\|x_n - x_{n-1}\| + \frac{1}{a}|\nu_n - \nu_{n-1}|\|w_n - x_n\|\right].\end{aligned}$$

This implies that

$$\|w_n - w_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{a}|\nu_n - \nu_{n-1}|\|w_n - x_n\|. \tag{3.41}$$

Moreover, from the resolvent identity (2.2) and (3.41), we induce

$$\begin{aligned}&\|J^B_{\lambda_n} w_n - J^B_{\lambda_{n-1}} w_{n-1}\| \\ &= \|J^B_{\lambda_{n-1}}\left(\frac{\lambda_{n-1}}{\lambda_n} w_n + \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right) J^B_{\lambda_n} w_n\right) - J^B_{\lambda_{n-1}} w_{n-1}\| \\ &\leq \|\frac{\lambda_{n-1}}{\lambda_n}(w_n - w_{n-1}) + \left(1 - \frac{\lambda_{n-1}}{\lambda_n}\right)(J^B_{\lambda_n} w_n - w_{n-1})\| \\ &\leq \|w_n - w_{n-1}\| + \frac{|\lambda_n - \lambda_{n-1}|}{a}\|J^B_{\lambda_n} w_n - w_n\| \\ &\leq \|x_n - x_{n-1}\| + |\nu_n - \nu_{n-1}|\frac{\|w_n - x_n\|}{a} + |\lambda_n - \lambda_{n-1}|\frac{\|J^B_{\lambda_n} w_n - w_n\|}{a}.\end{aligned} \tag{3.42}$$

Substituting (3.42) into (3.35), we derive

$$\begin{aligned}&\|y_n - y_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}|M_5 + \alpha_n \gamma l\|x_n - x_{n-1}\| \\ &\quad + (1 - \alpha_n \tau)\|J^B_{\lambda_n} w_n - J^B_{\lambda_{n-1}} w_{n-1}\| \\ &\leq |\alpha_n - \alpha_{n-1}|M_5 + \alpha_n \gamma l\|x_n - x_{n-1}\| + (1 - \alpha_n \tau)\|x_n - x_{n-1}\|\end{aligned} \tag{3.43}$$

$$
\begin{aligned}
&+|\nu_n-\nu_{n-1}|\frac{\|w_n-x_n\|}{a}+|\lambda_n-\lambda_{n-1}|\frac{\|J^B_{\lambda_n}w_n-w_n\|}{a}\\
&\le (1-\alpha_n(\tau-\gamma l))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_5\\
&\quad+|\nu_n-\nu_{n-1}|M_6+|\lambda_n-\lambda_{n-1}|M_7,
\end{aligned}
$$

where M_6 and $M_7>0$ are appropriate constants.

On the other hand, let $z_n=T_{r_n}y_n$. Then, since $z_{r_{n-1}}=T_{r_{n-1}}y_{n-1}$, we have

$$\langle y-z_n,Tz_n\rangle-\frac{1}{r_n}\langle y-z_n,(1+r_n)z_n-y_n\rangle\le 0,\quad \forall y\in C \tag{3.44}$$

and

$$\langle y-z_{n-1},Tz_{n-1}\rangle-\frac{1}{r_{n-1}}\langle y-z_{n-1},(1+r_{n-1})z_{n-1}-y_{n-1}\rangle\le 0,\quad \forall y\in C. \tag{3.45}$$

Putting $y:=z_{n-1}$ in (3.44) and $y:=z_n$ in (3.45), we get

$$\langle z_{n-1}-z_n,Tz_n\rangle-\frac{1}{r_n}\langle z_{n-1}-z_n,(1+r_n)z_n-y_n\rangle\le 0 \tag{3.46}$$

and

$$\langle z_n-z_{n-1},Tz_{n-1}\rangle-\frac{1}{r_{n-1}}\langle z_n-z_{n-1},(1+r_{n-1})z_{n-1}-y_{n-1}\rangle\le 0. \tag{3.47}$$

Adding up (3.46) and (3.47), we obtain

$$
\begin{aligned}
&\langle z_{n-1}-z_n,Tz_n-Tz_{n-1}\rangle\\
&-\langle z_{n-1}-z_n,\frac{(1+r_n)z_n-y_n}{r_n}-\frac{(1+r_{n-1})z_{n-1}-y_{n-1}}{r_{n-1}}\rangle\le 0.
\end{aligned}
\tag{3.48}
$$

Using the fact that T is pseudocontractive, we have by (3.48)

$$\langle z_{n-1}-z_n,\frac{z_n-y_n}{r_n}-\frac{z_{n-1}-y_{n-1}}{r_{n-1}}\rangle\ge 0,$$

and hence

$$\langle z_{n-1}-z_n,z_n-z_{n-1}+z_{n-1}-y_n-\frac{r_n}{r_{n-1}}(z_{n-1}-y_{n-1})\rangle\ge 0. \tag{3.49}$$

From (3.49), we derive

$$
\begin{aligned}
\|z_n-z_{n-1}\|^2&\le\langle z_{n-1}-z_n,y_{n-1}-y_n+(1-\frac{r_n}{r_{n-1}})(z_{n-1}-y_{n-1})\rangle\\
&\le\|z_{n-1}-z_n\|\left(\|y_{n-1}-y_n\|+\frac{|r_n-r_{n-1}|}{a}\|z_{n-1}-y_{n-1}\|\right).
\end{aligned}
$$

Thus we obtain

$$\|z_n-z_{n-1}\|\le\|y_{n-1}-y_n\|+\frac{|r_n-r_{n-1}|}{a}\|z_{n-1}-y_{n-1}\|. \tag{3.50}$$

Substituting (3.43) into (3.50) yields

$$
\begin{aligned}
\|z_n-z_{n-1}\|&\le(1-\alpha_n(\tau-\gamma l))\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_5\\
&\quad+|\nu_n-\nu_{n-1}|M_6+|\lambda_n-\lambda_{n-1}|M_7+|r_n-r_{n-1}|M_8
\end{aligned}
\tag{3.51}
$$

$$\leq \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_5 + |\nu_n - \nu_{n-1}|M_6 + |\lambda_n - \lambda_{n-1}|M_7 + |r_n - r_{n-1}|M_8,$$

where $M_8 > 0$ is an appropriate constant. In view of conditions (C1), (C4), (C5) and (C6), we find from (3.51)

$$\limsup_{n\to\infty}(\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|) \leq 0.$$

Thus, by Lemma 2.2, we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.52}$$

Since $x_{n+1} - x_n = (1 - \beta_n)(z_n - x_n)$, by (3.52) and condition (3), we conclude

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Step 3. We show that $\lim_{n\to\infty} \|x_n - w_n\| = 0$, where $w_n = A_{\nu_n} x_n$. To show this, let $p \in \Omega$. Then, since $p = A_{\nu_n} p$, we deduce

$$\begin{aligned}\|w_n - p\|^2 &= \|A_{\nu_n} x_n - A_{\nu_n} p\|^2 \\ &\leq \langle w_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|x_n - p\|^2 + \|w_n - p\|^2 - \|x_n - w_n\|^2),\end{aligned}$$

and hence

$$\|w_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - w_n\|^2.$$

Thus we have

$$\begin{aligned}\|T_{r_n} y_n - p\|^2 &\leq \|y_n - p\|^2 \\ &= \|\alpha_n(\gamma V x_n - \mu G p) + (I - \alpha_n \mu G) J^B_{\lambda_n} w_n - (I - \alpha_n \mu G)p\|^2 \\ &\leq (\alpha_n \|\gamma V x_n - \mu G p\| + (1 - \alpha_n \tau)\|w_n - p\|)^2 \\ &\leq (\alpha_n \|\gamma V x_n - \mu G p\| + \|w_n - p\|)^2 \\ &\leq \alpha_n M_9 + \|w_n - p\|^2 \\ &\leq \alpha_n M_9 + \|x_n - p\|^2 - \|x_n - w_n\|^2,\end{aligned}$$

where M_9 is an appropriate constant. This implies

$$\begin{aligned}\|x_n - w_n\|^2 &\leq \alpha_n M_9 + \|x_n - p\|^2 - \|T_{r_n} y_n - p\|^2 \\ &= \alpha_n M_9 + (\|x_n - p\| + \|T_{r_n} y_n - p\|)(\|x_n - p\| - \|T_{r_n} y_n - p\|) \\ &\leq \alpha_n M_9 + (\|x_n - p\| + \|T_{r_n} y_n - p\|)\|x_n - T_{r_n} y_n\| \\ &= \alpha_n M_9 + \frac{\|x_{n+1} - x_n\|}{1 - \beta_n} M_{10},\end{aligned}$$

where M_{10} is an appropriate constant. Hence, by conditions (C1) and (C3), and Step 2, we obtain

$$\lim_{n\to\infty} \|x_n - w_n\| = 0.$$

Step 4. We show that $\lim_{n\to\infty} \|J^B_{\lambda_n} w_n - x_n\| = 0$. To this end, let $p \in \Omega$. First, we observe

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n \gamma V x_n + (I - \alpha_n \mu G) J^B_{\lambda_n} w_n - p\|^2 \\ &\le (\alpha_n \|\gamma V x_n - \mu G p\| + \|(I - \alpha_n \mu G) J^B_{\lambda_n} w_n - (I - \alpha_n \mu G) p\|)^2 \\ &\le (\alpha_n \|\gamma V x_n - \mu G p\| + (1 - \alpha_n \tau) \|J^B_{\lambda_n} w_n - p\|)^2 \\ &\le (\alpha_n \|\gamma V x_n - \mu G p\| + \|J^B_{\lambda_n} w_n - p\|)^2 \\ &\le \alpha_n M_{11} + \|J^B_{\lambda_n} w_n - p\|^2, \end{aligned} \tag{3.53}$$

where M_{11} is an appropriate constant. Next, since $J^B_{\lambda_n}$ is firmly nonexpansive (see (2.1)) and $J^B_{\lambda_n} p = p$, we derive from (2.3)

$$\begin{aligned} \|J^B_{\lambda_n} w_n - p\|^2 &\le \langle J^B_{\lambda_n} w_n - p, w_n - p\rangle \\ &\le \frac{1}{2}(\|J^B_{\lambda_n} w_n - p\|^2 + \|w_n - p\|^2 - \|(J^B_{\lambda_n} w_n - p) - (w_n - p)\|^2) \\ &= \frac{1}{2}(\|J^B_{\lambda_n} w_n - p\|^2 + \|w_n - p\|^2 - \|J^B_{\lambda_n} w_n - x_n + x_n - w_n\|^2) \\ &\le \frac{1}{2}(\|J^B_{\lambda_n} w_n - p\|^2 + \|w_n - p\|^2 \\ &\quad - \|J^B_{\lambda_n} w_n - x_n\|^2 - \|x_n - w_n\|^2 + 2\|J^B_{\lambda_n} w_n - x_n\|\|x_n - w_n\|), \end{aligned}$$

and so

$$\begin{aligned} \|J^B_{\lambda_n} w_n - p\|^2 \le \|w_n - p\|^2 - \|J^B_{\lambda_n} w_n - x_n\|^2 - \|x_n - w_n\|^2 \\ + 2\|J^B_{\lambda_n} w_n - x_n\|\|x_n - w_n\|. \end{aligned} \tag{3.54}$$

Thus, by (3.33), (3.53) and (3.54), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|T_{r_n} y_n - p\|^2 \\ &\le \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|y_n - p\|^2 \\ &\le \beta_n \|x_n - p\|^2 + (1 - \beta_n)(\alpha_n M_{11} + \|J^B_{\lambda_n} w_n - p\|^2) \\ &\le \beta_n \|x_n - p\|^2 + (1 - \beta_n)(\alpha_n M_{11} + \|w_n - p\|^2 \\ &\quad - \|J^B_{\lambda_n} w_n - x_n\|^2 - \|x_n - w_n\|^2 + 2\|J^B_{\lambda_n} w_n - x_n\|\|x_n - w_n\|) \\ &\le \beta_n \|x_n - p\|^2 + (1 - \beta_n)(\alpha_n M_{11} + \|x_n - p\|^2 \\ &\quad - \|J^B_{\lambda_n} w_n - x_n\|^2 + 2\|J^B_{\lambda_n} w_n - x_n\|\|x_n - w_n\|) \\ &\le \|x_n - p\|^2 - (1 - \beta_n)\|J^B_{\lambda_n} w_n - x_n\|^2 + (1 - \beta_n)\alpha_n M_{11} \\ &\quad + 2(1 - \beta_n)\|J^B_{\lambda_n} w_n - x_n\|\|x_n - w_n\|. \end{aligned}$$

This implies

$$\begin{aligned} (1 - \beta_n)\|J^B_{\lambda_n} w_n - x_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1 - \beta_n)\alpha_n M_{11} \\ &\quad + 2(1 - \beta_n)\|J^B_{\lambda_n} w_n - x_n\|\|x_n - w_n\| \\ &\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \end{aligned}$$

$$+ \alpha_n M_{11} + \|x_n - w_n\| M_{12},$$

where M_{12} is an appropriate constants. Thus, by conditions (C1) and (C3) and Steps 2 and 3, we have

$$\lim_{n\to\infty} \|J_{\lambda_n}^B w_n - x_n\| = 0.$$

Step 5. We show that $\lim_{n\to\infty} \|w_n - J_{\lambda_n}^B w_n\| = 0$. Indeed, since

$$\|w_n - J_{\lambda_n}^B w_n\| \le \|w_n - x_n\| + \|x_n - J_{\lambda_n}^B w_n\|$$

by Steps 3 and 4, we have

$$\lim_{n\to\infty} \|w_n - J_{\lambda_n}^B w_n\| = 0.$$

Step 6. We show that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. In fact, since

$$\begin{aligned}\|x_n - y_n\| &= \|x_n - (\alpha_n \gamma V x_n + (I - \alpha_n \mu G) J_{\lambda_n}^B w_n)\| \\ &\le \alpha_n \|\mu G x_n - \gamma V x_n\| + \|(I - \alpha_n \mu G) x_n - (I - \alpha_n \mu G) J_{\lambda_n}^B w_n\| \\ &\le \alpha_n M_{13} + (1 - \alpha_n \tau)\|x_n - J_{\lambda_n}^B w_n\|,\end{aligned}$$

where $M_{13} > 0$ is an appropriate constant, by condition (C1) and Step 4, we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Step 7. We show that $\lim_{n\to\infty} \|x_n - T_{r_n} x_n\| = 0$. In fact, observing

$$\begin{aligned}&\|x_n - T_{r_n} x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_{r_n} y_n\| + \|T_{r_n} y_n - T_{r_n} x_n\| \\ &\le \|x_n - x_{n+1}\| + \beta_n \|x_n - T_{r_n} y_n\| + \|y_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + \beta_n (\|x_n - T_{r_n} x_n\| + \|T_{r_n} x_n - T_{r_n} y_n\|) \\ &\quad + \|y_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + \beta_n \|x_n - T_{r_n} x_n\| + (1 + \beta_n)\|y_n - x_n\|,\end{aligned}$$

we get

$$\|x_n - T_{r_n} x_n\| \le \frac{1}{1 - \beta_n}(\|x_n - x_{n+1}\| + (1 + \beta_n)\|y_n - x_n\|).$$

Hence, by condition (C3), Steps 2 and 6, we obtain

$$\lim_{n\to\infty} \|x_n - T_{r_n} x_n\| = 0.$$

Step 8. We show that $\lim_{n\to\infty} \|y_n - T_{r_n} y_n\| = 0$. Indeed, since

$$\begin{aligned}\|y_n - T_{r_n} y_n\| &\le \|y_n - x_n\| + \|x_n - T_{r_n} x_n\| + \|T_{r_n} x_n - T_{r_n} y_n\| \\ &\le 2\|x_n - y_n\| + \|x_n - T_{r_n} x_n\|\end{aligned}$$

by Steps 6 and 7, we have

$$\lim_{n\to\infty} \|y_n - T_{r_n} y_n\| = 0.$$

Step 9. We show that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, y_n-q\rangle\le 0.$$

For this purpose, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\lim_{i\to\infty}\langle(\gamma V-\mu G)q, y_{n_i}-q\rangle=\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, y_n-q\rangle.$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to some point z. Without loss of generality, we can assume that $y_{n_i}\rightharpoonup z$. Then, by using Steps 3, 4, 5, 6, 7 and 8 and argument similar to those of Steps 1, 2 and 3 in the proof of Theorem 3.2, we obtain $z\in\Omega$. Thus we have

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma V-\mu G)q, y_n-q\rangle&=\lim_{i\to\infty}\langle(\gamma V-\mu G)q, y_{n_i}-q\rangle\\&=\langle(\gamma V-\mu G)q, z-q\rangle\le 0.\end{aligned}$$

Step 10. We show that $\lim_{n\to\infty}\|x_n-q\|=0$. Indeed, from Lemma 2.1, we derive

$$\begin{aligned}\|y_n-q\|^2&=\|\alpha_n\gamma Vx_n+(I-\alpha_n\mu G)J^B_{\lambda_n}A_{\nu_n}x_n-q\|^2\\&=\|\alpha_n(\gamma Vx_n-\gamma Vq)+\alpha_n(\gamma Vq-\mu Gq)\\&\qquad+(I-\alpha_n\mu G)J^B_{\lambda_n}A_{\nu_n}x_n-(I-\alpha_n\mu G)q\|^2\\&\le\|\alpha_n(\gamma Vx_n-\gamma Vq)+(I-\alpha_n\mu G)J^B_{\lambda_n}A_{\nu_n}x_n-(I-\alpha_n\mu G)q\|^2\\&\quad+2\alpha_n\langle\gamma Vq-\mu Gq, y_n-q\rangle\\&\le(\alpha_n\gamma l\|x_n-q\|+(1-\alpha_n\tau)\|x_n-q\|)^2\\&\quad+2\alpha_n\langle\gamma Vq-\mu Gq, y_n-q\rangle\\&=((1-(\tau-\gamma l)\alpha_n)\|x_n-q\|)^2+2\alpha_n\langle\gamma Vq-\mu Gq, y_n-q\rangle.\end{aligned}\tag{3.55}$$

Thus, by (3.33) and (3.55), we obtain

$$\begin{aligned}\|x_{n+1}-q\|^2&\le\beta_n\|x_n-q\|^2+(1-\beta_n)\|T_{r_n}y_n-q\|^2\\&\le\beta_n\|x_n-q\|^2+(1-\beta_n)\|y_n-q\|^2\\&\le\beta_n\|x_n-q\|^2+(1-\beta_n)((1-(\tau-\gamma l)\alpha_n)\|x_n-q\|)^2\\&\quad+2(1-\beta_n)\alpha_n\langle\gamma Vq-\mu Gq, y_n-q\rangle\\&\le\beta_n\|x_n-q\|^2+(1-\beta_n)(1-(\tau-\gamma l)\alpha_n)\|x_n-q\|^2\\&\quad+2(1-\beta_n)\alpha_n\langle\gamma Vq-\mu Gq, y_n-q\rangle\\&=(1-(1-\beta_n)(\tau-\gamma l)\alpha_n)\|x_n-q\|^2\\&\quad+2(1-\beta_n)(\tau-\gamma l)\alpha_n\frac{\langle\gamma Vq-\mu Gq, y_n-q\rangle}{\tau-\gamma l}\\&=(1-\xi_n)\|x_n-q\|^2+\xi_n\delta_n,\end{aligned}$$

where $\xi_n = (1-\beta_n)(\tau-\gamma l)\alpha_n$ and $\delta_n = \frac{2\langle \gamma Vq-\mu G, y_n-q\rangle}{\tau-\gamma l}$. From conditions (C1), (C2) and (C3) and Step 9, it is easy to see that $\xi_n \to 0$, $\sum_{n=1}^{\infty} \xi_n = \infty$ and $\limsup_{n\to\infty} \delta_n \le 0$. Hence, by Lemma 2.3, we conclude

$$\lim_{n\to\infty} \|x_n - q\| = 0.$$

This completes the proof. □

By taking $V \equiv 0$, $G \equiv I$, $\mu = 1$ in Theorem 3.4, we obtain the following result.

Corollary 3.5. *Let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T_{r_n}((1-\alpha_n)J_{r_n}^B A_{r_n} x_n),\ n \ge 0.$$

Let $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *and* $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ *satisfy the conditions* (C1), (C2), (C3), (C4), (C5) *and* (C6) *in Theorem 3.4. Then* $\{x_n\}$ *converges strongly to a point* $q \in \Omega$*, which is the minimum-norm element of* Ω*.*

If in Theorem 3.4, we take $T \equiv I$, the identity mapping on C, then we obtain the following result.

Corollary 3.6. *Suppose that* $\Omega_1 = VI(C,A) \cap B^{-1}0 \ne \emptyset$*. Let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)(\alpha_n \gamma V x_n + (I-\alpha_n \mu G)J_{\lambda_n}^B A_{\nu_n} x_n),\ n \ge 0.$$

Let $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *and* $\{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ *satisfy the conditions* (C1), (C2), (C3), (C5) *and* (C6) *in Theorem 3.4. Then* $\{x_n\}$ *converges strongly to a point* $q \in \Omega_1$*, which is the unique solution of the following variational inequality*:

$$\langle (\mu G - \gamma V)q, p-q\rangle \ge 0, \quad \forall p \in \Omega_1.$$

If in Theorem 3.4, we have $C \equiv H$, then we have the following corollary.

Corollary 3.7. *Suppose that* $\Omega_2 = Fix(T) \cap A^{-1}0 \cap B^{-1}0 \ne \emptyset$*. Let* $T : H \to H$ *be a continuous pseudocontractive mapping and let* $A : H \to H$ *be a continuous monotone mapping. Let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)T_{r_n}(\alpha_n \gamma V x_n + (I-\alpha_n \mu G)J_{\lambda_n}^B A_{\nu_n} x_n),\ n \ge 0.$$

Let $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *and* $\{r_n\}, \{\lambda_n\}, \{\nu_n\} \subset (0,\infty)$ *satisfy the conditions* (C1), (C2), (C3), (C4), (C5) *and* (C6) *in Theorem 3.4. Then* $\{x_n\}$ *converges strongly to a point* $q \in \Omega_2$*, which is the unique solution of the following variational inequality*:

$$\langle (\mu G - \gamma V)q, p-q\rangle \ge 0, \quad \forall p \in \Omega_2.$$

Proof. Since $D(A) = H$, we note that $VI(H,A) = A^{-1}0$. So the result follows from Theorem 3.4. □

Remark 3.8. 1) It is worth pointing out that implicit and explicit iterative algorithms are new ones different from those announced by several authors; see, for instance, [7, 8, 15] and the references therein. In particular, we use the variable parameters r_t, λ_t, ν_t and r_n, λ_n, ν_n in comparison with the corresponding iterative algorithms in [7,8,15] and the references therein.

2) We know that $Fix(T) \cap VI(C, A) \cap B^{-1}0 \subset Fix(T) \cap (A + B)^{-1}0$ (see [8]). Thus, as results for finding a common element of the fixed point set of continuous pseudocontractive mapping more general than nonexpansive mapping and strictly pseudocontractive mapping and the zero point set of sum of maximal monotone operator and continuous monotone mapping more general than α-inverse strongly monotone mapping, Theorem 3.2 and Theorem 3.4 are new results, which develop and improve the corresponding results in [7, 12, 15] and the references therein.

3) Corollary 3.3 and Corollary 3.5 are also new results for finding a minimum-norm point of $Fix(T) \cap VI(C, A) \cap B^{-1}0$, where T is a continuous pseudocontractive mapping, A is a continuous monotone mapping and B is a maximal monotone operator.

4) By taking $V \equiv 0$, $G \equiv I$ and $\mu = 1$ in Corollary 3.6 and Corollary 3.7, we can obtain new results for finding the minimum-norm point of $VI(C, A) \cap B^{-1}0$ and $Fix(T) \cap A^{-1}0 \cap B^{-1}0$, respectively.

5) As applications in [15], by using Theorem 3.2 and Theorem 3.4, we can propose implicit and explicit iterative algorithms for the equilibrium problems coupled with fixed point problem for continuous pseudocontractive mapping.

Acknowledgment. The author would like to thank the anonymous reviewers for their careful reading and valuable comments along with providing a recent related paper, which improved the presentation of this manuscript.

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