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HOMOTOPY PROPERTIES OF map($\Sigma^n \mathbb{C}P^2, S^m$)

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Abstract. For given spaces X and Y, let $map(X,Y)$ and $map_*(X,Y)$ be the unbased and based mapping spaces from X to Y, equipped with compact-open topology respectively. Then let $map(X,Y;f)$ and $map_*(X,Y;g)$ be the path component of $map(X,Y)$ containing f and $map_*(X,Y)$ containing g, respectively. In this paper, we compute cohomotopy groups of suspended complex plane $\pi^{n+m}(\Sigma^n\mathbb{C}P^2)$ for $m = 6, 7$. Using these results, we classify path components of the spaces $map(\Sigma^n\mathbb{C}P^2, S^m)$ up to homotopy equivalence. We also determine the generalized Gottlieb groups $G_n(\mathbb{C}P^2, S^m)$. Finally, we compute homotopy groups of mapping spaces $map(\Sigma^n\mathbb{C}P^2, S^m; f)$ for all generators $[f]$ of $[\Sigma^n\mathbb{C}P^2, S^m]$, and Gottlieb groups of mapping components containing constant map $map(\Sigma^n\mathbb{C}P^2, S^m; *)$.

1. Introduction

Let X and Y be based topological spaces. A major object of homotopy theory is to study $[X,Y]$, the set of homotopy classes of based maps. In general, if Y is a co-H-group, $[X,Y]$ has a group structure. Let ΣX be the suspension of X. Since every suspended space ΣX is co-group, $[\Sigma X, Y]$ has group structure. If ΣX is a sphere S^n, $[S^n, Y]$ is the n-th homotopy group of Y. On the other hand, $[X, S^n]$ is called the n-th cohomotopy set of X and denoted by $\pi^n(X)$. If X is a co-H-group, the cohomotopy set is a group and called cohomotopy group. Homotopy groups and cohomotopy groups have been studied by many authors and are the major object in algebraic topology.

Another major object of homotopy theory is to investigate the set of (unbased) maps $f : X \to Y$. We denote $map(X,Y)$ to be the set of continuous maps from X to Y equipped with compact-open topology. Then we write $map(X,Y;f)$ for the path-component of $map(X,Y)$ containing f. Important cases are $map(X,Y;*)$, the space of null-homotopic maps and $map(X,X;1)$, the identity path-component. It is proved that every topological space appears as a quotient of a paracompact Hausdorff space in a natural way [7]. Thus it

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is important to study path components of mapping spaces to investigate topological spaces. In general, the mapping space $map(X,Y)$ does not have CW homotopy type. Even if X and Y are finite CW complexes, $map(X,Y)$ may not be of CW homotopy type. By Milnor [16], when X is a compact metric space and Y is a CW complex, the path components $map(X,Y;f)$ are of CW homotopy type. Also according to Kahn [9], $map(X,Y)$ is of CW type when X is a CW complex and Y only has finitely many non-trivial homotopy groups.

Lang proved that if X is a suspended space, then all path components of $map_*(X,Y)$ have the same homotopy type [12, Theorem 2.1]. Whitehead proved that $map(S^n,S^m;f)$ is homotopy equivalent to $map(S^n,S^m;0)$ if and only if w_f has a section, where $0: S^n \to S^m$ is a constant map [21, Theorem 2.8]. Lupton and Smith proved that

$$map(X,Y;f) \simeq map(X,Y;f+d)$$

for a CW co-H-space X and any CW complex Y, where $d: X \to Y$ is a cyclic map [10, Theorem 3.10]. Recently, Gatsinzi [2] proved that the dimension of the rational Gottlieb group of the universal cover $\widetilde{map}(X,S^{2n};f)$ of the function space $map(X,S^{2n};f)$ is at least equal to the dimension of $\tilde{H}^*(X;\mathbb{Q})$ under several assumptions. Lupton and Smith [10] showed that

$$G_n(map_*(X,Y;*)) \cong G_n(Y) \oplus G_n(X,Y).$$

Maruyama and Oshima [13] determined homotopy groups of

$$map_*(SU(3),SU(3)),\ map_*(Sp(2),Sp(2)) \text{ and } map_*(G_2,G_2).$$

In Section 2, we present some basic knowledge of composition methods [19]. We review a mapping cone sequence and Puppe sequence related to the suspended complex projective plane and discuss the concept of cyclic maps and its properties. Also we recall the Toda brackets and their properties related to suspended complex planes.

In Sections 3 and 4, we compute $\pi^n(\Sigma^{n+k}\mathbb{C}P^2)$ for $k=6,7$. As a result, we obtain the results (see Tables 1, 2).

In Section 5, we computer homotopy groups of $map_*(\mathbb{C}P^2,\mathbb{C}P^2,*)$ by the result of Sections 3, 4 and [8].

In Section 6, we apply our computation to the classification of path components of mapping spaces up to homotopy equivalent and evaluation fibrations up to fibre homotopy equivalent. Hansen proved that the evaluation fibration $w_f: (\Sigma X,\Sigma Y;f) \to \Sigma Y$ has a section if and only if $[f, id_{\Sigma Y}]=0$, where $[\ ,\ ]$ is the generalized Whitehead product [6]. Lupton and Smith proved that the following statements are all equivalent: (1) a map $f: X \to Y$ is cyclic, (2) w_f has a section, and (3) two fibrations w_f and w_0 are fiber homotopy equivalent, where 0 is a constant map [10]. Also, we apply our results to the formulation of generalized Gottlieb groups from suspended complex plane to sphere and Gottlieb groups of path components of constant map.

We use the notation of [8, 19] freely.

Table 1.

case n	2	3	4	5	6	7
$\pi^n(\Sigma^{n+6}\mathbb{C}P^2)$	$2+15$	$2+3$	$8+2+3^2+5$	$4+9$	4^2+9+3	$4+3$
case n	8	9	10	11	$n \geq 12$	
$\pi^n(\Sigma^{n+6}\mathbb{C}P^2)$	4^2+3^2	$4+3$	$2+3$	$2+3$	2	

Table 2.

case n	2	3	4	5	6
$\pi^n(\Sigma^{n+7}\mathbb{C}P^2)$	$2+3$	2^2+21	2^3	$4+2+63$	$4+2^2+63$
case n	7	8	9	10	11
$\pi^n(\Sigma^{n+7}\mathbb{C}P^2)$	$8+2^2$	$8+2^2$	$8+2^2$	$\infty+8+2+63$	$8+2+63$
case n	12	$n \geq 13$			
$\pi^n(\Sigma^{n+7}\mathbb{C}P^2)$	$\infty+8+2+63$	$8+2+63$			

where the integer n denotes the cyclic group $\mathbb{Z}_n$, ∞ denotes the groups of integers $\mathbb{Z}$, "+" denotes the direct sum of abelian groups, and $(s)^k$ denotes the k-times direct sum of $\mathbb{Z}_s$.

2. Preliminaries

The complex projective plane $\mathbb{C}P^2$ is defined by the mapping cone $S^2 \cup_{\eta_2} e^4$, where $\eta_2 : S^3 \to S^2$ is the Hopf fibering. Consider a Puppe sequence

$$S^3 \xrightarrow{\eta_2} S^2 \xrightarrow{i} \mathbb{C}P^2 \xrightarrow{p} S^4 \xrightarrow{\eta_3} S^3 \xrightarrow{\Sigma i} \cdots,$$

where $i : S^2 \to \mathbb{C}P^2$ is the inclusion map, $p : \mathbb{C}P^2 \to S^4$ is the collapsing map of S^2 to a point $*$, and $\eta_k = \Sigma^{k-2}\eta_2$ for $k \geq 2$. Then, we have a long exact sequence of homotopy sets

$$\pi_{n+3}(S^m) \xrightarrow{\eta^*_{n+3}} \pi_{n+4}(S^m) \xrightarrow{\Sigma^n p^*} [\Sigma^n\mathbb{C}P^2, S^m]$$
$$\xrightarrow{\Sigma^n i^*} \pi_{n+2}(S^m) \xrightarrow{\eta^*_{n+2}} \pi_{n+3}(S^m).$$

Therefore, we have the short exact sequence

$$0 \to \mathrm{Coker}\eta^*_{n+3} \xrightarrow{\Sigma^n p^*} [\Sigma^n\mathbb{C}P^2, S^m] \xrightarrow{\Sigma^n i^*} \mathrm{Ker}\eta^*_{n+2} \to 0. \tag{2.1}$$

When G is an abelian group and $p \geq 2$ is a prime number, we denote the p-primary parts of G by $G_{(p)}$.

For $p \geq 3$, we have an isomorphism

$$[\Sigma^n\mathbb{C}P^2, S^k]_{(p)} \cong \pi_{n+2}(S^k)_{(p)} \oplus \pi_{n+4}(S^k)_{(p)},$$

since $\pi_{n+1}(S^n)$ is of order 2 for $n \geq 3$ [19, Proposition 5.1].

It is well known that the Hopf fibrations $\eta_2 : S^3 \to S^2$, $\nu_4 : S^7 \to S^4$, and $\sigma_8 : S^{15} \to S^8$ induce isomorphisms

$$[X, S^3] \to [X, S^2], \qquad \alpha \mapsto \eta_2 \circ \alpha, \tag{2.2}$$

$$[X,S^3]\oplus[\Sigma X,S^7]\to[\Sigma X,S^4],\quad (\alpha,\beta)\mapsto\Sigma\alpha+\nu_4\circ\beta, \tag{2.3}$$

$$[X,S^7]\oplus[\Sigma X,S^{15}]\to[\Sigma X,S^8],\quad (\alpha,\beta)\mapsto\Sigma\alpha+\sigma_8\circ\beta. \tag{2.4}$$

Consider elements $\alpha\in[Y,Z]$, $\beta\in[X,Y]$, and $\gamma\in[W,X]$ which satisfy $\alpha\circ\beta=0$ and $\beta\circ\gamma=0$. Let C_β be the mapping cone of β, and $i:Y\to C_\beta$, $p:C_\gamma\to\Sigma X$ be the inclusion and the shrinking map, respectively. We denote an extension of α satisfying $i^*(\overline{\alpha})=\alpha$ by $\overline{\alpha}\in[C_\beta,Z]$, and a coextension of γ satisfying $p_*(\widetilde{\gamma})=\Sigma\gamma$ by $\widetilde{\gamma}\in[\Sigma W,C_\beta]$ [19].

We recall some relations between (co)extensions and Toda brackets [19].

Proposition 1. *Let $\alpha\in[Y,Z]$, $\beta\in[X,Y]$, and $\gamma\in[W,X]$ be elements which satisfy $\alpha\circ\beta=0$ and $\beta\circ\gamma=0$. Let $\{\alpha,\beta,\gamma\}$ be the Toda bracket and $p:C_\gamma\to\Sigma W$ be the shrinking map, respectively. Then, we have*
(a) $\overline{\alpha}\circ\widetilde{\gamma}\in\{\alpha,\beta,\gamma\}$,
(b) $\alpha\circ\overline{\beta}\in\{\alpha,\beta,\gamma\}\circ p$.

The following is useful for determining 2-primary parts of the class $[\Sigma^n\mathbb{C}P^2, S^m]$ [8].

Proposition 2. *Let $\iota_{\mathbb{C}}:\mathbb{C}P^2\to\mathbb{C}P^2$ and $\iota_4:S^4\to S^4$ be identity maps on $\mathbb{C}P^2$ and S^4, respectively. Let $\overline{2\iota_3}:\Sigma\mathbb{C}P^2\to S^3$ and $\widetilde{2\iota_4}:S^5\to\Sigma\mathbb{C}P^2$ be extension and coextension of $2\iota_3$ and $2\iota_4$, respectively. Then we have*

$$2\Sigma\iota_{\mathbb{C}}=\Sigma i\circ\overline{2\iota_3}+\widetilde{2\iota_4}\circ\Sigma p$$

on $[\Sigma\mathbb{C}P^2,\Sigma\mathbb{C}P^2]$.

Here, we recall the concept of a cyclic map and Gottlieb groups of a space X, denoted by $G_n(X)$ [4,20].

Definition 1. A map $f:Y\to X$ is cyclic if there is a map $F:X\times Y\to X$, called an affiliated map of f, such that the diagram homotopy commutative:

$$\begin{tikzcd} X\times Y \arrow[r, "F"] & X\times X \\ X\vee Y \arrow[u, "i"] \arrow[r, "1\vee f"] & X\vee X. \arrow[u, "\nabla"'] \end{tikzcd}$$

Let $G(Y,X)$ denote the set of all homotopy classes of cyclic maps from Y to X. Varadarajan showed that $G(Y,X)$ has a group structure for any co-H-space Y [20]. For an integer $i\geq 1$, the set of homotopy classes of cyclic maps $\Sigma^nX\to Y$ we denote by $G_n(X,Y)$, and call the n-th generalized Gottlieb group of (X,Y). When $Y=S^n$, $G(Y,X)=G_n(X)$ is the n-th Gottlieb group of X. In [4], Gottlieb introduced and studied the evaluation subgroups

$$G_n(X)=w_*(\pi_n(map(X,X;1))),$$

where $w_*:\pi_n(map(X,X;1))\to\pi_n(X)$. Note that the $G_n(X)$ can alternatively be described as homotopy classes of maps $f:S^n\to X$ such that $(f|1):S^n\vee X\to X$ admits an extension $F:S^n\times X\to X$ up to homotopy.

3. $[\Sigma^{n+6}\mathbb{C}P^2, S^n]$ for $n \geq 2$

In this section, we compute the n-th cohomotopy group of $(n+6)$-fold suspended complex projective plane by using (2.1).

Proposition 3. (1) $[\Sigma^8\mathbb{C}P^2, S^2] = \mathbb{Z}_2\{\eta_2 \circ \mu_3 \circ \Sigma^8 p\} \oplus \mathbb{Z}_{15}$.
(2) $[\Sigma^9\mathbb{C}P^2, S^3] = \mathbb{Z}_2\{\varepsilon' \circ \Sigma^9 p\} \oplus \mathbb{Z}_3$.
(3) $[\Sigma^{10}\mathbb{C}P^2, S^4] = \mathbb{Z}_8\{\nu_4 \circ \sigma' \circ \Sigma^{10} p\} \oplus \mathbb{Z}_2\{E\varepsilon' \circ \Sigma^{10} p\} \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5$.
(4) $[\Sigma^{11}\mathbb{C}P^2, S^5] = \mathbb{Z}_4\{\nu_5 \circ \sigma_8 \circ \Sigma^{11} p\} \oplus \mathbb{Z}_9$.

Proof. (1) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{11}^* \xrightarrow{\Sigma^8 p^*} [\Sigma^8\mathbb{C}P^2, S^2] \xrightarrow{\Sigma^8 i^*} \mathrm{Ker}\eta_{10}^* \to 0,$$

where $\eta_{11}^* : \pi_{11}(S^2) \to \pi_{12}(S^2)$ or, more precisely

$$\eta_{11}^* : \mathbb{Z}_2\{\eta_2 \circ \varepsilon_3\} \to \mathbb{Z}_2^2\{\eta_2^2 \circ \varepsilon_4, \eta_2 \circ \mu_3\}$$

and $\eta_{10}^* : \pi_{10}(S^2) \to \pi_{11}(S^2)$ or, more precisely

$$\eta_{10}^* : \mathbb{Z}_{15} \to \mathbb{Z}_2.$$

Then we have

$$\mathrm{Coker}\eta_{11}^* = \mathbb{Z}_2\{\eta_2 \circ \mu_3\}$$

and

$$\mathrm{Ker}\eta_{10}^* = \mathbb{Z}_{15}$$

by [19, (7.5)]. Now we have a short exact sequence

$$0 \to \mathbb{Z}_2\{\eta_2 \circ \mu_3\} \xrightarrow{\Sigma^8 p^*} [\Sigma^8\mathbb{C}P^2, S^2] \xrightarrow{\Sigma^8 i^*} \mathbb{Z}_{15} \to 0.$$

Thus we have

$$[\Sigma^8\mathbb{C}P^2, S^2] = \mathbb{Z}_2\{\eta_2 \circ \mu_3 \circ \Sigma^8 p\} \oplus \mathbb{Z}_{15}.$$

(2) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{12}^* \xrightarrow{\Sigma^9 p^*} [\Sigma^9\mathbb{C}P^2, S^3] \xrightarrow{\Sigma^9 i^*} \mathrm{Ker}\eta_{11}^* \to 0,$$

where $\eta_{12}^* : \pi_{12}(S^3) \to \pi_{13}(S^3)$ or, more precisely

$$\eta_{12}^* : \mathbb{Z}_2^2\{\mu_3, \eta_3 \circ \varepsilon_4\} \to \mathbb{Z}_4\{\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3 \circ \mu_4\} \oplus \mathbb{Z}_3$$

and $\eta_{11}^* : \pi_{11}(S^3) \to \pi_{12}(S^3)$ or, more precisely

$$\eta_{11}^* : \mathbb{Z}_2\{\varepsilon_3\} \to \mathbb{Z}_2^2\{\mu_3, \eta_3 \circ \varepsilon_4\}.$$

Then we have $\eta_{12}^*(\mu_3) = \eta_3 \circ \mu_4$, $\eta_{12}^*(\eta_3 \circ \varepsilon_4) = \eta_3 \circ \varepsilon_4 \circ \eta_{12} = \eta_3^2 \circ \varepsilon_5 = 2\varepsilon'$ and $\eta_{11}^*(\varepsilon_3) = \eta_3 \circ \varepsilon_4$ by [18, (2.2)], [19, Lemma 6.6, (7.5)]. Thus we have

$$\mathrm{Coker}\eta_{12}^* = \mathbb{Z}_2\{\varepsilon'\} \oplus \mathbb{Z}_3$$

and

$$\mathrm{Ker}\eta_{11}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_2\{\varepsilon'\} \oplus \mathbb{Z}_3 \xrightarrow{\Sigma^9 p^*} [\Sigma^9\mathbb{C}P^2, S^3] \xrightarrow{\Sigma^9 i^*} 0.$$

Thus we have

$$[\Sigma^9\mathbb{C}P^2, S^3] = \mathbb{Z}_2\{\varepsilon' \circ \Sigma^9 p\} \oplus \mathbb{Z}_3.$$

(3) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{13}^* \xrightarrow{\Sigma^{10}p^*} [\Sigma^{10}\mathbb{C}P^2, S^4] \xrightarrow{\Sigma^{10}i^*} \mathrm{Ker}\eta_{12}^* \to 0,$$

where $\eta_{13}^* : \pi_{13}(S^4) \to \pi_{14}(S^4)$ or, more precisely

$$\eta_{13}^* : \mathbb{Z}_2^3\{\nu_4^3, \mu_4, \eta_4 \circ \varepsilon_5\} \to \mathbb{Z}_8\{\nu_4 \circ \sigma'\} \oplus \mathbb{Z}_4\{E\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_4 \circ \mu_5\} \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5$$

and $\eta_{12}^* : \pi_{12}(S^4) \to \pi_{13}(S^4)$ or, more precisely

$$\eta_{12}^* : \mathbb{Z}_2\{\varepsilon_4\} \to \mathbb{Z}_2^3\{\nu_4^3, \mu_4, \eta_4 \circ \varepsilon_5\}.$$

Then we have $\eta_{13}^*(\nu_4^3) = 0$, $\eta_{13}^*(\mu_4) = \eta_4 \circ \mu_5$, $\eta_{13}^*(\eta_4 \circ \varepsilon_5) = \eta_4^2 \circ \varepsilon_6 = 2E\varepsilon'$ and $\eta_{12}^*(\varepsilon_4) = \eta_4 \circ \varepsilon_5$ by [18, (2.2)], [19, Lemma 6.6, (5.9), (7.5)]. Thus we have

$$\mathrm{Coker}\eta_{13}^* = \mathbb{Z}_8\{\nu_4 \circ \sigma'\} \oplus \mathbb{Z}_2\{E\varepsilon'\} \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5$$

and

$$\mathrm{Ker}\eta_{12}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_8\{\nu_4 \circ \sigma'\} \oplus \mathbb{Z}_2\{E\varepsilon'\} \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5 \xrightarrow{\Sigma^{10}p^*} [\Sigma^{10}\mathbb{C}P^2, S^4] \xrightarrow{\Sigma^{10}i^*} 0.$$

Thus we have

$$[\Sigma^{10}\mathbb{C}P^2, S^4] = \mathbb{Z}_8\{\nu_4 \circ \sigma' \circ \Sigma^{10}p\} \oplus \mathbb{Z}_2\{E\varepsilon' \circ \Sigma^{10}p\} \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5.$$

(4) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{14}^* \xrightarrow{\Sigma^{11}p^*} [\Sigma^{11}\mathbb{C}P^2, S^5] \xrightarrow{\Sigma^{11}i^*} \mathrm{Ker}\eta_{13}^* \to 0,$$

where $\eta_{14}^* : \pi_{14}(S^5) \to \pi_{15}(S^5)$ or, more precisely

$$\eta_{14}^* : \mathbb{Z}_2^3\{\nu_5^3, \mu_5, \eta_5 \circ \varepsilon_6\} \to \mathbb{Z}_8\{\nu_5 \circ \sigma_8\} \oplus \mathbb{Z}_2\{\eta_5 \circ \mu_6\} \oplus \mathbb{Z}_9$$

and $\eta_{13}^* : \pi_{13}(S^5) \to \pi_{14}(S^5)$ or, more precisely

$$\eta_{13}^* : \mathbb{Z}_2\{\varepsilon_5\} \to \mathbb{Z}_2^3\{\nu_5^3, \mu_5, \eta_5 \circ \varepsilon_6\}.$$

Then we have $\eta_{14}^*(\nu_5^3) = 0$, $\eta_{14}^*(\mu_5) = \eta_5 \circ \mu_6$, $\eta_{14}^*(\eta_5 \circ \varepsilon_6) = \eta_5^2 \circ \varepsilon_7 = 2E^2\varepsilon' = 2(2\nu_5 \circ \sigma_8) = 4\nu_5 \circ \sigma_8$ and $\eta_{13}^*(\varepsilon_5) = \eta_5 \circ \varepsilon_6$ by [18, (2.2)], [19, Lemma 6.6, (5.9), (7.10)]. Thus we have

$$\mathrm{Coker}\eta_{14}^* = \mathbb{Z}_4\{\nu_5 \circ \sigma_8\} \oplus \mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{13}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\nu_5 \circ \sigma_8\} \oplus \mathbb{Z}_9 \xrightarrow{\Sigma^{11}p^*} [\Sigma^{11}\mathbb{C}P^2, S^5] \xrightarrow{\Sigma^{11}i^*} 0.$$

Thus we have

$$[\Sigma^{11}\mathbb{C}P^2, S^5] = \mathbb{Z}_4\{\nu_5 \circ \sigma_8 \circ \Sigma^{11}p\} \oplus \mathbb{Z}_9. \qquad \square$$

Proposition 4. (1) $[\Sigma^{13}\mathbb{C}P^2, S^7] = \mathbb{Z}_4\{\nu_7 \circ \sigma_{10} \circ \Sigma^{13}p\} \oplus \mathbb{Z}_3$.
(2) $[\Sigma^{12}\mathbb{C}P^2, S^6] = \mathbb{Z}_4^2\{\nu_6 \circ \sigma_9 \circ \Sigma^{12}p, \overline{2\overline{\nu}_6}\} \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3$.

Proof. (1) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{16}^* \xrightarrow{\Sigma^{13}p^*} [\Sigma^{13}\mathbb{C}P^2, S^7] \xrightarrow{\Sigma^{13}i^*} \mathrm{Ker}\eta_{15}^* \to 0,$$

where $\eta_{16}^* : \pi_{16}(S^7) \to \pi_{17}(S^7)$ or, more precisely

$$\eta_{16}^* : \mathbb{Z}_2^4\{\sigma' \circ \eta_{14}^2, \nu_7^3, \mu_7, \eta_7 \circ \varepsilon_8\} \to \mathbb{Z}_8\{\nu_7 \circ \sigma_{10}\} \oplus \mathbb{Z}_2\{\eta_7 \circ \mu_8\} \oplus \mathbb{Z}_3$$

and $\eta_{15}^* : \pi_{15}(S^7) \to \pi_{16}(S^7)$ or, more precisely

$$\eta_{15}^* : \mathbb{Z}_2^3\{\sigma' \circ \eta_{14}, \overline{\nu}_7, \varepsilon_7\} \to \mathbb{Z}_2^4\{\sigma' \circ \eta_{14}^2, \nu_7^3, \mu_7, \eta_7 \circ \varepsilon_8\}.$$

Then we have $\eta_{16}^*(\sigma' \circ \eta_{14}^2) = \sigma' \circ \eta_{14}^3 = \sigma' \circ (4\nu_{14}) = 4(\sigma' \circ \nu_{14}) = 4(x\nu_7 \circ \sigma_{10}) = 4(\nu_7 \circ \sigma_{10})$ where x odd, $\eta_{16}^*(\nu_7^3) = 0$, $\eta_{16}^*(\mu_7) = \eta_7 \circ \mu_8$, $\eta_{16}^*(\eta_7 \circ \varepsilon_8) = \eta_7^2 \circ \varepsilon_9 = 2E^2(E^2\varepsilon') = 2E^2(2\nu_5 \circ \sigma_8) = 4\nu_7 \circ \sigma_{10}$, $\eta_{15}^*(\sigma' \circ \eta_{14}) = \sigma' \circ \eta_{14}^2$, $\eta_{15}^*(\overline{\nu}_7) = \nu_7^3$ and $\eta_{15}^*(\varepsilon_7) = \eta_7 \circ \varepsilon_8$ by [18, (2.2)], [19, (5.5), (5.9), (7.10), (7.19), Lemma 6.6]. Thus we have

$$\mathrm{Coker}\eta_{16}^* = \mathbb{Z}_4\{\nu_7 \circ \sigma_{10}\} \oplus \mathbb{Z}_3$$

and

$$\mathrm{Ker}\eta_{15}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\nu_7 \circ \sigma_{10}\} \oplus \mathbb{Z}_3 \xrightarrow{\Sigma^{13}p^*} [\Sigma^{13}\mathbb{C}P^2, S^7] \xrightarrow{\Sigma^{13}i^*} 0.$$

Thus we have

$$[\Sigma^{13}\mathbb{C}P^2, S^7] = \mathbb{Z}_4\{\nu_7 \circ \sigma_{10} \circ \Sigma^{13}p\} \oplus \mathbb{Z}_3.$$

(2) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{15}^* \xrightarrow{\Sigma^{12}p^*} [\Sigma^{12}\mathbb{C}P^2, S^6] \xrightarrow{\Sigma^{12}i^*} \mathrm{Ker}\eta_{14}^* \to 0,$$

where $\eta_{15}^* : \pi_{15}(S^6) \to \pi_{16}(S^6)$ or, more precisely

$$\eta_{15}^* : \mathbb{Z}_2^3\{\nu_6^3, \mu_6, \eta_6 \circ \varepsilon_7\} \to \mathbb{Z}_8\{\nu_6 \circ \sigma_9\} \oplus \mathbb{Z}_2\{\eta_6 \circ \mu_7\} \oplus \mathbb{Z}_9$$

and $\eta_{14}^* : \pi_{14}(S^6) \to \pi_{15}(S^6)$ or, more precisely

$$\eta_{14}^* : \mathbb{Z}_8\{\overline{\nu}_6\} \oplus \mathbb{Z}_2\{\varepsilon_6\} \oplus \mathbb{Z}_3 \to \mathbb{Z}_2^3\{\nu_6^3, \mu_6, \eta_6 \circ \varepsilon_7\}.$$

Then we have $\eta_{15}^*(\nu_6^3) = 0$, $\eta_{15}^*(\mu_6) = \eta_6 \circ \mu_7$, $\eta_{15}^*(\eta_6 \circ \varepsilon_7) = \eta_6^2 \circ \varepsilon_8 = 2E(E^2\varepsilon') = 2E(2\nu_5 \circ \sigma_8) = 4\nu_6 \circ \sigma_9$ $\eta_{14}^*(\overline{\nu}_6) = \nu_6^3$ and $\eta_{14}^*(\varepsilon_6) = \eta_6 \circ \varepsilon_7$ by [18, (2.2)], [19, Lemma 6.6, 6.3, (5.9), (7.10)]. Thus we have

$$\mathrm{Coker}\eta_{15}^* = \mathbb{Z}_4\{\nu_6 \circ \sigma_9\} \oplus \mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{14}^* = \mathbb{Z}_4\{2\overline{\nu}_6\} + \mathbb{Z}_3.$$

So we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\nu_6 \circ \sigma_9\} \oplus \mathbb{Z}_9 \xrightarrow{\Sigma^{12}p^*} [\Sigma^{12}\mathbb{C}P^2, S^6] \xrightarrow{\Sigma^{12}i^*} \mathbb{Z}_4\{2\overline{\nu}_6\} \oplus \mathbb{Z}_3 \to 0.$$

We consider a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_4\{\nu_6\circ\sigma_9\} \arrow[r, "\Sigma^{12}p^*"] \arrow[d, "\Sigma_1"] & {[\Sigma^{12}\mathbb{C}P^2, S^6]_{(2)}} \arrow[r, "\Sigma^{12}i^*"] \arrow[d, "\Sigma_2"] & \mathbb{Z}_4\{2\overline{\nu}_6\} \arrow[r] \arrow[d, "\Sigma_3"] & 0 \\
0 \arrow[r] & \mathbb{Z}_4\{\nu_7\circ\sigma_{10}\} \arrow[r, "\Sigma^{13}p^*"] & {[\Sigma^{13}\mathbb{C}P^2, S^7]_{(2)}} \arrow[r, "\Sigma^{12}i^*"] & 0 \arrow[r] & 0
\end{tikzcd}$$

Since Σ_1 and $\Sigma^{13}p^*$ are isomorphisms, $\Sigma^{12}p^*$ has left inverse. This implies that the first row splits. Thus we have

$$[\Sigma^{12}\mathbb{C}P^2, S^6] = \mathbb{Z}_4^2\{\nu_6\circ\sigma_9\circ\Sigma^{12}p, \overline{2\overline{\nu}_6}\}\oplus\mathbb{Z}_9\oplus\mathbb{Z}_3. \qquad \square$$

Proposition 5. (1) $[\Sigma^{14}\mathbb{C}P^2, S^8] = \mathbb{Z}_4^2\{\sigma_8\circ\nu_{15}\circ\Sigma^{14}p, \nu_8\circ\sigma_{11}\circ\Sigma^{14}p\}\oplus\mathbb{Z}_3^2$.
(2) $[\Sigma^{15}\mathbb{C}P^2, S^9] = \mathbb{Z}_4\{\sigma_9\circ\nu_{16}\circ\Sigma^{15}p\}\oplus\mathbb{Z}_3$.
(3) $[\Sigma^{16}\mathbb{C}P^2, S^{10}] = \mathbb{Z}_2\{\sigma_{10}\circ\nu_{17}\circ\Sigma^{16}p\}\oplus\mathbb{Z}_3$.
(4) $[\Sigma^{17}\mathbb{C}P^2, S^{11}] = \mathbb{Z}_2\{\sigma_{11}\circ\nu_{18}\circ\Sigma^{17}p\}\oplus\mathbb{Z}_3$.
(5) $[\Sigma^{n+6}\mathbb{C}P^2, S^n] = \mathbb{Z}_3\{\beta_1(n)\circ\Sigma^{n+6}p\}$ *for* $n\geq 12$.

Proof. (1) Consider the following short exact sequence

$$0\to \mathrm{Coker}\eta_{17}^* \xrightarrow{\Sigma^{14}p^*} [\Sigma^{14}\mathbb{C}P^2, S^8] \xrightarrow{\Sigma^{14}i^*} \mathrm{Ker}\eta_{16}^*\to 0,$$

where $\eta_{17}^*:\pi_{17}(S^8)\to\pi_{18}(S^8)$ or, more precisely

$$\eta_{17}^*:\mathbb{Z}_2^5\{\sigma_8\circ\eta_{15}^2, (E\sigma')\circ\eta_{15}^2, \nu_8^3, \mu_8, \eta_8\circ\varepsilon_9\}\to\mathbb{Z}_8^2\{\sigma_8\circ\nu_{15}, \nu_8\circ\sigma_{11}\}\oplus\mathbb{Z}_2\{\eta_8\circ\mu_9\}\oplus\mathbb{Z}_3^2$$

and $\eta_{16}^*:\pi_{16}(S^8)\to\pi_{17}(S^8)$ or, more precisely

$$\eta_{16}^*:\mathbb{Z}_2^4\{\sigma_8\circ\eta_{15}, (E\sigma')\circ\eta_{15}, \overline{\nu}_8, \varepsilon_8\}\to\mathbb{Z}_2^5\{\sigma_8\circ\eta_{15}^2, (E\sigma')\circ\eta_{15}^2, \nu_8^3, \mu_8, \eta_8\circ\varepsilon_9\}.$$

Then we have $\eta_{17}^*(\sigma_8\circ\eta_{15}^2) = \sigma_8\circ(4\nu_{15}) = 4(\sigma_8\circ\nu_{15})$, $\eta_{17}^*((E\sigma')\circ\eta_{15}^2) = E(\eta_{17}^*(\sigma'\circ\eta_{14}^2)) = 4\nu_8\circ\sigma_{11}$, $\eta_{17}^*(\nu_8^3) = 0$, $\eta_{17}^*(\mu_8) = \eta_8\circ\mu_9$, $\eta_{17}^*(\eta_8\circ\varepsilon_9) = 4\nu_8\circ\sigma_{11}$, $\eta_{16}^*(\sigma_8\circ\eta_{15}) = \sigma_8\circ\eta_{15}^2$, $\eta_{16}^*((E\sigma')\circ\eta_{15}) = (E\sigma')\circ\eta_{15}^2$, $\eta_{16}^*(\overline{\nu}_8) = \nu_8^3$ and $\eta_{16}^*(\varepsilon_8) = \eta_8\circ\varepsilon_9$ by [18, (2.2)], [19, (5.5), (5.9), (7.10), (7.19), Lemma 6.6]. Thus we have

$$\mathrm{Coker}\eta_{17}^* = \mathbb{Z}_4^2\{\sigma_8\circ\nu_{15}, \nu_8\circ\sigma_{11}\}\oplus\mathbb{Z}_3^2$$

and

$$\mathrm{Ker}\eta_{16}^* = 0.$$

So we have a short exact sequence

$$0\to\mathbb{Z}_4^2\{\sigma_8\circ\nu_{15}, \nu_8\circ\sigma_{11}\}\oplus\mathbb{Z}_3^2 \xrightarrow{\Sigma^{14}p^*} [\Sigma^{14}\mathbb{C}P^2, S^8] \xrightarrow{\Sigma^{14}i^*} 0.$$

Thus we have

$$[\Sigma^{14}\mathbb{C}P^2, S^8] = \mathbb{Z}_4^2\{\sigma_8\circ\nu_{15}\circ\Sigma^{14}p, \nu_8\circ\sigma_{11}\circ\Sigma^{14}p\}\oplus\mathbb{Z}_3^2.$$

(2) Consider the following short exact sequence

$$0\to \mathrm{Coker}\eta_{18}^* \xrightarrow{\Sigma^{15}p^*} [\Sigma^{15}\mathbb{C}P^2, S^9] \xrightarrow{\Sigma^{15}i^*} \mathrm{Ker}\eta_{17}^*\to 0,$$

where $\eta_{18}^* : \pi_{18}(S^9) \to \pi_{19}(S^9)$ or, more precisely

$$\eta_{18}^* : \mathbb{Z}_2^4\{\sigma_9 \circ \eta_{16}^2, \nu_9^3, \mu_9, \eta_9 \circ \varepsilon_{10}\} \to \mathbb{Z}_8\{\sigma_9 \circ \nu_{16}\} \oplus \mathbb{Z}_2\{\eta_9 \circ \mu_{10}\} \oplus \mathbb{Z}_3$$

and $\eta_{17}^* : \pi_{17}(S^9) \to \pi_{18}(S^9)$ or, more precisely

$$\eta_{17}^* : \mathbb{Z}_2^3\{\sigma_9 \circ \eta_{16}, \overline{\nu}_9, \varepsilon_9\} \to \mathbb{Z}_2^4\{\sigma_9 \circ \eta_{16}^2, \nu_9^3, \mu_9, \eta_9 \circ \varepsilon_{10}\}.$$

Then we have $\eta_{18}^*(\sigma_9 \circ \eta_{16}^2) = 4\sigma_9 \circ \nu_{16}$, $\eta_{18}^*(\nu_9^3) = 0$, $\eta_{18}^*(\mu_9) = \eta_9 \circ \mu_{10}$, $\eta_{18}^*(\eta_9 \circ \varepsilon_{10}) = 4\nu_9 \circ \sigma_{12} = 0$, $\eta_{17}^*(\sigma_9 \circ \eta_{16}) = \sigma_9 \circ \eta_{16}^2$, $\eta_{17}^*(\overline{\nu}_9) = \nu_9^3$ by and $\eta_{17}^*(\varepsilon_9) = \eta_9 \circ \varepsilon_{10}$ by [18, (2.2)], [19, (5.5), (5.9), (7.10), (7.19), (7.20), Lemma 5.14, 6.6,]. Thus we have

$$\mathrm{Coker}\eta_{18}^* = \mathbb{Z}_4\{\sigma_9 \circ \nu_{16}\} \oplus \mathbb{Z}_3$$

and

$$\mathrm{Ker}\eta_{17}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\sigma_9 \circ \nu_{16}\} \oplus \mathbb{Z}_3 \xrightarrow{\Sigma^{15}p^*} [\Sigma^{15}\mathbb{C}P^2, S^9] \xrightarrow{\Sigma^{15}i^*} 0.$$

Thus we have

$$[\Sigma^{15}\mathbb{C}P^2, S^9] = \mathbb{Z}_4\{\sigma_9 \circ \nu_{16} \circ \Sigma^{15}p\} \oplus \mathbb{Z}_3.$$

(3) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{19}^* \xrightarrow{\Sigma^{16}p^*} [\Sigma^{16}\mathbb{C}P^2, S^{10}] \xrightarrow{\Sigma^{16}i^*} \mathrm{Ker}\eta_{18}^* \to 0,$$

where $\eta_{19}^* : \pi_{19}(S^{10}) \to \pi_{20}(S^{10})$ or, more precisely

$$\eta_{19}^* : \mathbb{Z}\{P(\iota_{21})\} \oplus \mathbb{Z}_2^3\{\nu_{10}^3, \mu_{10}, \eta_{10} \circ \varepsilon_{11}\} \to \mathbb{Z}_4\{\sigma_{10} \circ \nu_{17}\} \oplus \mathbb{Z}_2\{\eta_{10} \circ \mu_{11}\} \oplus \mathbb{Z}_3$$

and $\eta_{18}^* : \pi_{18}(S^{10}) \to \pi_{19}(S^{10})$ or, more precisely

$$\eta_{18}^* : \mathbb{Z}_2^2\{\overline{\nu}_{10}, \varepsilon_{10}\} \to \mathbb{Z}\{\triangle(\iota_{21})\} \oplus \mathbb{Z}_2^3\{\nu_{10}^3, \mu_{10}, \eta_{10} \circ \varepsilon_{12}\}.$$

Then we have $\eta_{19}^*(P(\iota_{21})) = P(\eta_{21}) = 2\sigma_{10} \circ \nu_{17}$, $\eta_{19}^*(\nu_{10}^3) = 0$, $\eta_{19}^*(\mu_{10}) = \eta_9 \circ \mu_{10}$, $\eta_{19}^*(\eta_{10} \circ \varepsilon_{11}) = 4\nu_{10} \circ \sigma_{13} = 0$, $\eta_{18}^*(\overline{\nu}_{10}) = \nu_{10}^3$ and $\eta_{18}^*(\varepsilon_{10}) = \eta_{10} \circ \varepsilon_{11}$ by [18, (2.2)], [19, (5.9),(7.5),(7.10),(7.21), Lemma 6.3]. Thus we have

$$\mathrm{Coker}\eta_{19}^* = \mathbb{Z}_2\{\sigma_{10} \circ \nu_{17}\} \oplus \mathbb{Z}_3$$

and

$$\mathrm{Ker}\eta_{18}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_2\{\sigma_{10} \circ \nu_{17}\} \oplus \mathbb{Z}_3 \xrightarrow{\Sigma^{16}p^*} [\Sigma^{16}\mathbb{C}P^2, S^{10}] \xrightarrow{\Sigma^{16}i^*} 0.$$

Thus we have

$$[\Sigma^{16}\mathbb{C}P^2, S^{10}] = \mathbb{Z}_2\{\sigma_{10} \circ \nu_{17} \circ \Sigma^{16}p\} \oplus \mathbb{Z}_3.$$

(4) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{20}^* \xrightarrow{\Sigma^{17}p^*} [\Sigma^{17}\mathbb{C}P^2, S^{11}] \xrightarrow{\Sigma^{17}i^*} \mathrm{Ker}\eta_{19}^* \to 0,$$

where $\eta_{20}^* : \pi_{20}(S^{11}) \to \pi_{21}(S^{11})$ or, more precisely

$$\eta_{20}^* : \mathbb{Z}_2^3\{\nu_{11}^3, \mu_{11}, \eta_{11} \circ \varepsilon_{12}\} \to \mathbb{Z}_2^2\{\sigma_{11} \circ \nu_{18}, \eta_{11} \circ \mu_{12}\} \oplus \mathbb{Z}_3$$

and $\eta_{19}^* : \pi_{19}(S^{11}) \to \pi_{20}(S^{11})$ or, more precisely

$$\eta_{19}^* : \mathbb{Z}_2^2\{\overline{\nu}_{11}, \varepsilon_{11}\} \to \mathbb{Z}_2^3\{\nu_{11}^3, \mu_{11}, \eta_{11} \circ \varepsilon_{12}\}.$$

Then we have $\eta_{20}^*(\nu_{11}^3) = 0$, $\eta_{20}^*(\mu_{11}) = \eta_{10} \circ \mu_{11}$, $\eta_{20}^*(\eta_{11} \circ \varepsilon_{12}) = 4\nu_{11} \circ \sigma_{14} = 0$, $\eta_{19}^*(\overline{\nu}_{11}) = \nu_{11}^3$ and $\eta_{19}^*(\varepsilon_{11}) = \eta_{11} \circ \varepsilon_{12}$ by [18, (2.2)], [19, (5.9), (7.5), (7.10), Lemma 6.3]. Thus we have

$$\mathrm{Coker}\eta_{20}^* = \mathbb{Z}_2\{\sigma_{11} \circ \nu_{18}\} \oplus \mathbb{Z}_3$$

and

$$\mathrm{Ker}\eta_{19}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_2\{\sigma_{11} \circ \nu_{18}\} \oplus \mathbb{Z}_3 \xrightarrow{\Sigma^{17}p^*} [\Sigma^{17}\mathbb{C}P^2, S^{11}] \xrightarrow{\Sigma^{17}i^*} 0.$$

Thus we have

$$[\Sigma^{17}\mathbb{C}P^2, S^{11}] = \mathbb{Z}_2\{\sigma_{11} \circ \nu_{18} \circ \Sigma^{17}p\} \oplus \mathbb{Z}_3.$$

(5) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{21}^* \xrightarrow{\Sigma^{18}p^*} [\Sigma^{18}\mathbb{C}P^2, S^{12}] \xrightarrow{\Sigma^{18}i^*} \mathrm{Ker}\eta_{20}^* \to 0,$$

where $\eta_{21}^* : \pi_{21}(S^{12}) \to \pi_{22}(S^{12})$ or, more precisely

$$\eta_{21}^* : \mathbb{Z}_2^3\{\nu_{12}^3, \mu_{12}, \eta_{12} \circ \varepsilon_{13}\} \to \mathbb{Z}_2\{\eta_{12} \circ \mu_{13}\} \oplus \mathbb{Z}_3$$

and $\eta_{20}^* : \pi_{20}(S^{12}) \to \pi_{21}(S^{12})$ or, more precisely

$$\eta_{20}^* : \mathbb{Z}_2^2\{\overline{\nu}_{12}, \varepsilon_{12}\} \to \mathbb{Z}_2^3\{\nu_{12}^3, \mu_{12}, \eta_{12} \circ \varepsilon_{13}\}.$$

Then we have $\eta_{21}^*(\nu_{12}^3) = 0$, $\eta_{21}^*(\mu_{12}) = \eta_{12} \circ \mu_{13}$, $\eta_{21}^*(\eta_{12} \circ \varepsilon_{13}) = 4\nu_{12} \circ \sigma_{15} = 0$, $\eta_{20}^*(\overline{\nu}_{12}) = \nu_{12}^3$ and $\eta_{20}^*(\varepsilon_{12}) = \eta_{12} \circ \varepsilon_{13}$ by [18, (2.2)], [19, (5.9), (7.5), (7.10), Lemma 6.3]. Thus we have

$$\mathrm{Coker}\eta_{21}^* = \mathbb{Z}_3$$

and

$$\mathrm{Ker}\eta_{20}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_3 \xrightarrow{\Sigma^{18}p^*} [\Sigma^{18}\mathbb{C}P^2, S^{12}] \xrightarrow{\Sigma^{18}i^*} 0.$$

Thus we have

$$[\Sigma^{18}\mathbb{C}P^2, S^{12}] = \mathbb{Z}_3\{\beta_1(12) \circ \Sigma^{18}p\}.$$

By the Freudenthal suspension theorem, the suspension homomorphism

$$\Sigma : [\Sigma^{n+6}\mathbb{C}P^2, S^n] \to [\Sigma^{n+7}\mathbb{C}P^2, S^{n+1}]$$

is an isomorphism for $n \geq 12$. Thus we have

$$[\Sigma^{n+6}\mathbb{C}P^2, S^n] = \mathbb{Z}_3\{\beta_1(n) \circ \Sigma^{n+6}p\}. \qquad \square$$

From the above propositions, we have the following theorem.

Theorem 1. *For $n \geq 2$, the n-th cohomotopy group of $(n+6)$-fold suspended complex projective plane has the following group structure.*

case n	2	3	4	5	6	7
$[\Sigma^{n+6}\mathbb{C}P^2, S^n]$	$2+15$	$2+3$	$8+2+3^2+5$	$4+9$	4^2+9+3	$4+3$
case n	8	9	10	11	$n \geq 12$	
$[\Sigma^{n+6}\mathbb{C}P^2, S^n]$	4^2+3^2	$4+3$	$2+3$	$2+3$	2	

4. $[\Sigma^{n+7}\mathbb{C}P^2, S^n]$ for $n \geq 2$

In this section, we compute the n-th cohomotopy groups of $(n+7)$-fold suspended complex projective plane by using (2.1).

Proposition 6. (1) $[\Sigma^9\mathbb{C}P^2, S^2] = \mathbb{Z}_2\{\eta_2 \circ \varepsilon' \circ \Sigma^9 p\} \oplus \mathbb{Z}_3$.
(2) $[\Sigma^{10}\mathbb{C}P^2, S^3] = \mathbb{Z}_2^2\{\mu' \circ \Sigma^{10}p, \varepsilon_3 \circ \nu_{11} \circ \Sigma^{10}p\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$.

Proof. (1) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{12}^* \xrightarrow{\Sigma^9 p^*} [\Sigma^9\mathbb{C}P^2, S^2] \xrightarrow{\Sigma^9 i^*} \mathrm{Ker}\eta_{11}^* \to 0,$$

where $\eta_{12}^* : \pi_{12}(S^2) \to \pi_{13}(S^2)$ or, more precisely

$$\eta_{12}^* : \mathbb{Z}_2^2\{\eta_2^2 \circ \varepsilon_4, \eta_2 \circ \mu_3\} \to \mathbb{Z}_4\{\eta_2 \circ \varepsilon'\} \oplus \mathbb{Z}_2^2\{\eta_2^2 \circ \mu_4\} \oplus \mathbb{Z}_3$$

and $\eta_{11}^* : \pi_{11}(S^2) \to \pi_{12}(S^2)$ or, more precisely

$$\eta_{11}^* : \mathbb{Z}_2\{\eta_2 \circ \varepsilon_3\} \to \mathbb{Z}_2^2\{\eta_2^2 \circ \varepsilon_4, \eta_2 \circ \mu_3\}.$$

Then we have $\eta_{12}^*(\eta_2^2 \circ \varepsilon_4) = \eta_2 \circ \eta_3^2 \circ \varepsilon_5 = \eta_2 \circ (2\varepsilon') = 2(\eta_2 \circ \varepsilon')$, $\eta_{12}^*(\eta_2 \circ \mu_3) = \eta_2^2 \circ \mu_4$, $\eta_{11}^*(\eta_2 \circ \varepsilon_3) = \eta_2^2 \circ \varepsilon_4$ by [18, (2.2)], [19, (7.5), (7.10)]. Thus we have

$$\mathrm{Coker}\eta_{12}^* = \mathbb{Z}_2\{\eta_2 \circ \varepsilon'\} \oplus \mathbb{Z}_3$$

and

$$\mathrm{Ker}\eta_{11}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_2\{\eta_2 \circ \varepsilon'\} \oplus \mathbb{Z}_3 \xrightarrow{\Sigma^9 p^*} [\Sigma^9\mathbb{C}P^2, S^2] \xrightarrow{\Sigma^9 i^*} 0.$$

Thus we have

$$[\Sigma^9\mathbb{C}P^2, S^2] = \mathbb{Z}_2\{\eta_2 \circ \varepsilon' \circ \Sigma^9 p\} \oplus \mathbb{Z}_3.$$

(2) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{13}^* \xrightarrow{\Sigma^{10} p^*} [\Sigma^{10}\mathbb{C}P^2, S^3] \xrightarrow{\Sigma^{10} i^*} \mathrm{Ker}\eta_{12}^* \to 0,$$

where $\eta_{13}^* : \pi_{13}(S^3) \to \pi_{14}(S^3)$ or, more precisely

$$\eta_{13}^* : \mathbb{Z}_4\{\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3 \circ \mu_4\} \to \mathbb{Z}_4\{\mu'\} \oplus \mathbb{Z}_2^2\{\varepsilon_3 \circ \nu_{11}, \nu' \circ \varepsilon_6\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$$

and $\eta_{12}^*: \pi_{11}(S^2) \to \pi_{12}(S^2)$ or, more precisely

$$\eta_{12}^*: \mathbb{Z}_2^2\{\mu_3, \eta_3 \circ \varepsilon_4\} \to \mathbb{Z}_4\{\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_3 \circ \mu_4\}.$$

Then we have $\eta_{13}^*(\varepsilon') = \nu' \circ \varepsilon_6$, $\eta_{13}^*(\eta_3 \circ \mu_4) = 2\mu'$, $\eta_{12}^*(\mu_3) = \eta_3 \circ \mu_4$, and $\eta_{12}^*(\eta_3 \circ \varepsilon_4) = \eta_3^2 \circ \varepsilon_5 = 2\varepsilon'$ by [18, (2.2)], [19, (7.7), (7.10), (7.12)]. Thus we have

$$\mathrm{Coker}\eta_{13}^* = \mathbb{Z}_2^2\{\mu', \varepsilon_3 \circ \nu_{11}\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7$$

and

$$\mathrm{Ker}\eta_{12}^* = 0.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_2^2\{\mu'\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7 \xrightarrow{\Sigma^{10}p^*} [\Sigma^{10}\mathbb{C}P^2, S^3] \xrightarrow{\Sigma^{10}i^*} 0.$$

Thus we have

$$[\Sigma^{10}\mathbb{C}P^2, S^3] = \mathbb{Z}_2^2\{\mu' \circ \Sigma^{10}p, \varepsilon_3 \circ \nu_{11} \circ \Sigma^{10}p\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7. \qquad \square$$

Proposition 7.

$$[\Sigma^{11}\mathbb{C}P^2, S^4] = \mathbb{Z}_4\{\nu_4^2 \circ g_{10}(\mathbb{C})\} \oplus \mathbb{Z}_2^3\{E\mu' \circ \Sigma^{11}p, \varepsilon_4 \circ \nu_{12} \circ \Sigma^{11}p, \nu_4 \circ \overline{\nu}_7 \circ \Sigma^{11}p\} \oplus \mathbb{Z}_{21}.$$

There is a relation $\nu_4 \circ \varepsilon_7 \circ \Sigma^{11}p = 2\nu_4^2 \circ 2g_{10}(\mathbb{C})$.

Proof. Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{14}^* \xrightarrow{\Sigma^{11}p^*} [\Sigma^{11}\mathbb{C}P^2, S^4] \xrightarrow{\Sigma^{11}i^*} \mathrm{Ker}\eta_{13}^* \to 0,$$

where $\eta_{14}^*: \pi_{14}(S^4) \to \pi_{15}(S^4)$ or, more precisely

$$\eta_{14}^*: \mathbb{Z}_8\{\nu_4 \circ \sigma'\} \oplus \mathbb{Z}_4\{E\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_4 \circ \mu_5\} \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5 \to$$
$$\mathbb{Z}_4\{E\mu'\} \oplus \mathbb{Z}_2^5\{E\mu', \nu_4 \circ \sigma' \circ \eta_{14}, \nu_4 \circ \overline{\nu}_7, \nu_4 \circ \varepsilon_7, \varepsilon_4 \circ \nu_{12}, (E\nu') \circ \varepsilon_7\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$$

and $\eta_{13}^*: \pi_{13}(S^4) \to \pi_{14}(S^4)$ or, more precisely,

$$\eta_{13}^*: \mathbb{Z}_2^3\{\nu_4^3, \mu_4, \eta_4 \circ \varepsilon_5\} \to \mathbb{Z}_8\{\nu_4 \circ \sigma'\} \oplus \mathbb{Z}_4\{E\varepsilon'\} \oplus \mathbb{Z}_2\{\eta_4 \circ \mu_5\} \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5.$$

Then we have $\eta_{14}^*(\nu_4 \circ \sigma') = \nu_4 \circ \sigma' \circ \eta_{14}$, $\eta_{14}^*(E\varepsilon') = E(\varepsilon' \circ \eta_{13}) = E(\nu' \circ \varepsilon_6) = (E\nu') \circ \varepsilon_7$, $\eta_{14}^*(\eta_4 \circ \mu_5) = E(\eta_3^2 \circ \mu_5) = E(2\mu') = 2E\mu'$, $\eta_{13}^*(\nu_4^3) = 0$, $\eta_{13}^*(\mu_4) = \eta_4 \circ \mu_5$ and $\eta_{13}^*(\eta_4 \circ \varepsilon_5) = E(\eta_3^2 \circ \varepsilon_5) = E(2\varepsilon') = 2E\varepsilon'$ by [18, (2.2)], [19, (5.9), (7.7), (7.10), (7.12)]. Thus we have

$$\mathrm{Coker}\eta_{14}^* = \mathbb{Z}_2^4\{E\mu', \nu_4 \circ \overline{\nu}_7, \nu_4 \circ \varepsilon_7, \varepsilon_4 \circ \nu_{12}\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$$

and

$$\mathrm{Ker}\eta_{13}^* = \mathbb{Z}_2\{\nu_4^3\}.$$

So we have a short exact sequence

$$0 \to \mathbb{Z}_2^4\{E\mu', \nu_4 \circ \overline{\nu}_7, \nu_4 \circ \varepsilon_7, \varepsilon_4 \circ \nu_{12}\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$$
$$\xrightarrow{\Sigma^{11}p^*} [\Sigma^{11}\mathbb{C}P^2, S^4] \xrightarrow{\Sigma^{11}i^*} \mathbb{Z}_2\{\nu_4^3\} \to 0.$$

By [8, Proposition 3.3, 3.6], we have $\nu_5^2\circ\overline{2\iota_{11}} = \varepsilon_5\circ\Sigma^9 p$. This implies a relation

$$2\nu_4^2\circ g_{10}(\mathbb{C}) = \nu_4^3\circ\overline{2\iota_{13}} = \nu_4\circ\varepsilon_7\circ\Sigma^{11}p.$$

Thus we have

$$[\Sigma^{11}\mathbb{C}P^2, S^4] = \mathbb{Z}_4\{\nu_4^2\circ g_{10}(\mathbb{C})\} \oplus \mathbb{Z}_2^3\{E\mu'\circ\Sigma^{11}p, \varepsilon_4\circ\nu_{12}\circ\Sigma^{11}p, \nu_4\circ\overline{\nu}_7\circ\Sigma^{11}p\}\oplus\mathbb{Z}_{21}. \quad \square$$

Proposition 8.

$$[\Sigma^{12}\mathbb{C}P^2, S^5] = \mathbb{Z}_4\{\zeta_5\circ\Sigma^{12}p\} \oplus \mathbb{Z}_2^2\{\nu_5\circ\overline{\nu}_8\circ\Sigma^{12}p, \nu_5^2\circ g_{11}(\mathbb{C})\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9.$$

Proof. Consider the following short exact sequence

$$0\to \mathrm{Coker}\eta_{15}^* \xrightarrow{\Sigma^{12}p^*} [\Sigma^{12}\mathbb{C}P^2, S^5] \xrightarrow{\Sigma^{12}i^*} \mathrm{Ker}\eta_{14}^*\to 0,$$

where $\eta_{15}^*:\pi_{15}(S^5)\to\pi_{16}(S^5)$ or, more precisely

$$\eta_{15}^*:\mathbb{Z}_8\{\nu_5\circ\sigma_8\}\oplus\mathbb{Z}_2\{\eta_5\circ\mu_6\}\oplus\mathbb{Z}_9\to\mathbb{Z}_8\{\zeta_5\}\oplus\mathbb{Z}_2^2\{\nu_5\circ\overline{\nu}_8, \nu_5\circ\varepsilon_8\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$$

and $\eta_{14}^*:\pi_{14}(S^5)\to\pi_{15}(S^5)$ or, more precisely

$$\eta_{14}^*:\mathbb{Z}_2^3\{\nu_5^3, \mu_5, \eta_5\circ\varepsilon_6\}\to\mathbb{Z}_8\{\nu_5\circ\sigma_8\}\oplus\mathbb{Z}_2\{\eta_5\circ\mu_6\}\oplus\mathbb{Z}_9.$$

Then we have $\eta_{15}^*(\nu_5\circ\sigma_8) = \nu_5\circ\varepsilon_8$, $\eta_{15}^*(\eta_5\circ\mu_6) = 4\zeta_5$, $\eta_{14}^*(\nu_5^3) = 0$, $\eta_{14}^*(\mu_5) = \eta_5\circ\mu_6$ and $\eta_{14}^*(\eta_5\circ\varepsilon_6) = 4(\nu_5\circ\sigma_8)$ by [18, (2.2)], [19, p. 152, (5.9), (7.10), (7.14)]. Thus we have

$$\mathrm{Coker}\eta_{15}^* = \mathbb{Z}_4\{\zeta_5\}\oplus\mathbb{Z}_2\{\nu_5\circ\overline{\nu}_8\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{14}^* = \mathbb{Z}_2\{\nu_5^3\}.$$

So we have a short exact sequence

$$\begin{aligned}(4.1)\qquad & 0\to\mathbb{Z}_4\{\zeta_5\}\oplus\mathbb{Z}_2\{\nu_5\circ\overline{\nu}_8\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9\\ &\xrightarrow{\Sigma^{12}p^*}[\Sigma^{12}\mathbb{C}P^2, S^5]\xrightarrow{\Sigma^{12}i^*}\mathbb{Z}_2\{\nu_5^3\}\to 0.\end{aligned}$$

Consider an EHP sequence

$$[\Sigma^{13}\mathbb{C}P^2, S^9]\xrightarrow{\Delta}[\Sigma^{11}\mathbb{C}P^2, S^4]\xrightarrow{E}[\Sigma^{12}\mathbb{C}P^2, S^5] \xrightarrow{H}[\Sigma^{12}\mathbb{C}P^2, S^9]\xrightarrow{\Delta}[\Sigma^{10}\mathbb{C}P^2, S^4],$$

where $[\Sigma^{13}\mathbb{C}P^2, S^9] = \mathbb{Z}_4\{\nu_9\circ g_{12}(\mathbb{C})\}\oplus\mathbb{Z}_2\{\overline{\nu}_9\circ\Sigma^{13}p\}$, $[\Sigma^{11}\mathbb{C}P^2, S^4] = \mathbb{Z}_4\{\nu_4^2\circ g_{10}\}\oplus\mathbb{Z}_2^3\{E\mu'\circ\Sigma^{11}p, \varepsilon_4\circ\nu_{12}\circ\Sigma^{11}p, \nu_4\circ\overline{\nu}_7\circ\Sigma^{11}p\}$, $[\Sigma^{12}\mathbb{C}P^2, S^9] = \mathbb{Z}_{16}\{\sigma_9\circ\Sigma^{12}p\}$ and $[\Sigma^{10}\mathbb{C}P^2, S^4] = \mathbb{Z}_8\{\nu_4\circ\sigma'\circ\Sigma^{10}p\}\oplus\mathbb{Z}_2\{E\varepsilon'\circ\Sigma^{10}p\}\oplus\mathbb{Z}_3^2\oplus\mathbb{Z}_5$. Then we have

$$\Delta(\nu_9\circ g_{12}(\mathbb{C})) = 2\nu_4^2\circ g_{10}(\mathbb{C}),$$
$$\Delta(\overline{\nu}_9\circ\Sigma^{13}p) = \varepsilon_4\circ\nu_{12}\Sigma^{11}p$$

and

$$\Delta(\sigma_9\circ\Sigma^{12}p) = (x\nu_4\circ\sigma'\pm E\varepsilon')\circ\Sigma^{10}p,$$

where x is odd. Then we have a short exact sequence

$$0 \to \mathbb{Z}_2^4\{\nu_4^2 \circ g_{10}(\mathbb{C}), E\mu' \circ \Sigma^{11}p, \varepsilon_4 \circ \nu_{12} \circ \Sigma^{11}p, \nu_4 \circ \overline{\nu} \circ \Sigma^{11}p\}$$
$$\xrightarrow{E} [\Sigma^{12}\mathbb{C}P^2, S^5] \xrightarrow{H} \mathbb{Z}_2\{8\sigma_9 \circ \Sigma^{12}p\} \to 0.$$

By [19, Lemma 6.7, (7.14)], we have $H(\zeta_5) = 8\sigma_9$ and $E^2\mu' = 2\zeta_5$. Thus we obtain

$$[\Sigma^{12}\mathbb{C}P^2, S^5] = \mathbb{Z}_4\{\zeta_5 \circ \Sigma^{12}p\} \oplus \mathbb{Z}_2^2\{\nu_5 \circ \overline{\nu}_8 \circ \Sigma^{12}p, \nu_5^2 \circ g_{11}(\mathbb{C})\}. \qquad \square$$

Proposition 9.

$$[\Sigma^{13}\mathbb{C}P^2, S^6] = \mathbb{Z}_4\{\zeta_6 \circ \Sigma^{13}p\}$$
$$\oplus \mathbb{Z}_2^2\{\overline{\nu}_6 \circ \nu_{14} \circ \Sigma^{13}p, \nu_6^2 \circ g_{12}(\mathbb{C})\} \oplus \mathbb{Z}_{63}.$$

Proof. Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{16}^* \xrightarrow{\Sigma^{13}p^*} [\Sigma^{13}\mathbb{C}P^2, S^6] \xrightarrow{\Sigma^{13}i^*} \mathrm{Ker}\eta_{15}^* \to 0,$$

where $\eta_{16}^* : \pi_{16}(S^6) \to \pi_{17}(S^6)$ or, more precisely

$$\eta_{16}^* : \mathbb{Z}_8\{\nu_6 \circ \sigma_9\} \oplus \mathbb{Z}_2\{\eta_6 \circ \mu_7\} \oplus \mathbb{Z}_9 \to \mathbb{Z}_8\{\zeta_6\} \oplus \mathbb{Z}_4\{\overline{\nu}_6 \circ \nu_{14}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$$

and $\eta_{15}^* : \pi_{15}(S^6) \to \pi_{16}(S^6)$ or, more precisely

$$\eta_{15}^* : \mathbb{Z}_2^3\{\nu_6^3, \mu_6, \eta_6 \circ \varepsilon_7\} \to \mathbb{Z}_8\{\nu_6 \circ \sigma_9\} \oplus \mathbb{Z}_2\{\eta_6 \circ \mu_7\} \oplus \mathbb{Z}_9.$$

Then we have $\eta_{16}^*(\nu_6 \circ \sigma_9) = \nu_6 \circ \varepsilon_9 = 2\overline{\nu}_6 \circ \nu_{14}$, $\eta_{16}^*(\eta_6 \circ \mu_7) = 4\zeta_6$, $\eta_{15}^*(\nu_6^3) = 0$, $\eta_{15}^*(\mu_6) = \eta_6 \circ \mu_7$ and $\eta_{15}^*(\eta_6 \circ \varepsilon_7) = \eta_6^2 \circ \varepsilon_8 = 4(\nu_6 \circ \sigma_9)$ by [18, (2.2)], [19, p. 70, p. 152, (5.9), (7.10), (7.14)]. Thus we have

$$\mathrm{Coker}\eta_{16}^* = \mathbb{Z}_4\{\zeta_6\} \oplus \mathbb{Z}_2\{\overline{\nu}_6 \circ \nu_{14}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{15}^* = \mathbb{Z}_2\{\nu_6^3\}.$$

Now we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\zeta_6\} \oplus \mathbb{Z}_2\{\overline{\nu}_6 \circ \nu_{14}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9 \xrightarrow{\Sigma^{13}p^*} [\Sigma^{13}\mathbb{C}P^2, S^6] \xrightarrow{\Sigma^{13}i^*} \mathbb{Z}_2\{\nu_6^3\} \to 0.$$

Thus we have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_4\{\zeta_5\} \oplus \mathbb{Z}_2\{\nu_5 \circ \overline{\nu}_8\} \arrow[r, "\Sigma^{12}p^*"] \arrow[d, "\Sigma_1"] & {[\Sigma^{12}\mathbb{C}P^2, S^5]} \arrow[r, "\Sigma^{12}i^*"] \arrow[d, "\Sigma_2"'] & \mathbb{Z}_2\{\nu_5^3\} \arrow[r] \arrow[d, "\Sigma_3"'] & 0 \\
0 \arrow[r] & \mathbb{Z}_4\{\zeta_6\} \oplus \mathbb{Z}_2\{\overline{\nu}_6 \circ \nu_{14}\} \arrow[r, "\Sigma^{13}p^*"] & {[\Sigma^{13}\mathbb{C}P^2, S^6]} \arrow[r, "\Sigma^{13}i^*"] & \mathbb{Z}_2\{\nu_6^3\} \arrow[r] & 0
\end{tikzcd}$$

Since the first row is split and Σ_3 is an isomorphism, the second row also split.

$$[\Sigma^{13}\mathbb{C}P^2, S^6] = \mathbb{Z}_4\{\zeta_6 \circ \Sigma^{13}p\} \oplus \mathbb{Z}_2^2\{\overline{\nu}_6 \circ \nu_{14} \circ \Sigma^{13}p, \nu_6^2 \circ g_{12}(\mathbb{C})\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9.$$
$\square$

Lemma 1.

$$[\Sigma^{14}\mathbb{C}P^2, S^7]_{(2)} \cong [\Sigma^{15}\mathbb{C}P^2, S^8]_{(2)} \cong [\Sigma^{16}\mathbb{C}P^2, S^9]_{(2)}.$$

Proof. By (2.4) and Proposition 3.2 of [8], we have

$$[\Sigma^{14}\mathbb{C}P^2, S^7] \cong [\Sigma^{15}\mathbb{C}P^2, S^8].$$

Consider the following EHP sequence

$$[\Sigma^{17}\mathbb{C}P^2, S^{17}]_{(2)} \xrightarrow{\triangle} [\Sigma^{15}\mathbb{C}P^2, S^8]_{(2)} \xrightarrow{E} [\Sigma^{16}\mathbb{C}P^2, S^9]_{(2)}$$
$$\xrightarrow{H} [\Sigma^{16}\mathbb{C}P^2, S^{17}]_{(2)} \xrightarrow{\triangle} [\Sigma^{14}\mathbb{C}P^2, S^8]_{(2)},$$

where $[\Sigma^{17}\mathbb{C}P^2, S^{17}] = 0$ [8, Proposition 3.2], $[\Sigma^{16}\mathbb{C}P^2, S^{17}]_{(2)} = \mathbb{Z}_4\{\nu_{17}\circ\Sigma^{16}p\}$ [8, Proposition 3.1] and $[\Sigma^{14}\mathbb{C}P^2, S^8]_{(2)} = \mathbb{Z}_4^2\{\sigma_8\circ\nu_{15}\circ\Sigma^{14}p, \nu_8\circ\sigma_{11}\circ\Sigma^{14}p\}$ by Proposition 5. By exactness we have $E : [\Sigma^{15}\mathbb{C}P^2, S^8]_{(2)} \to [\Sigma^{16}\mathbb{C}P^2, S^9]_{(2)}$ is injective. By (7.19) of [19] we have

$$\triangle(\nu_{17}\circ\Sigma^{16}p) = \triangle(\nu_{17})\circ\Sigma^{14}p = 2\sigma_8\circ\nu_{15}\circ\Sigma^{14}p - x\nu_8\circ\sigma_{11}\circ\Sigma^{14}p,$$

where x is odd. Thus $\triangle : [\Sigma^{16}\mathbb{C}P^2, S^{17}]_{(2)} \to [\Sigma^{14}\mathbb{C}P^2, S^8]_{(2)}$ is injective, so that $E : [\Sigma^{15}\mathbb{C}P^2, S^8] \to [\Sigma^{16}\mathbb{C}P^2, S^9]$ is surjective. Therefore the homomorphism $E : [\Sigma^{15}\mathbb{C}P^2, S^8]_{(2)} \to [\Sigma^{16}\mathbb{C}P^2, S^9]_{(2)}$ is an isomorphism. □

Proposition 10. (A) $[\Sigma^{14}\mathbb{C}P^2, S^7] = \mathbb{Z}_8\{\overline{\sigma'\circ\eta_{14}^2 + \eta_7\circ\varepsilon_8}\} \oplus \mathbb{Z}_2^2\{\bar{\nu}_7\circ\nu_{15}\circ\Sigma^{14}p, \nu_7^2\circ g_{13}(\mathbb{C})\}$.
Relation: $2\overline{\sigma'\circ\eta_{14}^2 + \eta_7\circ\varepsilon_8} = a\zeta_7\circ\Sigma^{14}p$ *for* a *odd.*

(B) $[\Sigma^{15}\mathbb{C}P^2, S^8] = \mathbb{Z}_8\{\overline{(E\sigma')\circ\eta_{15}^2 + \eta_8\circ\varepsilon_9}\} \oplus \mathbb{Z}_2^2\{\bar{\nu}_8\circ\nu_{16}\circ\Sigma^{15}p, \nu_8^2\circ g_{14}(\mathbb{C})\}$.
Relation: $2\overline{(E\sigma')\circ\eta_{15}^2 + \eta_8\circ\varepsilon_9} = a\zeta_8\circ\Sigma^{15}p$ *for* a *odd.*

(C) $[\Sigma^{16}\mathbb{C}P^2, S^9] = \mathbb{Z}_8\{\overline{\eta_9\circ\varepsilon_{10}}\} \oplus \mathbb{Z}_2^2\{\bar{\nu}_9\circ\nu_{17}\circ\Sigma^{16}p, \nu_9^2\circ g_{15}(\mathbb{C})\}$.
Relation: $2\overline{\eta_9\circ\varepsilon_{10}} = a\zeta_9\circ\Sigma^{16}p$ *for* a *odd.*

Proof. (A) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{17}^* \xrightarrow{\Sigma^{14}p^*} [\Sigma^{14}\mathbb{C}P^2, S^7] \xrightarrow{\Sigma^{14}i^*} \mathrm{Ker}\eta_{16}^* \to 0,$$

where $\eta_{17}^* : \pi_{17}(S^7) \to \pi_{18}(S^7)$ or, more precisely

$$\eta_{17}^* : \mathbb{Z}_8\{\nu_7\circ\sigma_{10}\} \oplus \mathbb{Z}_2\{\eta_7\circ\mu_8\} \oplus \mathbb{Z}_3 \to \mathbb{Z}_8\{\zeta_7\} \oplus \mathbb{Z}_2\{\bar{\nu}_7\circ\nu_{15}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$$

and $\eta_{16}^* : \pi_{16}(S^7) \to \pi_{17}(S^7)$ or, more precisely

$$\eta_{16}^* : \mathbb{Z}_2^4\{\sigma'\circ\eta_{14}^2, \nu_7^3, \mu_7, \eta_7\circ\varepsilon_8\} \to \mathbb{Z}_8\{\nu_7\circ\sigma_{10}\} \oplus \mathbb{Z}_2\{\eta_7\circ\mu_8\} \oplus \mathbb{Z}_3.$$

Then we have $\eta_{17}^*(\nu_7\circ\sigma_{10}) = \nu_7\circ\varepsilon_{10}$, $\eta_{17}^*(\eta_7\circ\mu_8) = 4\zeta_6$, $\eta_{16}^*(\sigma'\circ\eta_{14}^2) = \sigma'\circ\eta_{14}^3 = \sigma'\circ 4\nu_{14} = 4\sigma'\circ\nu_{14} = 4(\nu_7\circ\sigma_{10})$, $\eta_{16}^*(\nu_7^3) = 0$, $\eta_{16}^*(\mu_7) = \eta_7\circ\mu_8$ and $\eta_{16}^*(\eta_7\circ\varepsilon_8) = 4(\nu_7\circ\sigma_{10})$ by [18, (2.2)], [19, p. 152, (5.5), (7.10), (7.14), (7.19)]. Thus we have

$$\mathrm{Coker}\eta_{17}^* = \mathbb{Z}_4\{\zeta_7\} \oplus \mathbb{Z}_2\{\bar{\nu}_7\circ\nu_{15}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{16}^* = \mathbb{Z}_2^2\{\nu_7^3, \sigma'\circ\eta_{14}^2 + \eta_7\circ\varepsilon_8\}.$$

Thus we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\zeta_7\} \oplus \mathbb{Z}_2\{\bar{\nu}_7\circ\nu_{15}\} \oplus \mathbb{Z}_{63} \xrightarrow{\Sigma^{14}p^*} [\Sigma^{14}\mathbb{C}P^2, S^7]$$

$$\xrightarrow{\Sigma^{14}i^*} \mathbb{Z}_2^2\{\nu_7^3, \sigma'\circ\eta_{14}^2+\eta_7\circ\varepsilon_8\} \to 0.$$

(B) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{18}^* \xrightarrow{\Sigma^{15}p^*} [\Sigma^{15}\mathbb{C}P^2, S^8] \xrightarrow{\Sigma^{15}i^*} \mathrm{Ker}\eta_{17}^* \to 0,$$

where $\eta_{18}^* : \pi_{18}(S^8) \to \pi_{19}(S^8)$,

$\eta_{18}^* : \mathbb{Z}_8^2\{\sigma_8\circ\nu_{15}, \nu_8\circ\sigma_{11}\}\oplus\mathbb{Z}_2\{\eta_8\circ\mu_9\}\oplus\mathbb{Z}_3^2 \to \mathbb{Z}_8\{\zeta_8\}\oplus\mathbb{Z}_2\{\overline{\nu}_8\circ\nu_{16}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$

and $\eta_{17}^* : \pi_{17}(S^8) \to \pi_{18}(S^8)$,

$\eta_{17}^* : \mathbb{Z}_2^5\{\sigma_8\circ\eta_{15}^2, (E\sigma')\circ\eta_{15}^2, \nu_8^3, \mu_8, \eta_8\circ\varepsilon_9\} \to \mathbb{Z}_8^2\{\sigma_8\circ\nu_{15}, \nu_8\circ\sigma_{11}\}\oplus\mathbb{Z}_2\{\eta_8\circ\mu_9\}\oplus\mathbb{Z}_3^2.$

Then we have $\eta_{18}^*(\sigma_8\circ\nu_{15}) = 0$, $\eta_{18}^*(\nu_8\circ\sigma_{11}) = \nu_8\circ\sigma_{11}\circ\eta_{18} = \nu_8\circ\eta_{11}\circ\sigma_{12} = 0$, $\eta_{18}^*(\eta_8\circ\mu_9) = 4\zeta_8$, $\eta_{17}^*(\sigma_8\circ\eta_{15}^2) = \sigma_8\circ\eta_{15}^3 = \sigma_8\circ 4\nu_{15} = 4\sigma_8\circ\nu_{15}$, $\eta_{17}^*((E\sigma')\circ\eta_{15}^2) = 4\nu_8\circ\sigma_{11}$, $\eta_{17}^*(\nu_8^3) = 0$, $\eta_{17}^*(\mu_8) = \eta_8\circ\mu_9$ and $\eta_{17}^*(\eta_8\circ\varepsilon_9) = 4(\nu_8\circ\sigma_{11})$ by [18, (2.2)], [19, Lemma 6.4, (5.9), (7.5), (7.10), (7.14)]. Thus we have

$$\mathrm{Coker}\eta_{18}^* = \mathbb{Z}_4\{\zeta_8\}\oplus\mathbb{Z}_2\{\overline{\nu}_8\circ\nu_{16}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9.$$

and

$$\mathrm{Ker}\eta_{17}^* = \mathbb{Z}_2^2\{\nu_8^3, (E\sigma')\circ\eta_{15}^2+\eta_8\circ\varepsilon_9\}.$$

Thus we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\zeta_8\}\oplus\mathbb{Z}_2\{\overline{\nu}_8\circ\nu_{16}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9 \xrightarrow{\Sigma^{15}p^*} [\Sigma^{15}\mathbb{C}P^2, S^8]$$
$$\xrightarrow{\Sigma^{15}i^*} \mathbb{Z}_2^2\{\nu_8^3, (E\sigma')\circ\eta_{15}^2+\eta_8\circ\varepsilon_9\} \to 0.$$

(C) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{19}^* \xrightarrow{\Sigma^{16}p^*} [\Sigma^{16}\mathbb{C}P^2, S^9] \xrightarrow{\Sigma^{16}i^*} \mathrm{Ker}\eta_{18}^* \to 0,$$

where $\eta_{19}^* : \pi_{19}(S^9) \to \pi_{20}(S^9)$ or, more precisely

$$\eta_{19}^* : \mathbb{Z}_8\{\sigma_9\circ\nu_{16}\}\oplus\mathbb{Z}_2\{\eta_9\circ\mu_{10}\}\oplus\mathbb{Z}_3 \to \mathbb{Z}_8\{\zeta_8\}\oplus\mathbb{Z}_2\{\overline{\nu}_8\circ\nu_{16}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$$

and $\eta_{18}^* : \pi_{18}(S^9) \to \pi_{19}(S^9)$ or, more precisely

$$\eta_{18}^* : \mathbb{Z}_2^4\{\sigma_9\circ\eta_{16}^2, \nu_9^3, \mu_9, \eta_9\circ\varepsilon_{10}\} \to \mathbb{Z}_8\{\sigma_9\circ\nu_{16}\}\oplus\mathbb{Z}_2\{\eta_9\circ\mu_{10}\}\oplus\mathbb{Z}_3.$$

Then we have $\eta_{19}^*(\sigma_9\circ\nu_{16}) = 0$, $\eta_{19}^*(\eta_9\circ\mu_{10}) = 4\zeta_9$, $\eta_{18}^*(\sigma_9\circ\eta_{16}^2) = \sigma_9\circ\eta_{16}^3 = \sigma_9\circ 4\nu_{16} = 4\sigma_9\circ\nu_{16}$, $\eta_{18}^*(\nu_9^3) = 0$, $\eta_{18}^*(\mu_9) = \eta_9\circ\mu_{10}$ and $\eta_{18}^*(\eta_9\circ\varepsilon_{10}) = 4(\nu_9\circ\sigma_{12})$ by [18, (2.2)], [19, Lemma 6.4, (5.5), (5.9), (7.5), (7.10), (7.14)]. Thus we have

$$\mathrm{Coker}\eta_{19}^* = \mathbb{Z}_4\{\zeta_9\}\oplus\mathbb{Z}_2\{\overline{\nu}_9\circ\nu_{17}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{18}^* = \mathbb{Z}_2^2\{\nu_9^3, \eta_9\circ\varepsilon_{10}\}.$$

Thus we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\zeta_9\}\oplus\mathbb{Z}_2\{\overline{\nu}_9\circ\nu_{17}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9 \xrightarrow{\Sigma^{16}p^*} [\Sigma^{16}\mathbb{C}P^2, S^9]$$
$$\xrightarrow{\Sigma^{16}i^*} \mathbb{Z}_2^2\{\nu_9^3, \eta_9\circ\varepsilon_{10}\} \to 0.$$

In the proof of $[\Sigma^{15}\mathbb{C}P^2, S^8]$, there is an extension $\overline{(E\sigma')\circ\eta_{15}^2+\eta_8\circ\varepsilon_9}\in[\Sigma^{15}\mathbb{C}P^2, S^8]$ of $(E\sigma')\circ\eta_{15}^2+\eta_8\circ\varepsilon_9$. By [19, Lemma 5.14, (2.1)] we have

$$E((E\sigma')\circ\eta_{15}^2)=(E^2\sigma')\circ\eta_{16}^2=(2\sigma_9)\circ\eta_{16}^2=\sigma_9\circ(2\eta_{16}^2)=0.$$

Thus we obtain

$$E(\overline{(E\sigma')\circ\eta_{15}^2+\eta_8\circ\varepsilon_9})=\overline{\eta_9\circ\varepsilon_{10}}.$$

This implies that $2\overline{\eta_9\circ\varepsilon_{10}}=2\iota_9\circ\overline{\eta_9\circ\varepsilon_{10}}$ by [19, (2.1)]. By [19, Proposition 1.9, Lemma 9.1] we have

$$2\iota_9\circ\overline{\eta_9\circ\varepsilon_{10}}\in\{2\iota_9,\eta_9\circ\varepsilon_{10},\eta_{18}\}\circ\Sigma^{16}p\ni\zeta_9\circ\Sigma^{16}p \mod 2\zeta_9\circ\Sigma^{16}p,\ \overline{\nu}_9\circ\nu_{17}\circ\Sigma^{16}p$$

that is,

$$2\overline{\eta_9\circ\varepsilon_{10}}\equiv a\zeta_9\circ\Sigma^{16}p \mod \overline{\nu}_9\circ\nu_{17}\circ\Sigma^{16}p$$

for a odd. Since $2(\nu_9^2\circ\overline{\nu_{15}})=2(\nu_9^2\circ E\overline{\nu_{14}})=(2\nu_9^2)\circ\overline{\nu_{15}}=0$ by [19, (2.1)], $\nu_9^2\circ\overline{\nu_{15}}$ is of order 2.

Thus we have

$$[\Sigma^{16}\mathbb{C}P^2, S^9]=\mathbb{Z}_8\{\overline{\eta_9\circ\varepsilon_{10}}\}\oplus\mathbb{Z}_2^2\{\overline{\nu}_9\circ\nu_{17}\circ\Sigma^{16}p,\nu_9^2\circ g_{15}(\mathbb{C})\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9.$$

By Lemma 1, we have

$$[\Sigma^{14}\mathbb{C}P^2, S^7]=\mathbb{Z}_8\{\overline{\sigma'\circ\eta_{14}^2+\eta_7\circ\varepsilon_8}\}\oplus\mathbb{Z}_2^2\{\overline{\nu}_7\circ\nu_{15}\circ\Sigma^{14}p,\nu_7^2\circ g_{13}(\mathbb{C})\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$$

and

$$[\Sigma^{15}\mathbb{C}P^2, S^8]=\mathbb{Z}_8\{\overline{(E\sigma')\circ\eta_{15}^2+\eta_8\circ\varepsilon_9}\}\oplus\mathbb{Z}_2^2\{\overline{\nu}_8\circ\nu_{16}\circ\Sigma^{15}p,\nu_8^2\circ g_{14}(\mathbb{C})\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9.$$

□

Proposition 11. (1) $[\Sigma^{17}\mathbb{C}P^2, S^{10}]=\mathbb{Z}\{\overline{P(\iota_{21})}\}\oplus\mathbb{Z}_8\{\overline{\eta_{10}\circ\varepsilon_{11}}\}\oplus\mathbb{Z}_2\{\nu_{10}^2\circ\overline{\nu_{16}}\}\oplus\mathbb{Z}_{63}$.

(2) $[\Sigma^{18}\mathbb{C}P^2, S^{11}]=\mathbb{Z}_8\{\overline{\eta_{11}\circ\varepsilon_{12}}\}\oplus\mathbb{Z}_2\{\nu_{11}^2\circ\overline{\nu_{17}}\}\oplus\mathbb{Z}_{63}$.

(3) $[\Sigma^{19}\mathbb{C}P^2, S^{12}]=\mathbb{Z}\{P(\iota_{25})\circ\Sigma^{19}p\}\oplus\mathbb{Z}_8\{\overline{\eta_{12}\circ\varepsilon_{13}}\}\oplus\mathbb{Z}_2\{\nu_{12}^2\circ\overline{\nu_{18}}\}\oplus\mathbb{Z}_{63}$.

(4) *For* $n\geq 13$, $[\Sigma^{n+7}\mathbb{C}P^2, S^n]=\mathbb{Z}_8\{\overline{\eta_n\circ\varepsilon_{n+1}}\}\oplus\mathbb{Z}_2\{\nu_n^2\circ\overline{\nu_{n+6}}\}\oplus\mathbb{Z}_{63}$.

Proof. (1) Consider the following short exact sequence

$$0\to\mathrm{Coker}\eta_{20}^*\xrightarrow{\Sigma^{17}p^*}[\Sigma^{17}\mathbb{C}P^2, S^{10}]\xrightarrow{\Sigma^{17}i^*}\mathrm{Ker}\eta_{19}^*\to 0,$$

where $\eta_{20}^*:\pi_{20}(S^{10})\to\pi_{21}(S^{10})$ or, more precisely

$$\eta_{20}^*:\mathbb{Z}_4\{\sigma_{10}\circ\nu_{17}\}\oplus\mathbb{Z}_2\{\eta_{10}\circ\mu_{11}\}\oplus\mathbb{Z}_3\to\mathbb{Z}_8\{\zeta_{10}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$$

and $\eta_{19}^*:\pi_{19}(S^{10})\to\pi_{20}(S^{10})$ or, more precisely

$$\eta_{19}^*:\mathbb{Z}\{\triangle(\iota_{21})\}\oplus\mathbb{Z}_2^3\{\nu_{10}^3,\mu_{10},\eta_{10}\circ\varepsilon_{11}\}\to\mathbb{Z}_4\{\sigma_{10}\circ\nu_{17}\}\oplus\mathbb{Z}_2\{\eta_{10}\circ\mu_{11}\}\oplus\mathbb{Z}_3.$$

Then we have $\eta_{20}^*(\sigma_{10}\circ\nu_{17})=0$, $\eta_{20}^*(\eta_{10}\circ\mu_{11})=4\zeta_{10}$, $\eta_{19}^*(P(\iota_{21}))=P(\eta_{21})=2\sigma_{10}\circ\nu_{17}$, $\eta_{19}^*(\nu_{10}^3)=0$, $\eta_{19}^*(\mu_{10})=\eta_{10}\circ\mu_{11}$ and $\eta_{19}^*(\eta_{10}\circ\varepsilon_{11})=4(\nu_{10}\circ\sigma_{13})=0$ by [18, (2.2)], [19, Lemma 6.4, (5.5), (5.9), (7.5), (7.10), (7.14), (7.21)]. Thus we have

$$\mathrm{Coker}\eta_{20}^*=\mathbb{Z}_4\{\zeta_{10}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{19}^* = \mathbb{Z}\{2P(\iota_{21})\} \oplus \mathbb{Z}_2^2\{\nu_{10}^3, \eta_{10} \circ \varepsilon_{11}\}.$$

Thus we have a short exact sequence

$$\begin{aligned} 0 \to \mathbb{Z}_4\{\zeta_{10}\} \oplus \mathbb{Z}_{63} &\xrightarrow{\Sigma^{17}p^*} [\Sigma^{17}\mathbb{C}P^2, S^{10}] \\ &\xrightarrow{\Sigma^{17}i^*} \mathbb{Z}\{2P(\iota_{21})\} \oplus \mathbb{Z}_2^2\{\nu_{10}^3, \eta_{10} \circ \varepsilon_{11}\} \to 0. \end{aligned}$$

By Proposition 10, we have

$$2\overline{\eta_{10} \circ \varepsilon_{11}} = E(2\overline{\eta_9 \circ \varepsilon_{10}}) = a\zeta_{10} \circ \Sigma^{17}p,$$

where a odd.

(2) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{21}^* \xrightarrow{\Sigma^{18}p^*} [\Sigma^{18}\mathbb{C}P^2, S^{11}] \xrightarrow{\Sigma^{18}i^*} \mathrm{Ker}\eta_{20}^* \to 0,$$

where $\eta_{21}^* : \pi_{21}(S^{11}) \to \pi_{22}(S^{11})$ or, more precisely

$$\eta_{21}^* : \mathbb{Z}_2^2\{\sigma_{11} \circ \nu_{18}, \eta_{11} \circ \mu_{12}\} \oplus \mathbb{Z}_3 \to \mathbb{Z}_8\{\zeta_{11}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$$

and $\eta_{20}^* : \pi_{20}(S^{11}) \to \pi_{21}(S^{11})$ or, more precisely

$$\eta_{20}^* : \mathbb{Z}_2^3\{\nu_{11}^3, \mu_{11}, \eta_{11} \circ \varepsilon_{12}\} \to \mathbb{Z}_2^2\{\sigma_{11} \circ \nu_{18}, \eta_{11} \circ \mu_{12}\} \oplus \mathbb{Z}_3.$$

Then we have $\eta_{21}^*(\sigma_{11} \circ \nu_{18}) = 0$, $\eta_{21}^*(\eta_{11} \circ \mu_{12}) = 4\zeta_{11}$, $\eta_{20}^*(\nu_{11}^3) = 0$, $\eta_{20}^*(\mu_{11}) = \eta_{11} \circ \mu_{12}$ and $\eta_{20}^*(\eta_{11} \circ \varepsilon_{12}) = 4(\nu_{11} \circ \sigma_{14}) = 0$ by [18, (2.2)], [19, (5.9), (7.10), (7.14), (7.20)]. Thus we have

$$\mathrm{Coker}\eta_{21}^* = \mathbb{Z}_4\{\zeta_{11}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{20}^* = \mathbb{Z}_2^2\{\nu_{11}^3, \eta_{11} \circ \varepsilon_{12}\}.$$

Thus we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\zeta_{11}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9 \xrightarrow{\Sigma^{18}p^*} [\Sigma^{18}\mathbb{C}P^2, S^{11}] \xrightarrow{\Sigma^{18}i^*} \mathbb{Z}_2^2\{\nu_{11}^3, \eta_{11} \circ \varepsilon_{12}\} \to 0.$$

By Proposition 10, we have $2\overline{\eta_{11} \circ \varepsilon_{12}} = a\zeta_{11} \circ \Sigma^{18}p$ where a odd. Since ν_{11}^2 has order 2, we have $\nu_{11}^2 \circ \overline{\nu_{17}}$ has order 2.

(3) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{22}^* \xrightarrow{\Sigma^{19}p^*} [\Sigma^{19}\mathbb{C}P^2, S^{12}] \xrightarrow{\Sigma^{19}i^*} \mathrm{Ker}\eta_{21}^* \to 0,$$

where $\eta_{22}^* : \pi_{22}(S^{12}) \to \pi_{23}(S^{12})$ or, more precisely

$$\eta_{22}^* : \mathbb{Z}_2\{\eta_{12} \circ \mu_{13}\} \oplus \mathbb{Z}_3 \to \mathbb{Z}\{P(\iota_{25})\} \oplus \mathbb{Z}_8\{\zeta_{12}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$$

and $\eta_{21}^* : \pi_{21}(S^{12}) \to \pi_{22}(S^{12})$ or, more precisely

$$\eta_{21}^* : \mathbb{Z}_2^3\{\nu_{12}^3, \mu_{12}, \eta_{12} \circ \varepsilon_{13}\} \to \mathbb{Z}_2\{\eta_{12} \circ \mu_{13}\} \oplus \mathbb{Z}_3.$$

Then we have $\eta_{21}^*(\eta_{12} \circ \mu_{13}) = 4\zeta_{12}$, $\eta_{20}^*(\nu_{12}^3) = 0$, $\eta_{20}^*(\mu_{12}) = \eta_{12} \circ \mu_{13}$ and $\eta_{20}^*(\eta_{12} \circ \varepsilon_{13}) = 4(\nu_{12} \circ \sigma_{15}) = 0$ by [18, (2.2)], [19, (5.9), (7.10), (7.14), (7.20)]. Thus we have

$$\mathrm{Coker}\eta_{22}^* = \mathbb{Z}\{P(\iota_{25})\} \oplus \mathbb{Z}_4\{\zeta_{12}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{21}^* = \mathbb{Z}_2^2\{\nu_{12}^3, \eta_{12}\circ\varepsilon_{13}\}.$$

Thus we have a short exact sequence

$$0 \to \mathbb{Z}\{P(\iota_{25})\}\oplus\mathbb{Z}_4\{\zeta_{11}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9 \xrightarrow{\Sigma^{19}p^*} [\Sigma^{19}\mathbb{C}P^2, S^{12}] \xrightarrow{\Sigma^{19}i^*} \mathbb{Z}_2^2\{\nu_{12}^3, \eta_{12}\circ\varepsilon_{13}\} \to 0.$$

By Proposition 10, we have $2\overline{\eta_{12}\circ\varepsilon_{13}} = a\zeta_{12}\circ\Sigma^{19}p$ where a odd. Since ν_{12}^2 has order 2, we have $\nu_{12}^2\circ\overline{\nu_{18}}$ has order 2.

(4) Consider the following short exact sequence

$$0 \to \mathrm{Coker}\eta_{23}^* \xrightarrow{\Sigma^{20}p^*} [\Sigma^{20}\mathbb{C}P^2, S^{13}] \xrightarrow{\Sigma^{20}i^*} \mathrm{Ker}\eta_{22}^* \to 0,$$

where $\eta_{23}^*: \pi_{23}(S^{13}) \to \pi_{24}(S^{13})$ or, more precisely

$$\eta_{23}^*: \mathbb{Z}_2\{\eta_{13}\circ\mu_{14}\}\oplus\mathbb{Z}_3 \to \mathbb{Z}_8\{\zeta_{13}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$$

and $\eta_{22}^*: \pi_{22}(S^{13}) \to \pi_{23}(S^{13})$ or, more precisely

$$\eta_{22}^*: \mathbb{Z}_2^3\{\nu_{13}^3, \mu_{13}, \eta_{13}\circ\varepsilon_{14}\} \to \mathbb{Z}_2\{\eta_{13}\circ\mu_{14}\}\oplus\mathbb{Z}_3.$$

Then we have $\eta_{22}^*(\eta_{13}\circ\mu_{14}) = 4\zeta_{13}$, $\eta_{21}^*(\nu_{13}^3) = 0$, $\eta_{21}^*(\mu_{13}) = \eta_{13}\circ\mu_{14}$ and $\eta_{21}^*(\eta_{13}\circ\varepsilon_{14}) = 4(\nu_{13}\circ\sigma_{16}) = 0$ by [18, (2.2)], [19, (5.9), (7.10), (7.14), (7.20)]. Thus we have

$$\mathrm{Coker}\eta_{23}^* = \mathbb{Z}_4\{\zeta_{13}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9$$

and

$$\mathrm{Ker}\eta_{22}^* = \mathbb{Z}_2^2\{\nu_{13}^3, \eta_{13}\circ\varepsilon_{14}\}.$$

Thus we have a short exact sequence

$$0 \to \mathbb{Z}_4\{\zeta_{13}\}\oplus\mathbb{Z}_7\oplus\mathbb{Z}_9 \xrightarrow{\Sigma^{20}p^*} [\Sigma^{20}\mathbb{C}P^2, S^{13}] \xrightarrow{\Sigma^{20}i^*} \mathbb{Z}_2^2\{\nu_{13}^3, \eta_{13}\circ\varepsilon_{14}\} \to 0.$$

By (3), we see that $2\overline{\eta_{12}\circ\varepsilon_{13}} = a\zeta_{12}\circ\Sigma^{19}p$ where a odd. Thus we have $2\overline{\eta_{13}\circ\varepsilon_{14}} = a\zeta_{13}\circ\Sigma^{20}p$ where a odd. Since ν_{13}^2 has order 2, we have $\nu_{13}^2\circ\overline{\nu_{19}}$ has order 2. By the Freudenthal suspension theorem, the suspension homomorphism

$$\Sigma: [\Sigma^{n+7}\mathbb{C}P^2, S^n] \to [\Sigma^{n+8}\mathbb{C}P^2, S^{n+1}]$$

is an isomorphism for $n \geq 13$. □

From the above propositions, we have the following theorem.

Theorem 2. *For $n \geq 2$, the n-th cohomotopy group of $(n+7)$-fold suspended complex projective plane has the following group structure.*

case n	2	3	4	5	6
$[\Sigma^{n+7}\mathbb{C}P^2, S^n]$	$2+3$	2^2+21	2^3	$4+2+63$	$4+2^2+63$
case n	7	8	9	10	11
$[\Sigma^{n+7}\mathbb{C}P^2, S^n]$	$8+2^2$	$8+2^2$	$8+2^2$	$\infty+8+2+63$	$8+2+63$
case n	12	$n \geq 13$			
$[\Sigma^{n+6}\mathbb{C}P^2, S^n]$	$\infty+8+2+63$	$8+2+63$			

5. Homotopy groups of $map_*(\mathbb{C}P^2, \mathbb{C}P^2, *)$

It is well known that $\mathbb{C}P^2$ is a base space of a S^1-bundle

$$S^1 \to S^5 \xrightarrow{p} \mathbb{C}P^2.$$

Thus we have an isomorphism

$$p_* : [\Sigma^n \mathbb{C}P^2, S^5] \to [\Sigma^n \mathbb{C}P^2, \mathbb{C}P^2].$$

Also we have an isomorphism

$$\pi_n(map_*(\mathbb{C}P^2, \mathbb{C}P^2; *)) \cong [\Sigma^n \mathbb{C}P^2, \mathbb{C}P^2]$$

by adjointness [12]. McGibbon [15] showed that homotopy class $[\mathbb{C}P^n, \mathbb{C}P^n] \cong \mathbb{Z}$ which is determined by using homomorphism between homology groups. We denote the homotopy group $\pi_n(map_*(\mathbb{C}P^2, \mathbb{C}P^2; *))$ by $\pi_n(map_*(\mathbb{C}P^2, \mathbb{C}P^2))$. By the results of Sections 3 and 4 and [8] we obtain the following:

Theorem 3.

(1) $\pi_4(map_*(\mathbb{C}P^2, \mathbb{C}P^2)) \cong \mathbb{Z}_4\{\nu_5 \circ \Sigma^4 p\} \oplus \mathbb{Z}_3\{\alpha_1(5) \circ \Sigma^4 p\}$.
(2) $\pi_5(map_*(\mathbb{C}P^2, \mathbb{C}P^2)) \cong 0$.
(3) $\pi_6(map_*(\mathbb{C}P^2, \mathbb{C}P^2)) \cong \mathbb{Z}_4\{\nu_5 \circ \overline{2\iota_8}\} \oplus \mathbb{Z}_3\{\overline{\alpha_1(5)}\}$.
(4) $\pi_7(map_*(\mathbb{C}P^2, \mathbb{C}P^2)) \cong \mathbb{Z}_2\{\nu_5^2 \circ \Sigma^7 p\}$.
(5) $\pi_8(map_*(\mathbb{C}P^2, \mathbb{C}P^2)) \cong \mathbb{Z}_4\{\overline{\nu_5 \circ \eta_8^2}\} \oplus \mathbb{Z}_3\{\alpha_2(5) \circ \Sigma^8 p\} \oplus \mathbb{Z}_5\{\alpha_1'(5) \circ \Sigma^8 p\}$.
(6) $\pi_9(map_*(\mathbb{C}P^2, \mathbb{C}P^2)) \cong \mathbb{Z}_4\{\nu_5 \circ \overline{\nu_8}\}$.
(7) $\pi_{10}(map_*(\mathbb{C}P^2, \mathbb{C}P^2)) \cong \mathbb{Z}_2\{\nu_5^3 \circ \Sigma^{10} p\} \oplus \mathbb{Z}_4\{\overline{\sigma'''}\} \oplus \mathbb{Z}_3\{\overline{\alpha_2(5)}\} \oplus \mathbb{Z}_5\{\overline{\alpha_1'(5)}\}$.
(8) $\pi_{11}(map_*(\mathbb{C}P^2, \mathbb{C}P^2)) \cong \mathbb{Z}_4\{\nu_5 \circ \sigma_8 \circ \Sigma^{11} p\} \oplus \mathbb{Z}_9\{\beta_1(5) \circ \Sigma^{11} p\}$.
(9) $\pi_{12}(map_*(\mathbb{C}P^2, \mathbb{C}P^2)) \cong \mathbb{Z}_4\{\zeta_5 \circ \Sigma^{12} p\} \oplus \mathbb{Z}_2^2\{\nu_5 \circ \overline{\nu}_8 \circ \Sigma^{12} p, \nu_5^2 \circ \overline{\nu_{11}}\}$.

$$\oplus\, \mathbb{Z}_9\{\alpha_3'(5) \circ \Sigma^{12} p\} \oplus \mathbb{Z}_7$$

6. Applications: Classifying path-components of mapping spaces and cyclic maps

In this section, we apply the results obtained in Sections 3 and 4 to the classification of components of mapping spaces and the computation of generalized Gottlieb groups.

The term fibration is used for a Hurewicz fibration, that is a (not necessarily surjective) map with the homotopy lifting property with respect to all spaces [14, p. 49]. It is well known that the evaluation map $w_f : map(X, Y; f) \to Y$, $w_f(g) = g(*)$, is a fibration [5, Lemma 8.15]. For fibrations $p : E_1 \to B$ and

$q : E_2 \to B$, p and q are said to be *fiber homotopy equivalent* if there is a homotopy equivalence $h : E_1 \to E_2$ such that $q \circ h = p$ [14, p. 52].

Here we remind several results of the generalized Whitehead product. If α and β are homotopy classes, then the whitehead product of α and β is denoted by $[\alpha, \beta]$.

A map $f : X \to Y$ is *cyclic* if there is a map $F : X \times Y \to Y$, called *affiliated map*, such that $F(x, *) = f(x)$ and $F(*, y) = y$. We denote the set of cyclic map from X to Y by $G(X, Y)$ and it has group structure if X is a co-H-group [20].

We recall the following equivalent statements due to [6, Lemma 2] and [10, Theorem 3.7]:

Theorem 4. *Let ΣX and ΣY be CW complexes with non-degenerate basepoints and ΣX is finite CW complex. Then the following are equivalent.*
(A) *the map $f : \Sigma X \to \Sigma Y$ is cyclic.*
(B) $[f, id_{\Sigma Y}] = 0$ *where $[\ ,\ ]$ is the generalized Whitehead product.*
(C) *the evaluation fibration $w_f : map(\Sigma X, \Sigma Y; f) \to \Sigma Y$ has a section.*
(D) *the evaluation fibration $w_f : map(\Sigma X, \Sigma Y; f) \to \Sigma Y$ is fibre-homotopy equivalent to $w_0 : map(\Sigma X, \Sigma Y; *) \to Y$.*

Here is a connection of path components of mapping spaces and cyclic maps.

Theorem 5 ([11, Theorem 3.10])**.** *Suppose X is a CW co-H-space and Y is any CW complex. Let $d \in G(X, Y)$ be any cyclic map. Then for each map $f : X \to Y$, we have $map(X, Y; f) \simeq map(X, Y; f + d)$. If X is a finite co-H-space then the corresponding evaluation fibrations w_f and w_{f+d} are fibre-homotopy equivalent.*

The following theorem shows a relation between the generalized Whitehead product and evaluation fibration [6, Theorem 1].

Theorem 6. *Given a pair of homotopy classes $\alpha = [f], \beta = [g] \in [\Sigma A, \Sigma B]$ such that at least one of the identities $[\alpha, \iota_{\Sigma B}] = \pm[\beta, \iota_{\Sigma B}]$ holds. Then the evaluation fibrations $w_f : map(\Sigma A, \Sigma B; f) \to \Sigma B)$ and $w_g : map(\Sigma A, \Sigma B; g) \to \Sigma B)$ are fibre homotopy equivalent.*

The following is useful to compute generalized Whitehead product [17].

Remark 1. Let $\alpha \in [\Sigma K, X]$, $\beta \in [\Sigma L, X]$, $\gamma \in [P, X]$ and $\delta \in [Q, L]$ where K, L, P and Q are polyhedra. Then we have

$$[\alpha \circ \Sigma\gamma, \beta \circ \Sigma\delta] = [\alpha, \beta] \circ \Sigma(\gamma \wedge \delta).$$

We also recall a property of the generalized Whitehead product for H-spaces.

Theorem 7 ([1, Proposition 3.1])**.** *If X is an H-space, then $[\alpha, \beta] = 0$ for all $\alpha \in [\Sigma A, X]$ and $\beta \in [\Sigma B, X]$.*

We recall Proposition 4.4 of [8].

Proposition 12. (1) $[\Sigma\mathbb{C}P^2, S^2] \cong \mathbb{Z}\{\eta_2 \circ \overline{2\iota_3}\}$.
(2) $[\Sigma^2\mathbb{C}P^2, S^3] \cong \mathbb{Z}_2\{\nu' \circ \Sigma^2 p\} \oplus \mathbb{Z}_3\{\alpha_1(3) \circ \Sigma^2 p\}$.
(3) $[\Sigma^3\mathbb{C}P^2, S^4] \cong \mathbb{Z}\{\nu_4 \circ \Sigma^3 p\} \oplus \mathbb{Z}_2\{\Sigma\nu' \circ \Sigma^3 p\} \oplus \mathbb{Z}_3\{\alpha_1(4) \circ \Sigma^3 p\}$.
(4) $[\Sigma^n\mathbb{C}P^2, S^{n+1}] \cong \mathbb{Z}_4\{\nu_{n+1} \circ \Sigma^n p\} \oplus \mathbb{Z}_3\{\alpha_1(n+1) \circ \Sigma^n p\}$ *for* $n \geq 4$.

Let G be an abelian group and S be a subset of G. Let $\langle S\rangle$ denote the subgroup of G that is the smallest subgroup containing S. Then we have the following theorem.

Theorem 8. (1) *For each* $[f],[g] \in [\Sigma\mathbb{C}P^2, S^2]$, *the evaluation fibrations* $w_f : map(\Sigma\mathbb{C}P^2, S^2; f) \to S^2$ *and* $w_g : map(\Sigma\mathbb{C}P^2, S^2; g) \to S^2$ *are fibre homotopy equivalent.*

(2) *For each* $[f],[g] \in [\Sigma^2\mathbb{C}P^2, S^3]$, *the evaluation fibrations* $w_f : map(\Sigma^2\mathbb{C}P^2, S^3; f) \to S^3$ *and* $w_g : map(\Sigma^2\mathbb{C}P^2, S^3; g) \to S^3$ *are fibre homotopy equivalent.*

(3) *The mapping space* $map(\Sigma^3\mathbb{C}P^2, S^4)$ *has three different path components up to homotopy equivalent as follows*:
  (a) *For each* $[f],[g] \in \langle 2\nu_4 \circ \Sigma^3 p, \Sigma\nu' \circ \Sigma^3 p\rangle$, *the evaluation fibrations* $w_f : map(\Sigma^3\mathbb{C}P^2, S^4; f) \to S^4$ *and* $w_g : map(\Sigma^3\mathbb{C}P^2, S^4; g) \to S^4$ *are fibre homotopy equivalent.*
  (b) *For each* $[f] \in \{(2n+1)\nu_4 \circ \Sigma^3 p \mid n \in \mathbb{Z}\}$ *and* $[g] \in \langle 2\nu_4 \circ \Sigma^3 p, \Sigma\nu' \circ \Sigma^3 p\rangle$, *the evaluation fibrations* $w_f : map(\Sigma^3\mathbb{C}P^2, S^4; f) \to S^4$ *and* $w_{f+g} : map(\Sigma^3\mathbb{C}P^2, S^4; f+g) \to S^4$ *are fibre homotopy equivalent.*
  (c) *For each* $[f] \in \langle 2\nu_4 \circ \Sigma^3 p, \Sigma\nu' \circ \Sigma^3 p\rangle$ *and* $[h] \in \langle \alpha_1(4) \circ \Sigma^3 p\rangle$, *the evaluation fibrations* $w_h : map(\Sigma^3\mathbb{C}P^2, S^4; f) \to S^4$ *and* $w_{h+f} : map(\Sigma^3\mathbb{C}P^2, S^4; h+f) \to S^4$ *are fibre homotopy equivalent.*

(4) *For each* $[f],[g] \in [\Sigma^4\mathbb{C}P^2, S^5]$, *the evaluation fibrations* $w_f : map(\Sigma^4\mathbb{C}P^2, S^5; f) \to S^5$ *and* $w_g : map(\Sigma^4\mathbb{C}P^2, S^5; g) \to S^5$ *are fibre homotopy equivalent.*

(5) *The mapping space* $map(\Sigma^5\mathbb{C}P^2, S^6)$ *has three different path components up to homotopy equivalent as follows*:
  (a) *The evaluation fibrations* $w_* : map(\Sigma^5\mathbb{C}P^2, S^6; *) \to S^6$ *and* $w_f : map(\Sigma^5\mathbb{C}P^2, S^6; f) \to S^6$ *are fiber homotopy equivalent where* $[f] = 2\nu_6 \circ \Sigma^5 p$.
  (b) *For each* $[f] \in \{\nu_6 \circ \Sigma^5 p, 3\nu_6 \circ \Sigma^5 p\}$ *and* $[g] = 2\nu_6 \circ \Sigma^5 p$, *the evaluation fibrations* $w_f : (\Sigma^5\mathbb{C}P^2, S^6; f) \to S^6$ *and* $w_{f+g} : (\Sigma^5\mathbb{C}P^2, S^6; f+g) \to S^6$ *are fibre homotopy equivalent.*
  (c) *For each* $[f] \in \langle \alpha_1(6) \circ \Sigma^5 p\rangle$ *and* $[g] = 2\nu_6 \circ \Sigma^5 p$, *the evaluation fibrations* $w_f : (\Sigma^5\mathbb{C}P^2, S^6; f) \to S^6$ *and* $w_{f+g} : (\Sigma^5\mathbb{C}P^2, S^6; f+g) \to S^6$ *are fibre homotopy equivalent.*

(6) *For each* $[f],[g] \in [\Sigma^6\mathbb{C}P^2, S^7]$, *the evaluation fibrations* $w_f : map(\Sigma^6\mathbb{C}P^2, S^7; f) \to S^7$ *and* $w_g : map(\Sigma^6\mathbb{C}P^2, S^7; g) \to S^7$ *are fibre homotopy equivalent.*

(7) *The mapping space $map(\Sigma^7\mathbb{C}P^2, S^8)$ has four different path components up to homotopy equivalent as follows*:
   (a) $w_* : map(\Sigma^7\mathbb{C}P^2, S^8; *) \to S^8$.
   (b) *Two evaluation fibrations* $w_f : map(\Sigma^7\mathbb{C}P^2, S^8; f) \to S^8$ *and* $w_g : map(\Sigma^7\mathbb{C}P^2, S^8; g) \to S^8$ *are fibre homotopy equivalent where* $[f] = \nu_8 \circ \Sigma^7 p$ *and* $[g] = 3\nu_8 \circ \Sigma^7 p$.
   (c) $w_h : map(\Sigma^7\mathbb{C}P^2, S^8; h) \to S^8$ *where* $[h] = 2\nu_8 \circ \Sigma^7 p$.
   (d) *Two evaluation fibrations* $w_f : map(\Sigma^7\mathbb{C}P^2, S^8; f) \to S^8$ *and* $w_g : map(\Sigma^7\mathbb{C}P^2, S^8; g) \to S^8$ *are fibre homotopy equivalent where* $[f] = \alpha_1(9) \circ \Sigma^7 p$ *and* $[g] = 2\alpha_1(9) \circ \Sigma^7 p$.

(8) *The mapping space $map(\Sigma^8\mathbb{C}P^2, S^9)$ has two different path components up to homotopy equivalent as follows*:
   (a) *For each* $[f], [g] \in \langle \alpha_1(9) \circ \Sigma^8 p, 2\nu_9 \circ \Sigma^8 p \rangle$,
   the evaluation fibrations $w_f : map(\Sigma^7\mathbb{C}P^2, S^8; f) \to S^8$ *and* $w_g : map(\Sigma^7\mathbb{C}P^2, S^8; g) \to S^8$ *are fibre homotopy equivalent.*
   (b) *For each* $[f] \in \{\nu_9 \circ \Sigma^8 p, 3\nu_9 \circ \Sigma^8 p\}$ *and*
   $[g] \in \langle \alpha_1(9) \circ \Sigma^8 p, 2\nu_9 \circ \Sigma^8 p \rangle$, *the evaluation fibrations*
   $w_f : map(\Sigma^7\mathbb{C}P^2, S^8; f) \to S^8$ *and* $w_{f+g} : map(\Sigma^7\mathbb{C}P^2, S^8; g) \to S^8$ *are fibre homotopy equivalent.*

Proof. (1) By [3, (2.1)] we have $[\iota_2, \eta_2] = 0$. By [3, Lemma 1.1] we have $[\iota_2, \eta_2 \circ \overline{2\iota_3}] = 0$. By Theorem 4 we have $\eta_2 \circ \overline{2\iota_3}$ is cyclic.

(2) Since S^3 is an H-space, we have $G_2(\mathbb{C}P^2, S^3) = [\Sigma^2\mathbb{C}P^2, S^3]$.

(3) We have

$$[\iota_4, \nu_4 \circ \Sigma^3 p] = [\iota_4, \nu_4] \circ \Sigma(\iota_3 \wedge \Sigma^2 p) = 2\nu_4^2 \circ \Sigma^6 p \neq 0$$

and

$$[\iota_4, \Sigma\nu' \circ \Sigma^3 p] = [\iota_4, \Sigma\nu'] \circ \Sigma(\iota_3 \wedge \Sigma^2 p) = 4\nu_4^2 \circ \Sigma^6 p = 0$$

by (2.5) of [3] and Proposition 3.4 of [8]. By Theorem 4, Theorem 5 and [10, Theorem 3.10] we have (a) and (b). Finally we have

$$[\iota_4, \alpha_1(4) \circ \Sigma^3 p] = [\iota_4, \alpha_1(4)] \circ \Sigma(\iota_3 \wedge \Sigma^2 p) = [\iota_4, \iota_4] \circ \alpha_1(7) \circ \Sigma^6 p.$$

By [8, Proposition 3.4] $[\iota_4, \iota_4] \circ \alpha_1(7) \circ \Sigma^6 p$ has order 3. By biaddivitity of generalized Whitehead product we have

$$[\iota_4, 2\alpha_1(4) \circ \Sigma^3 p] = [\iota_4, -\alpha_1(4) \circ \Sigma^3 p] = -[\iota_4, \alpha_1(4) \circ \Sigma^3 p].$$

By Theorem 5 and [10, Theorem 3.10] we have (c).

(4) We have

$$[\iota_5, \nu_5 \circ \Sigma^4 p] = [\iota_5, \nu_5] \circ \Sigma(\iota_4 \wedge \Sigma^3 p) = 0$$

by [3, (2.6)] and Theorem 5. Also we have

$$[\iota_5, \alpha_1(5) \circ \Sigma^4 p] = [\iota_5, \alpha_1(5)] \circ \Sigma(\iota_4 \wedge \circ\Sigma^3 p) = 0$$

since $G_8(S^5) = \pi_8(S^5)$ [3, p. 428] and Theorem 5.

(5) We have

$$[\iota_6, \nu_6 \circ \Sigma^5 p] = [\iota_6, \nu_6] \circ \Sigma(\iota_5 \wedge \Sigma^4 p) = 2\overline{\nu}_6 \circ \Sigma^{10} p \neq 0$$

by [19, Lemma 6.2] and [8, Proposition 3.7]. Since $\overline{\nu}_6 \circ \Sigma^{10} p$ has order 4, we have $[\iota_6, 3\nu_6 \circ \Sigma^5 p] = [\iota_6, -\nu_6 \circ \Sigma^5 p] = -[\iota_6, \nu_6 \circ \Sigma^5 p]$ and $[\iota_6, 2\nu_6 \circ \Sigma^5 p] = 0 = [\iota_6, *]$. Also we have that

$$[\iota_6, \alpha_1(6) \circ \Sigma^5 p] = [\iota_6, \alpha_1(6)] \circ \Sigma(\iota_5 \wedge \Sigma^4 p) = [\iota_6, \iota_6] \circ \alpha_1(11) \circ \Sigma^{10} p$$

has order 3 in $[\Sigma^{10}\mathbb{C}P^2, S^6]$ [8, Proposition 3.7]. By biadditivity of generalized Whitehead product we have

$$[\iota_6, 2\alpha_1(6) \circ \Sigma^5 p] = [\iota_6, -\alpha_1(6) \circ \Sigma^5 p] = -[\iota_6, \alpha_1(6) \circ \Sigma^5 p].$$

Thus we conclude that

$$\begin{aligned}
\mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; \nu_6 \circ \Sigma^5 p) &\simeq_f \mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; 3\nu_6 \circ \Sigma^5 p) \\
\mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; 2\nu_6 \circ \Sigma^5 p) &\simeq_f \mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; *) \\
\mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; \alpha_1(6) \circ \Sigma^5 p) &\simeq_f \mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; \alpha_1(6) \circ \Sigma^5 p + 2\nu_6 \circ \Sigma^5 p)
\end{aligned}$$

by Theorem 6.

(6) Since S^7 is an H-space, we have $G(\Sigma^2\mathbb{C}P^2, S^3) = [\Sigma^2\mathbb{C}P^2, S^3]$.

(7) We have

$$[\iota_8, \nu_8 \circ \Sigma^7 p] = [\iota_8, \nu_8] \circ \Sigma(\iota_7 \wedge \Sigma^6 p) = 2\sigma_8 \circ \nu_{15} \circ \Sigma^{14} p - x\nu_8 \circ \sigma_{11} \circ \Sigma^{14} p \neq 0,$$

where x is an odd by [3, (2.9)] and Proposition 5. By Proposition 12, we have $[\iota_8, \nu_8 \circ \Sigma^7 p]$ is of order 4. By Proposition 5 we have $[\iota_8, \nu_8 \circ \Sigma^7 p] \neq 0$ and $\pm[\iota_8, 2\nu_8 \circ \Sigma^7 p] \neq 0$ and

$$[\iota_8, 3\nu_8 \circ \Sigma^7 p] = [\iota_8, -\nu_8 \circ \Sigma^7 p] = -[\iota_8, 3\nu_8 \circ \Sigma^7 p].$$

So we conclude that $\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; *)$, $\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; 2\nu_8 \circ \Sigma^7 p)$ and $\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; \nu_8 \circ \Sigma^7 p) \simeq_f \mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; 3\nu_8 \circ \Sigma^7 p)$ have different fibre homotopy types each others by Theorem 6. Also we have that

$$[\iota_8, \alpha_1(8) \circ \Sigma^7 p] = [\iota_8, \alpha_1(8)] \circ \Sigma(\iota_7 \wedge \Sigma^6 p)$$

has order 3 by Proposition 5. By biadditivity of generalized Whitehead product we have

$$[\iota_8, 2\alpha_1(8) \circ \Sigma^7 p] = [\iota_8, -\alpha_1(8) \circ \Sigma^7 p] = -[\iota_8, \alpha_1(8) \circ \Sigma^7 p].$$

By Theorem 6 we have $\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; \alpha_1(8) \circ \Sigma^7 p)$ and $\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; -\alpha_1(8) \circ \Sigma^7 p)$ have same fibre homotopy type.

(8) We have

$$[\iota_9, \nu_9 \circ \Sigma^8 p] = [\iota_9, \nu_9] \circ \Sigma(\iota_8 \wedge \Sigma^7 p) = \overline{\nu}_9 \circ \nu_{17} \circ \Sigma^{16} p \neq 0$$

by [3, (2.10)] and Proposition 10. Since $\overline{\nu}_9 \circ \nu_{17} \circ \Sigma^{16} p$ has order 2, we have $[\iota_9, 2\nu_9 \circ \Sigma^8 p] = 0 = [\iota_9, *]$. By biadditivity of generalized Whitehead product we have

$$[\iota_9, 3\nu_9 \circ \Sigma^8 p] = [\iota_9, -\nu_9 \circ \Sigma^8 p] = -[\iota_9, \nu_9 \circ \Sigma^8 p].$$

Also we have

$$[\iota_9, \alpha_1(9) \circ \Sigma^8 p] = [\iota_9, \alpha_1(9)] \circ \Sigma(\iota_8 \wedge \Sigma^7 p) = 0$$

since $\alpha_1(9) \in G_{12}(S^9)$. By Theorem 6 we have map$(\Sigma^8\mathbb{C}P^2, S^9; f)$ has same fibre homotopy type for $[f] \in \langle 2\nu_9 \circ \Sigma^8 p, \alpha_1(9) \circ \Sigma^8 p\rangle$ and map$(\Sigma^8\mathbb{C}P^2, S^9; \nu_9 \circ \Sigma^8 p) \simeq_f$ map$(\Sigma^8\mathbb{C}P^2, S^9; 3\nu_9 \circ \Sigma^8 p)$. □

Corollary 1. (1) $G_1(\mathbb{C}P^2, S^2) = \mathbb{Z}\{\eta_2 \circ \overline{2\iota_3}\}$.
(2) $G_2(\mathbb{C}P^2, S^3) = \mathbb{Z}_2\{\nu' \circ \Sigma^2 p\} \oplus \mathbb{Z}_3\{\alpha_1(3) \circ \Sigma^2 p\}$.
(3) $G_3(\mathbb{C}P^2, S^4) = \mathbb{Z}\{2\nu_4 \circ \Sigma^3 p\} \oplus \mathbb{Z}_2\{\Sigma\nu' \circ \Sigma^3 p\}$.
(4) *For* $n = 4, 5, 6$, $G_n(\mathbb{C}P^2, S^{n+1}) \cong \mathbb{Z}_4\{\nu_{n+1} \circ \Sigma^n p\} \oplus \mathbb{Z}_3\{\alpha_1(n+1) \circ \Sigma^n p\}$.
(5) $G_7(\mathbb{C}P^2, S^8) = 0$.
(6) $G_8(\mathbb{C}P^2, S^9) = \mathbb{Z}_2\{2\nu_9 \circ \Sigma^8 p\} \oplus \mathbb{Z}_3\{\alpha_1(9) \circ \Sigma^8 p\}$.

We recall a long exact sequence [12, Theorem 2.7]:

Theorem 9. *There is a long exact sequence*

$$\cdots \to [\Sigma^r A, map(\Sigma B, X; f)] \xrightarrow{w_*} [\Sigma^r A, X]$$
$$\xrightarrow{P_\alpha} [\Sigma(\Sigma^{r-1} A \wedge B), X] \xrightarrow{i'_*} [\Sigma^{r-1} A, map(\Sigma B, X; f)] \to \cdots,$$

where $\alpha = [f] \in [\Sigma B, X]$ *and* $P_\alpha(\beta) = [\beta, \alpha]$ *is the generalized Whitehead product and* $w : map(\Sigma B, X; f) \to B$ *is the evaluation map.*

Corollary 2. $i'_* : [\Sigma^{r+m}\mathbb{C}P^2, S^n] \to \pi_r(map(\Sigma^m\mathbb{C}P^2, S^n; f))$ *is an isomorphism for* $r \leq n-2$ *and* $m \geq 1$.

Theorem 10. (1) $\pi_1(map(\Sigma\mathbb{C}P^2, S^2; f)) \cong \mathbb{Z}_2\{\eta_2 \circ \nu' \circ \Sigma^2 p\} \oplus \mathbb{Z}_3\{[\iota_2, \iota_2] \circ \alpha_1(3)\}$ *for all* $f : \Sigma\mathbb{C}P^2 \to S^2$.
(2) $\pi_2(map(\Sigma^2\mathbb{C}P^2, S^3; f)) \cong \mathbb{Z}_2\{\nu' \circ \overline{2\iota_6}\} \oplus \mathbb{Z}_3\{\overline{\alpha_1(3)}\}$ *for all* $f : \Sigma^2\mathbb{C}P^2 \to S^3$.
(3) (a) *For* $[f] \in \langle 2\nu_4 \circ \Sigma^3 p, \Sigma\nu' \circ \Sigma^3 p\rangle$,

$$\pi_3(map(\Sigma^3\mathbb{C}P^2, S^4; f))$$
$$\cong \mathbb{Z}_4\{\nu_4^2 \circ \Sigma^6 p\} \oplus \mathbb{Z}_3^2\{\alpha_1(4) \circ \alpha_1(7) \circ \Sigma^6 p, [\iota_4, \iota_4] \circ \alpha_1(7) \circ \Sigma^6 p\}.$$

(b) *For* $[f] \in \{(2n+1)\nu_4 \circ \Sigma^3 p \mid n \in \mathbb{Z}\}$ *and* $[g] \in \langle 2\nu_4 \circ \Sigma^3 p, \Sigma\nu' \circ \Sigma^3 p\rangle$,

$$\pi_3(map(\Sigma^3\mathbb{C}P^2, S^4; f+g))$$
$$\cong \mathbb{Z}_2\{\nu_4^2 \circ \Sigma^6 p\} \oplus \mathbb{Z}_3^2\{\alpha_1(4) \circ \alpha_1(7) \circ \Sigma^6 p, [\iota_4, \iota_4] \circ \alpha_1(7) \circ \Sigma^6 p\}.$$

(c) *For* $[f] \in \{\alpha_1(4) \circ \Sigma^3 p, 3\alpha_1(4) \circ \Sigma^3 p\}$ *and* $[g] \in \langle 2\nu_4 \circ \Sigma^3 p, \Sigma\nu' \circ \Sigma^3 p\rangle$,

$$\pi_3(map(\Sigma^3\mathbb{C}P^2, S^4; f+g)) \cong \mathbb{Z}_4\{\nu_4^2 \circ \Sigma^6 p\} \oplus \mathbb{Z}_3\{\alpha_1(4) \circ \alpha_1(7) \circ \Sigma^6 p\}.$$

(4) $$\pi_4(map(\Sigma^4\mathbb{C}P^2, S^5; f)) \cong \mathbb{Z}_4\{\overline{\nu_5 \circ \eta_8^2}\} \oplus \mathbb{Z}_3\{\alpha_2(5) \circ \Sigma^8 p\} \oplus \mathbb{Z}_5\{\alpha_{1,5}(5) \circ \Sigma^8 p\}$$
for all $f : \Sigma^4\mathbb{C}P^2 \to S^5$.

(5) (a) *For* $[f] \in \langle 2\nu_6 \circ \Sigma^5 p\rangle$,

$$\pi_5(map(\Sigma^5\mathbb{C}P^2, S^6; f)) \cong \mathbb{Z}_4^2\{\nu_6 \circ g_9, \overline{\nu}_6 \circ \Sigma^{10}p\} \oplus \mathbb{Z}_3\{[\iota_6, \iota_6] \circ \alpha_1(11) \circ \Sigma^{10}p\}.$$

(b) *For* $[f] \in \{\nu_6 \circ \Sigma^5 p, 3\nu_6 \circ \Sigma^5 p\}$ *and* $[g] \in \langle 2\nu_6 \circ \Sigma^5 p\rangle$,

$$\pi_5(map(\Sigma^5\mathbb{C}P^2, S^6; f)) \cong \mathbb{Z}_4\{\nu_6 \circ g_9\} \oplus \mathbb{Z}_2\{\overline{\nu}_6 \circ \Sigma^{10}p\} \oplus \mathbb{Z}_3\{[\iota_6, \iota_6] \circ \alpha_1(11) \circ \Sigma^{10}p\}.$$

(c) *For* $[f] \in \{\alpha_1(6) \circ \Sigma^5 p, 2\alpha_1(6) \circ \Sigma^5 p\}$ *and* $[g] \in \langle 2\nu_6 \circ \Sigma^5 p\rangle$,

$$\pi_5(map(\Sigma^5\mathbb{C}P^2, S^6; f)) \cong \mathbb{Z}_4^2\{\nu_6 \circ g_9, \overline{\nu}_6 \circ \Sigma^{10}p\}.$$

(6) $\pi_6(map(\Sigma^6\mathbb{C}P^2, S^7; f)) \cong \mathbb{Z}_8\{\sigma' \circ \overline{2\iota_{14}}\} \oplus \mathbb{Z}_3\{\overline{\alpha_2(7)}\} \oplus \mathbb{Z}_5\{\overline{\alpha_{1,5}(7)}\}$ *for all* $f : \Sigma^6 p \to S^7$.

(7) (a)

$$\pi_7(map(\Sigma^7\mathbb{C}P^2, S^8; *)) \cong \mathbb{Z}_4^2\{\sigma_8 \circ \nu_{15} \circ \Sigma^{14}p, \nu_8 \circ \sigma_{11} \circ \Sigma^{14}p\} \oplus \mathbb{Z}_3^2\{\beta_1(8) \circ \Sigma^{14}p, [\iota_8, \iota_8] \circ \alpha_1(15) \circ \Sigma^{14}p\}.$$

(b) *For* $[f] \in \{\nu_8 \circ \Sigma^7 p, 3\nu_8 \circ \Sigma^7 p\}$,

$$\pi_7(map(\Sigma^7\mathbb{C}P^2, S^8; f)) \cong \mathbb{Z}_2\{2\sigma_8 \circ \nu_{15} \circ \Sigma^{14}p\}.$$

(c) *For* $[f] = \nu_8 \circ \Sigma^7 p$,

$$\pi_7(map(\Sigma^7\mathbb{C}P^2, S^8; f)) \cong \mathbb{Z}_4\{\sigma_8 \circ \nu_{15} \circ \Sigma^{14}p\} \oplus \mathbb{Z}_2\{\nu_8 \circ \sigma_{11} \circ \Sigma^{14}p\} \oplus \mathbb{Z}_3^2\{\beta_1(8) \circ \Sigma^{14}p, [\iota_8, \iota_8] \circ \alpha_1(15) \circ \Sigma^{14}p\},$$

(d) *For* $[f] \in \{\alpha_1(9) \circ \Sigma^7 p, 2\alpha_1(9) \circ \Sigma^7 p\}$,

$$\pi_7(map(\Sigma^7\mathbb{C}P^2, S^8; f)) \cong \mathbb{Z}_4^2\{\sigma_8 \circ \nu_{15} \circ \Sigma^{14}p, \nu_8 \circ \sigma_{11} \circ \Sigma^{14}p\} \oplus \mathbb{Z}_3\{\beta_1(8) \circ \Sigma^{14}p\}.$$

(8) (a) *For* $[f] \in \langle \alpha_1(9) \circ \Sigma^8 p, 2\nu_9 \circ \Sigma^8 p\rangle$,

$$\pi_8(map(\Sigma^8\mathbb{C}P^2, S^9; f)) \cong \mathbb{Z}_8\{\nu_9^2 \circ g_{15}\} \oplus \mathbb{Z}_2^2\{\overline{\nu}_9 \circ \nu_{17} \circ \Sigma^{16}p, \overline{\eta_9 \circ \varepsilon_{10}}\} \oplus \mathbb{Z}_7\{\alpha_{1,7}(9) \circ \Sigma^{14}p\} \oplus \mathbb{Z}_9\{\alpha_3'(9) \circ \Sigma^{14}p\}.$$

(b) *For* $[f] \in \{\nu_9 \circ \Sigma^8 p, 3\nu_9 \circ \Sigma^8 p\}$ *and* $[g] \in \langle \alpha_1(9) \circ \Sigma^8 p, 2\nu_9 \circ \Sigma^8 p\rangle$,

$$\pi_8(map(\Sigma^8\mathbb{C}P^2, S^9; f+g)) \cong \mathbb{Z}_8\{\nu_9^2 \circ g_{15}\} \oplus \mathbb{Z}_2\{\overline{\eta_9 \circ \varepsilon_{10}}\} \oplus \mathbb{Z}_7\{\alpha_{1,7}(9) \circ \Sigma^{14}p\} \oplus \mathbb{Z}_9\{\alpha_3'(9) \circ \Sigma^{14}p\}.$$

Proof. (1) We apply $A = S^0$, $B = \mathbb{C}P^2$, $X = S^2$ and $r = 2$ to Theorem 9. Let $f : \Sigma\mathbb{C}P^2 \to S^2$ be any map and let $\alpha = [f]$. Then we have an exact sequence

$$\pi_2(S^2) \xrightarrow{P_\alpha} [\Sigma^2\mathbb{C}P^2, S^2] \xrightarrow{i'_*} \pi_1(\mathrm{map}(\Sigma\mathbb{C}P^2, S^2; f)) \to 0.$$

By Theorem 8 we have P_α is trivial, so that

$$i'_* : [\Sigma^2\mathbb{C}P^2, S^2] \to \pi_1(\mathrm{map}(\Sigma\mathbb{C}P^2, S^2; f))$$

is an isomorphism. By Proposition 3.2 of [8] this proof is complete.

(2) We apply $A = S^0$, $B = \Sigma\mathbb{C}P^2$, $X = S^3$ and $r = 3$ to Theorem 9. Let $f : \Sigma^2\mathbb{C}P^2 \to S^3$ be any map and let $\alpha = [f]$. Then we have an exact sequence

$$\pi_3(S^3) \xrightarrow{P_\alpha} [\Sigma^4\mathbb{C}P^2, S^3] \xrightarrow{i'_*} \pi_2(\mathrm{map}(\Sigma^2\mathbb{C}P^2, S^3; f)) \to 0.$$

Since S^3 is an H-space, we have P_α is trivial. Thus we have $i'_* : [\Sigma^2\mathbb{C}P^2, S^2] \to \pi_1(\mathrm{map}(\Sigma\mathbb{C}P^2, S^2; f))$ is an isomorphism. By Proposition 3.3 of [8] this proof is complete.

(3) We apply $A = S^0$, $B = \Sigma^2\mathbb{C}P^2$, $X = S^4$ and $r = 4$ to Theorem 9. Then we have an exact sequence

$$\pi_4(S^4) \xrightarrow{P_\alpha} [\Sigma^6\mathbb{C}P^2, S^4] \xrightarrow{i'_*} \pi_3(\mathrm{map}(\Sigma^3\mathbb{C}P^2, S^4; f)) \to 0.$$

First we consider $\alpha = [f] \in \langle 2\nu_4 \circ \Sigma^3 p, \Sigma\nu' \circ \Sigma^3 p\rangle$. By Theorem 8 P_α is trivial. Then we have $i_* : [\Sigma^6\mathbb{C}P^2, S^4] \to \pi_3(\mathrm{map}(\Sigma\mathbb{C}P^3, S^4; f))$ is an isomorphism. By Proposition 3.4 of [8] we have (a). Second we consider that $\alpha = [f] \in \{(2n+1)\nu_4 \circ \Sigma^3 p \,|\, n \in \mathbb{Z}\}$. By Theorem 8 we have

$$\mathrm{Coker}P_\alpha = \mathbb{Z}_2\{\nu_4^2 \circ \Sigma^6 p\} \oplus \mathbb{Z}_3^2\{\alpha_1(4) \circ \alpha_1(7) \circ \Sigma^3 p, [\iota_4, \iota_4] \circ \alpha_1(7) \circ \Sigma^3 p\}.$$

Since $i'_* : \mathrm{Coker}P_\alpha \to \pi_3(\mathrm{map}(\Sigma\mathbb{C}P^3, S^4; f))$ is an isomorphism, we have (b) by Proposition 3.4 of [8]. Finally we consider $\alpha = [f] \in \{\alpha_1(4) \circ \Sigma^3 p, 3\alpha_1(4) \circ \Sigma^3 p\}$. By Theorem 8 we have

$$\mathrm{Coker}P_\alpha = \mathbb{Z}_4\{\nu_4^2 \circ \Sigma^6 p\} \oplus \mathbb{Z}_3\{\alpha_1(4) \circ \alpha_1(7) \circ \Sigma^3 p\}.$$

Since $i'_* : \mathrm{Coker}P_\alpha \to \pi_3(\mathrm{map}(\Sigma\mathbb{C}P^3, S^4; f))$ is an isomorphism, we have (c) by Proposition 3.4 of [8].

(4) We apply $A = S^0$, $B = \Sigma^3\mathbb{C}P^2$, $X = S^5$ and $r = 5$ to Theorem 9. Let $f : \Sigma^4\mathbb{C}P^2 \to S^5$ be any map and let $\alpha = [f]$. Then we have an exact sequence

$$\pi_5(S^5) \xrightarrow{P_\alpha} [\Sigma^8\mathbb{C}P^2, S^5] \xrightarrow{i'_*} \pi_4(\mathrm{map}(\Sigma^4\mathbb{C}P^2, S^5; f)) \to 0.$$

By Theorem 8 we have P_α is trivial. Thus we have $i'_* : [\Sigma^8\mathbb{C}P^2, S^5] \to \pi_4(\mathrm{map}(\Sigma^4\mathbb{C}P^2, S^5; f))$ is an isomorphism. By Proposition 3.5 of [8] this proof is complete.

(5) We apply $A = S^0$, $B = \Sigma^5\mathbb{C}P^2$, $X = S^6$ and $r = 6$ to Theorem 9. Then we have an exact sequence

$$\pi_6(S^6) \xrightarrow{P_\alpha} [\Sigma^{10}\mathbb{C}P^2, S^6] \xrightarrow{i'_*} \pi_5(\mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; f)) \to 0.$$

First we consider $\alpha = [f] \in \langle 2\nu_6 \circ \Sigma^5 p\rangle$. By Theorem 8 we have P_α is trivial. Thus we have $i'_* : [\Sigma^{10}\mathbb{C}P^2, S^6] \to \pi_5(\mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; f))$ is an isomorphism. By Proposition 3.6 of [8], we have (a). Second we consider $\alpha = [f] \in \{\nu_6 \circ \Sigma^5 p, 3\nu_5 \circ \Sigma^5 p\}$. By Theorem 8 we have

$$\mathrm{Coker}P_\alpha = \mathbb{Z}_4\{\nu_6 \circ g_9\} \oplus \mathbb{Z}_2\{\overline{\nu}_6 \circ \Sigma^{10}\} \oplus \mathbb{Z}_3\{[\iota_6, \iota_6] \circ \alpha_1(11) \circ \Sigma^{10} p\}.$$

By the isomorphism $i'_* : \mathrm{Coker}P_\alpha \to \pi_5(\mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; f))$ we have (b). Finally we consider $\alpha = [f] \in \{\alpha_1(6) \circ \Sigma^5 p, 2\alpha_2(6) \circ \Sigma^5 p\}$. By Theorem 8 we have

$$\mathrm{Coker}P_\alpha = \mathbb{Z}_4^2\{\nu_6 \circ g_9, \overline{\nu}_6 \circ \Sigma^{10}\}.$$

By the isomorphism $i'_* : \mathrm{Coker}P_\alpha \to \pi_5(\mathrm{map}(\Sigma^5\mathbb{C}P^2, S^6; f))$ we have (c).

(6) We apply $A = S^0$, $B = \Sigma^6\mathbb{C}P^2$, $X = S^7$ and $r = 7$ to Theorem 9. Let $f : \Sigma^6\mathbb{C}P^2 \to S^7$ be any map and let $\alpha = [f]$. Then we have an exact sequence

$$\pi_7(S^7) \xrightarrow{P_\alpha} [\Sigma^{12}\mathbb{C}P^2, S^7] \xrightarrow{i'_*} \pi_6(\mathrm{map}(\Sigma^6\mathbb{C}P^2, S^7; f)) \to 0.$$

Since S^7 is an H-space, we have P_α is trivial. Thus we have $i'_* : [\Sigma^{12}\mathbb{C}P^2, S^7] \to \pi_6(\mathrm{map}(\Sigma^6\mathbb{C}P^2, S^7; f))$ is an isomorphism. By Proposition 3.7 of [8] this proof is complete.

(7) We apply $A = S^0$, $B = \Sigma^6\mathbb{C}P^2$, $X = S^7$ and $r = 7$ to Theorem 9. Then we have an exact sequence

$$\pi_8(S^8) \xrightarrow{P_\alpha} [\Sigma^{14}\mathbb{C}P^2, S^8] \xrightarrow{i'_*} \pi_7(\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; f)) \to 0.$$

First we consider that $f = *$ and $\alpha = [f]$. Then we have P_α is trivial. Thus we $i'_* : [\Sigma^{14}\mathbb{C}P^2, S^8] \to \pi_7(\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; f))$ is an isomorphism. Second we consider that $\alpha = [f] \in \{\nu_8 \circ \Sigma^7 p, 3\nu_8 \circ \Sigma^7 p\}$. Then we have

$$\mathrm{Coker}P_\alpha = \mathbb{Z}_2\{2\sigma_8 \circ \nu_{15} \circ \Sigma^{14} p\} \oplus \mathbb{Z}_3^2,$$

where $2\sigma_8 \circ \nu_{15} \circ \Sigma^{14} p = x\nu_8 \circ \sigma_{11}\Sigma^{14} p$ for some odd x. By isomorphism $i'_* : \mathrm{Coker}P_\alpha \to \pi_7(\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; f))$ we have (b). Third we consider that $\alpha = [f] = 2\nu_8 \circ \Sigma^7 p$. By Theorem 8 we have

$$\mathrm{Coker}P_\alpha = \mathbb{Z}_4\{\sigma_8 \circ \nu_{15} \circ \Sigma^{14} p\} \oplus \mathbb{Z}_2\{\nu_8 \circ \sigma_{11} \circ \Sigma^{14}\} \oplus \mathbb{Z}_3^2.$$

By isomorphism $i'_* : \mathrm{Coker}P_\alpha \to \pi_7(\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; f))$ we have (c). Finally we consider that $\alpha = [f] \in \{\alpha_1(9) \circ \Sigma^7 p, 2\alpha_1(9) \circ \Sigma^7 p\}$. By Theorem 8 we have

$$\mathrm{Coker}P_\alpha = \mathbb{Z}_4^2\{\sigma_8 \circ \nu_{15} \circ \Sigma^{14} p, \nu_8 \circ \sigma_{11} \circ \Sigma^{14}\} \oplus \mathbb{Z}_3\{\beta_1(8) \circ \Sigma^{14} p\}.$$

By isomorphism $i'_* : \mathrm{Coker}P_\alpha \to \pi_7(\mathrm{map}(\Sigma^7\mathbb{C}P^2, S^8; f))$ we have (d).

(8) We apply $A = S^0$, $B = \Sigma^7\mathbb{C}P^2$, $X = S^8$ and $r = 8$ to Theorem 9. Then we have an exact sequence

$$\pi_9(S^9) \xrightarrow{P_\alpha} [\Sigma^{16}\mathbb{C}P^2, S^9] \xrightarrow{i'_*} \pi_8(\mathrm{map}(\Sigma^8\mathbb{C}P^2, S^9; f)) \to 0.$$

First we consider $\alpha = [f] \in \langle \nu_9 \circ \Sigma^9 p, 3\nu_9 \circ \Sigma^9 p \rangle$. By Theorem 8 we have P_α is trivial. By Theorem 9 we have (a). Finally we consider $\alpha = [f] \in \{\nu_9 \circ \Sigma^8 p, 3\nu_9 \circ \Sigma^8\}$. By Theorem 8 we have

$$\begin{aligned}\mathrm{Coker}P_\alpha = {} & \mathbb{Z}_8\{\overline{(E\sigma') \circ \eta_{15}^2 + \eta_8 \circ \varepsilon_9}\} \oplus \mathbb{Z}_2\{\nu_9^2 \circ \overline{\nu_{15}}\} \\ & \oplus \mathbb{Z}_7\{\alpha_{1,7}(9) \circ \Sigma^{14} p\} \oplus \mathbb{Z}_9\{\alpha'_3(9) \circ \Sigma^{14} p\}.\end{aligned}$$

By an isomorphism $i'_* : \mathrm{Coker}P_\alpha \to \pi_8(\mathrm{map}(\Sigma^8\mathbb{C}P^2, S^9; f))$ we have (B). $\square$

Corollary 3. (1) *For* $[f],[g] \in [\Sigma\mathbb{C}P^2, S^2]$*,* $map(\Sigma\mathbb{C}P^2, S^2; f)$ *is homotopy equivalent to* $map(\Sigma\mathbb{C}P^2, S^2; g)$*.*

(2) *For* $[f],[g] \in [\Sigma^2\mathbb{C}P^2, S^3]$*,* $map(\Sigma^2\mathbb{C}P^2, S^3; f)$ *is homotopy equivalent to* $map(\Sigma^2\mathbb{C}P^2, S^3; g)$*.*

(3) *For* $[f] \in \langle 2\nu_4 \circ \Sigma^3 p, \Sigma\nu' \circ \Sigma^3 p\rangle$*,* $[g] \in \{(2n+1)\nu_4 \circ \Sigma^3 p \,|\, n \in \mathbb{Z}\}$ *and* $[h] \in \{\alpha_1(4)\circ\Sigma^3 p, 3\alpha_1(4)\circ\Sigma^3 p\}$*, four path-components* map_f*,* map_{f+g}*,* map_{h+f} *and* map_{g+h} *have different homotopy types each other.*

(4) *For* $[f],[g] \in [\Sigma^4\mathbb{C}P^2, S^5]$*,* $map(\Sigma^4\mathbb{C}P^2, S^5; f)$ *is homotopy equivalent to* $map(\Sigma^4\mathbb{C}P^2, S^5; g)$*.*

(5) *For* $[f] \in \langle 2\nu_6 \circ \Sigma^5 p\rangle$*,* $[g] \in \{\nu_6 \circ \Sigma^5 p, 3\nu_6 \circ \Sigma^5 p\}$ *and* $[h] \in \{\alpha_1(6) \circ \Sigma^5 p, 2\alpha_1(6)\circ\Sigma^5 p\}$*, four path-components* map_f*,* map_{f+g}*,* map_{h+f} *and* map_{g+h} *have different homotopy types each other.*

(6) *For* $[f],[g] \in [\Sigma^6\mathbb{C}P^2, S^7]$*,* $map(\Sigma^6\mathbb{C}P^2, S^7; f)$ *is homotopy equivalent to* $map(\Sigma^6\mathbb{C}P^2, S^7; g)$*.*

(7) *For* $[f] = \nu_8 \circ \Sigma^7 p$ *and* $[g] = \alpha_1(8) \circ \Sigma^7 p$*, six path-components* map_**,* map_f*,* map_{2f}*,* map_g*,* map_{f+g} *and* map_{2f+g} *have different homotopy types each other.*

(8) *For* $[g] \in \{\nu_9 \circ \Sigma^8 p, 3\nu_9 \circ \Sigma^8 p\}$ *and* $[f] \in \langle \alpha_1(9) \circ \Sigma^8 p, 2\nu_9 \circ \Sigma^8 p\rangle$*, two path-components* map_f *and* map_{f+g} *have different homotopy types.*

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