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SOME RESULTS ON THE UNIQUE RANGE SETS

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Abstract. In this paper, we exhibit the equivalence between different notions of unique range sets, namely, unique range sets, weighted unique range sets and weak-weighted unique range sets under certain conditions. Also, we present some uniqueness theorems which show how two meromorphic functions are uniquely determined by their two finite shared sets. Moreover, in the last section, we make some observations that help us to construct other new classes of unique range sets.

1. Introduction: Unique range sets

We use $M(\mathbb{C})$ to denote the field of all meromorphic functions. Let $f \in M(\mathbb{C})$ and $S \subset \mathbb{C} \cup \{\infty\}$ be a non-empty set with distinct elements. We set

$$E_f(S) = \bigcup_{a \in S} \{z : f(z) - a = 0\},$$

where a zero of $f - a$ with multiplicity m counts m times in $E_f(S)$. Let $\overline{E}_f(S)$ denote the collection of distinct elements in $E_f(S)$.

Let $g \in M(\mathbb{C})$. We say that two functions f and g share the set S CM (resp. IM) if $E_f(S) = E_g(S)$ (resp. $\overline{E}_f(S) = \overline{E}_g(S)$).

In 1968, F. Gross ([14]) first studied the uniqueness problem of meromorphic functions that share distinct sets instead of values. Since then, the uniqueness theory of meromorphic functions under the set sharing environment has become one of the important branches in the value distribution theory.

In the same paper ([14]), F. Gross proved that there exist three finite sets S_j $(j = 1, 2, 3)$ such that if two non-constant entire functions f and g share them, then $f \equiv g$. Later, in 1976, he asked the following question:

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Question 1.1 ([15])**.** Can one find two (or possible even one) finite sets S_j ($j = 1, 2$) such that if two non-constant entire functions f and g share them, then $f \equiv g$?

In 1982, F. Gross and C. C. Yang ([16]) first ensured the existence of such set. They proved that if two non-constant entire functions f and g share the set $S = \{z \in \mathbb{C} : e^z + z = 0\}$, then $f \equiv g$.

Moreover, this type of set was termed as a unique range set for entire functions. Later, similar definition for meromorphic functions was also introduced in the literature.

Definition 1.1 ([20])**.** Let $S \subset \mathbb{C} \cup \{\infty\}$; f and g be two non-constant meromorphic (resp. entire) functions. If $E_f(S) = E_g(S)$ implies $f \equiv g$, then S is called a unique range set for meromorphic (resp. entire) functions or in brief URSM (resp. URSE).

Here, we note that the set provided by Gross and Yang ([16]) was an infinite set. Thus after the introduction to the idea of unique range sets, many efforts were made to seek unique range sets with cardinalities as small as possible.

In 1994, H. X. Yi ([22]), settled the question of Gross by exhibiting a unique range set for entire functions with 15 elements.

In the next year, P. Li and C. C. Yang ([19]) exhibited a unique range set for meromorphic (resp. entire) functions with 15 (resp. 7) elements. They considered the zero set of the following polynomial:

$$P(z) = z^n + az^{n-m} + b, \tag{1.1}$$

where a and b are two non-zero constants such that $z^n + az^{n-m} + b = 0$ has no multiple roots. Also, $m \geq 2$ (resp. 1), $n > 2m + 10$ (resp. $2m + 4$) are integers with n and $n - m$ having no common factors.

In 1996, H. X. Yi ([24]) further improved the result of Li and Yang ([19]) and obtained a unique range set for meromorphic functions with 13 elements.

Also, in 2002, T. T. H. An ([2]) exhibited another new class of unique range set for meromorphic functions with 13 elements by considering the zero set of the following polynomial:

$$P(z) = z^n + az^{n-m} + bz^{n-2m} + c, \tag{1.2}$$

where it was assumed that $P(z) = 0$ has no multiple roots and $a, b, c \in \mathbb{C} \setminus \{0\}$ such that $a^2 \neq 4b$. Also, n and $2m$ are two positive integers such that n and $2m$ have no common factors and $n > 8 + 4m$.

As an attempt to reduce the cardinality of the unique range sets, in 1998, G. Frank and M. Reinders ([11]) studied the zero set of the following polynomial and obtained a unique range set with 11 elements.

$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c, \tag{1.3}$$

where $n \geq 11$ and $c \neq 0, 1$.

And, till today, this is the smallest available unique range set for meromorphic functions. But, in 2007, T. C. Alzahary ([1]) exhibited another new class of unique range set for meromorphic functions with 11 elements by considering the zero set of the following polynomial:

$$P(z) = az^n - n(n-1)z^2 + 2n(n-2)bz - (n-1)(n-2)b^2, \tag{1.4}$$

where a and b are two non-zero complex numbers satisfying $ab^{n-2} \neq 1, 2$ and $n \geq 11$.

Recently, another new class of unique range set for meromorphic functions with 11 elements were exhibited in ([7]) using the zero set of the following polynomial:

$$P(z) = z^n - \frac{2n}{n-m} z^{n-m} + \frac{n}{n-2m} z^{n-2m} + c, \tag{1.5}$$

where c is any complex number satisfying $|c| \neq \frac{2m^2}{(n-m)(n-2m)}$ and $c \neq 0$, $c \neq -\frac{1-\frac{2n}{n-m}+\frac{n}{n-2m}}{2}$ and $m \geq 1$, $n > \max\{2m+8, 4m+1\}$.

Until now, the best results were given by H. Fujimoto ([12]) in 2000. He gave a generic unique range set for meromorphic functions of at least 11 elements when multiplicities are counted. To state those results, we need to explain some definitions.

Definition 1.2 ([12, 13])**.** Let $P(z)$ be a non-constant monic polynomial. We call $P(z)$ as a "uniqueness polynomial in broad sense" if $P(f) \equiv P(g)$ implies $f \equiv g$ for any two non-constant meromorphic functions f, g; while a "uniqueness polynomial" if $P(f) \equiv cP(g)$ implies $f \equiv g$ for any two non-constant meromorphic functions f, g and non-zero constant c.

For a discrete subset $S = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{C}$ $(a_i \neq a_j)$, we consider the following polynomial

$$P(z) = (z - a_1)(z - a_2) \cdots (z - a_n). \tag{1.6}$$

Assume that the derivative $P'(z)$ has k distinct zeros $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$ respectively. Under the assumption that

$$P(d_{l_s}) \neq P(d_{l_t}) \ (1 \leq l_s < l_t \leq k), \tag{1.7}$$

H. Fujimoto ([12]) proved the following theorem:

Theorem 1.1 ([12])**.** *Let $P(z)$ be a "uniqueness polynomial" of the form* (1.6) *satisfying the condition* (1.7)*. Moreover, either $k \geq 3$ or $k = 2$ and $\min\{q_1, q_2\} \geq 2$.*

If S is the set of zeros of $P(z)$, then S is a unique range set for meromorphic (*resp. entire*) *function whenever $n > 2k + 6$* (*resp. $n > 2k + 2$*).

Remark 1.1. We note that Theorem 1.1 gives the best possible generic unique range set for meromorphic function when $k = 2$ (i.e., unique range set with 11 elements).

In 2017, in ([8]), the form of the polynomial when $k = 2$ was illustrated in more general settings. If $k = 2$, then the unique range set generating polynomial is the following polynomial:

$$P(z) = Q(z) + c, \tag{1.8}$$

where

$$Q(z) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} \frac{(-1)^{i+j}}{n+m+1-i-j} z^{n+m+1-i-j} a^j b^i,$$

$a \neq b$, $b \neq 0$, $c \notin \{0, -Q(a), -Q(b), -\frac{Q(a)+Q(b)}{2}\}$. Also, m, n are two integers such that $m + n > 9$, $\max\{m, n\} \geq 3$ and $\min\{m, n\} \geq 2$.

Remark 1.2. If we take $a = 0$ and $b = 1$ in (1.8), then we get the unique range set generating polynomial of degree at least 11. For details, see [5, 6].

2. Unique range sets with weight two

Let l be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_l(a; f)$, the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq l$ and $l + 1$ times if $m > l$.

If for two non-constant meromorphic functions f and g, we have

$$E_l(a; f) = E_l(a; g),$$

then we say that f *and* g *share the value* a *with weight* l. Thus the IM and CM sharing respectively correspond to the weight 0 and ∞. This idea of weighted sharing was first introduced in ([18]).

Let $S \subset \mathbb{C} \cup \{\infty\}$. We define $E_f(S, l)$ as

$$E_f(S, l) = \bigcup_{a \in S} E_l(a; f),$$

where l is a non-negative integer or infinity. Clearly $E_f(S) = E_f(S, \infty)$.

Let l be a non-negative integer or infinity. A set $S \subset \mathbb{C}$ is called a unique range set for meromorphic (resp. entire) functions with weight l, in short, URSM_l (resp. URSE_l) if for any two non-constant meromorphic (resp. entire) functions f and g, the condition

$$E_f(S, l) = E_g(S, l)$$

implies $f \equiv g$.

In the last few years, the notion of weighted sharing took place a major role in study of the uniqueness theory of meromorphic functions. As a result most of the existing unique range sets were relaxed to the unique range sets with weight two, but the cardinality of the respective unique range sets remained same when the sharing environment is relaxed from the CM sharing to the weighted sharing with weight two.

In this direction, in 2012, A. Banerjee and I. Lahiri made the following observations:

Theorem 2.1 ([10])**.** *Let $P(z) = a_n z^n + \sum_{j=2}^{m} a_j z^j + a_0$ be a polynomial of degree n, where $n - m \geq 3$ and $a_p a_m \neq 0$ for some positive integer p with $2 \leq p \leq m$ and $\gcd(p, 3) = 1$. Suppose further that $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of all distinct zeros of $P(z)$. Let k be the number of distinct zeros of the derivative $P'(z)$. If $n \geq 2k + 7$ (resp. $2k + 3$), then the following statements are equivalent*:

(i) *P is a "uniqueness polynomial" for meromorphic (resp. entire) function.*
(ii) *S is a URSM$_2$ (resp. URSE$_2$).*
(iii) *S is a URSM (resp. URSE).*
(iv) *P is a "uniqueness polynomial in broad sense" for meromorphic (resp. entire) function.*

To prove Theorem 2.1, the authors used the following lemma:

Lemma 2.1 ([10], Lemma 2.1)**.** *Let $P(z) = a_n z^n + \sum_{j=2}^{m} a_j z^j + a_0$ be a polynomial of degree n, where $n - m \geq 3$ and $a_p a_m \neq 0$ for some positive integer p with $2 \leq p \leq m$ and $\gcd(p, 3) = 1$. Suppose that*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1,$$

where f and g are non-constant meromorphic functions and $c_0 (\neq 0)$, c_1 are constants. If $n \geq 6$, then $c_1 = 0$.

It is noted that in Lemma 2.1, the condition $n - m \geq 3$ is necessary. Thus the polynomial considered in Theorem 2.1 is a specific polynomial (i.e., $n - m \geq 3$, i.e., a gap after the n-th degree term).

The main observation of this paper is that the *condition "$n - m \geq 3$" is not necessary* in order to show the equivalency between a unique range set with counting multiplicity and a unique range set with weight two.

Before going to state our main results, we need to introduce the deficiency functions ([17, 20]).

Let $a \in \mathbb{C} \cup \{\infty\}$, we set

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)},$$

$$\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

The quantity $\delta(a; f)$ is called the deficiency of the value a. Clearly

$$0 \leq \delta(a; f) \leq \Theta(a; f) \leq 1.$$

Now, we state the main result of this paper.

Theorem 2.2. *Let $P(z)$ be a polynomial of degree n such that $P(z) = a_0(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$; where $\alpha_i \neq \alpha_j$, $1 \leq i, j \leq n$. Further suppose that $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of all distinct zeros of $P(z)$. Let k be the number* *of distinct zeros of the derivative $P'(z)$. Let f and g be two non-constant meromorphic functions such that*

$$\Theta(\infty; f) + \Theta(\infty; g) + \frac{1}{2}\min\{\delta(0,f), \delta(0,g)\} > \frac{2k+6-n}{2}.$$

Then the following two statements are equivalent:

(a) *If $E_f(S,2) = E_g(S,2)$, then $f \equiv g$.*

(b) *If $E_f(S) = E_g(S)$, then $f \equiv g$.*

The following corollary is an immediate consequence of Theorem 2.2.

Corollary 2.1. *Let $P(z)$ be a polynomial of degree n such that $P(z) = a_0(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$; where $\alpha_i \neq \alpha_j$, $1 \leq i,j \leq n$. Further suppose that $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of all distinct zeros of $P(z)$. Let k be the number of distinct zeros of the derivative $P'(z)$. If $n \geq 2k+7$ (resp. $2k+3$), then the following two statements are equivalent*:

(a) *S is a URSM$_2$ (resp. URSE$_2$).*

(b) *S is a URSM (resp. URSE).*

Remark 2.1. Now, we consider the following polynomial:

$$P(z) = z^n + az^{n-m} + b, \tag{2.1}$$

where a and b are two non-zero constants such that $z^n + az^{n-m} + b = 0$ has no multiple roots; $m \geq 2$ (resp. 1), $n \geq 2m+9$ (resp. $2m+5$) are integers with n and $n-m$ having no common factors.

Here $k = m+1$ and this polynomial satisfies the assumptions of Corollary 2.1. Since the zero set of the polynomial gives URSM (resp. URSE) with 13 (resp. 7) elements ([24]), thus the zero set of the polynomial gives URSM$_2$ (resp. URSE$_2$) with 13 (resp. 7) elements.

Remark 2.2. The next polynomial was due to G. Frank and M. Reinders ([11]).

$$P(z) = \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - c, \tag{2.2}$$

where $n \geq 11$ and $c \neq 0, 1$.

Here $k = 2$ and this polynomial satisfies the assumptions of Corollary 2.1. Since the zero set of the corresponding polynomial gives URSM (resp. URSE) with 11 (resp. 7) elements ([11]), thus the zero set of the respective polynomial gives URSM$_2$ (resp. URSE$_2$) with 11 (resp. 7) elements.

Remark 2.3. The polynomial described in the equation (1.8) satisfies the assumptions of Corollary 2.1. Here also, $k = 2$. Since the zero set of the polynomial gives URSM (resp. URSE) with 11 (resp. 7) elements ([8]), thus the zero set of the polynomial gives URSM$_2$ (resp. URSE$_2$) with 11 (resp. 7) elements.

Now, we explain some definitions and notations which are used to proceed further.

Definition 2.1 ([20])**.** Let $a \in \mathbb{C} \cup \{\infty\}$ and $m \in \mathbb{N}$.

(i) We denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f.

(ii) We denote by $N(r, a; f \mid\leq m)$ (resp. $N(r, a; f \mid\geq m)$ by the counting function of those a-points of f whose multiplicities are not greater (resp. less) than m where each a-point is counted according to its multiplicity.

Similarly, $\overline{N}(r, a; f \mid\leq m)$ and $\overline{N}(r, a; f \mid\geq m)$ are the reduced counting function of $N(r, a; f \mid\leq m)$ and $N(r, a; f \mid\geq m)$ respectively.

Definition 2.2 ([20])**.** Let f and g be two non-constant meromorphic functions such that f and g share a IM. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q.

(i) We denote by $\overline{N}_L(r, a; f)$ the reduced counting function of those a-points of f and g where $p > q$.

(ii) We denote by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$.

(iii) We denote by $\overline{N}_E^{(2}(r, a; f)$ the reduced counting function of those a-points of f and g where $p = q \geq 2$.

In the same way, we can define $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$. When f and g share a with weight m, $m \geq 1$ then

$$N_E^{1)}(r, a; f) = N(r, a; f \mid= 1).$$

Definition 2.3 ([20])**.** Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly

$$\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f) \text{ and } \overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g).$$

Proof of Theorem 2.2. The case (a)⇒(b) is obvious. So, we only prove the case (b)⇒(a).

Let f and g be two non-constant meromorphic functions share the set

$$S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

with weight 2 and

$$\Theta(\infty; f) + \Theta(\infty; g) + \frac{1}{2}\min\{\delta(0, f), \delta(0, g)\} > \frac{2k + 6 - n}{2}.$$

In this case, our claim is to show that $f \equiv g$. For that, we put

$$P(z) = a_0(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n),$$

and

$$F(z) := \frac{1}{P(f(z))} \text{ and } G(z) := \frac{1}{P(g(z))}.$$

Let $S(r)$ be any function $S(r) : (0, \infty) \to \mathbb{R}$ satisfying $S(r) = o(T(r, F) + T(r, G))$ for $r \to \infty$ outside a set of finite Lebesgue measure.

Let

$$H(z) := \frac{F''(z)}{F'(z)} - \frac{G''(z)}{G'(z)},$$

and this function H was introduced by H. Fujimoto ([12]).

Now we consider two cases:

Case-I. First we assume that $H \not\equiv 0$. It is given that

$$2k + 6 - 2\Theta(\infty; f) - 2\Theta(\infty; g) - \min\{\delta(0, f), \delta(0, g)\} + \epsilon < n,$$

where ϵ is a small positive number.

Since $H(z)$ can be expressed as

$$H(z) = \frac{G'(z)}{F'(z)} \left(\frac{F'(z)}{G'(z)}\right)',$$

so all poles of H are simple. Also, *poles of H may occur* at

(1) poles of F and G.
(2) zeros of F' and G',

Now, by simple calculations, one can show that "simple poles" of F are the zeros of H. Thus

$$N(r, \infty; F| = 1) = N(r, \infty; G| = 1) \leq N(r, 0; H). \tag{2.3}$$

Now, using the lemma of logarithmic derivative and the first fundamental theorem, (2.3) can be written as

$$N(r, \infty; F| = 1) = N(r, \infty; G| = 1) \leq N(r, \infty; H) + S(r). \tag{2.4}$$

Let $\beta_1, \beta_2, \ldots, \beta_k$ be the k-distinct zeros of $P'(z)$. Since

$$F'(z) = -\frac{f'(z)P'(f(z))}{(P(f(z)))^2},\ G'(z) = -\frac{g'(z)P'(g(z))}{(P(g(z)))^2}$$

and f, g share S with weight 2, so by simple calculations, we can write

$$\begin{aligned} &N(r, \infty; H) \\ \leq &\sum_{j=1}^{k} \left(\overline{N}(r, \beta_j; f) + \overline{N}(r, \beta_j; g)\right) + \overline{N}_0(r, 0; f') + \overline{N}_0(r, 0; g') \\ &+ \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, \infty; F, G), \end{aligned} \tag{2.5}$$

where $\overline{N}_0(r, 0; f')$ denotes the reduced counting function of zeros of f', which are not zeros of $\prod_{i=1}^{n}(f - \alpha_i) \prod_{j=1}^{k}(f - \beta_j)$, similarly, $\overline{N}_0(r, 0; g')$ is defined.

Since

$$\begin{aligned} &\overline{N}(r, \infty; F| \geq 2) + \overline{N}_0(r, 0, g') + \overline{N}_*(r, \infty; F, G) \\ \leq &\overline{N}(r, 0; P(g)| \geq 2) + \overline{N}_0(r, 0, g') + \overline{N}(r, 0; P(g)| \geq 3) \\ \leq &N(r, 0; g'), \end{aligned} \tag{2.6}$$

so, using Lemma 3 of ([21]), we get

$$\overline{N}(r,\infty;F|\geq 2)+\overline{N}_0(r,0,g')+\overline{N}_*(r,\infty;F,G) \tag{2.7}$$
$$\leq N(r,0;g)+\overline{N}(r,\infty;g)+S(r,g).$$

Put $T(r)=\max\{T(r,f),T(r,g)\}$ and $\delta(0)=\min\{\delta(0,f),\delta(0,g)\}$. Now, for any $\varepsilon(>0)$, using the second fundamental theorem and (2.4), (2.5) and (2.7), we have

$$(n+k-1)T(r,f) \tag{2.8}$$
$$\begin{aligned}
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;P(f))+\sum_{j=1}^{k}\overline{N}(r,\beta_j;f)-\overline{N}_0(r,0;f')+S(r,f)\\
&\leq 2\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\sum_{j=1}^{k}\big(2\overline{N}(r,\beta_j;f)+\overline{N}(r,\beta_j;g)\big)\\
&\quad +\overline{N}(r,\infty;F|\geq 2)+\overline{N}_0(r,0;g')+\overline{N}_*(r,\infty;F,G)+S(r)\\
&\leq 2\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+2kT(r,f)+kT(r,g)+N(r,0;g)+S(r)\\
&\leq \ (3k+5-2\Theta(\infty;f)-2\Theta(\infty;g)-\delta(0)+\varepsilon)\,T(r)+S(r).
\end{aligned}$$

Similarly,

$$(n+k-1)T(r,g) \tag{2.9}$$
$$\leq \ (3k+5-2\Theta(\infty;g)-2\Theta(\infty;f)-\delta(0)+\varepsilon)\,T(r)+S(r).$$

Thus comparing (2.8) and (2.9), we have

$$(n+k-1)T(r) \tag{2.10}$$
$$\leq \ (3k+5-2\Theta(\infty;g)-2\Theta(\infty;f)-\delta(0)+\varepsilon)\,T(r)+S(r),$$

which contradicts the assumption that

$$\Theta(\infty;f)+\Theta(\infty;g)+\frac{1}{2}\min\{\delta(0,f),\delta(0,g)\}>\frac{2k+6-n}{2}.$$

Hence $H\equiv 0$.

Case-II. Next we assume that $H\equiv 0$. Then by integration, we have

$$\frac{1}{P(f(z))}\equiv\frac{c_0}{P(g(z))}+c_1,$$

i.e.,

$$\frac{1}{a_0(f-\alpha_1)(f-\alpha_2)\cdots(f-\alpha_n)}\equiv\frac{c_0}{a_0(g-\alpha_1)(g-\alpha_2)\cdots(g-\alpha_n)}+c_1,$$

where c_0 is a non-zero complex constant. If z_0 is an α_i point of f of multiplicity m, then it is a pole of $\frac{1}{P(f(z))}$ of order m, hence it is a pole of $\frac{1}{P(g(z))}$ of order m, i.e., z_0 is an α_j point of g of order m for some $j\in\{1,2,\ldots,n\}$.

Thus f and g share the set $S=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ in counting multiplicity. Since S is an URSM, so $f\equiv g$. This completes the proof. $\square$

3. Unique range sets with weak weight three

Let l be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_{l)}(a; f)$, the set of all a-points of f, whose multiplicities are not greater than l and each such a-points are counted according to its multiplicity.

If for two non-constant meromorphic functions f and g, we have

$$E_{l)}(a; f) = E_{l)}(a; g),$$

then we say that f and g share the value a with "weak-weight l".

Let $S \subset \mathbb{C} \cup \{\infty\}$. We put

$$E_{l)}(S, f) = \bigcup_{a \in S} E_{l)}(a; f),$$

where l is a positive integer or infinity.

A set $S \subset \mathbb{C}$ is called a unique range set for meromorphic (resp. entire) functions with weak weight l, in short URSM$_{l)}$ (resp. URSE$_{l)}$) if for any two non-constant meromorphic (resp. entire) functions f and g, $E_{l)}(S, f) = E_{l)}(S, g)$ implies $f \equiv g$.

In 2009, X. Bai, Q. Han and A. Chen ([4]) proved the following "weak-weighted" sharing version of Fujimoto's Theorem:

Theorem 3.1 ([4]). *In addition to the hypothesis of Theorem 1.1, further we suppose that $l \geq 3$ is a positive integer or ∞.*

If S is the set of zeros of $P(z)$ and $n > 2k + 6$ (resp. $n > 2k + 2$), then S is a URSM$_{l)}$ (resp. URSE$_{l)}$).

Using the concept of weighted sharing and weak-weighted sharing, Banerjee and Lahiri ([10]) gave some equivalence between the different notions of unique range sets and uniqueness polynomials as follows:

Theorem 3.2 ([10]). *Let $P(z) = a_n z^n + \sum_{j=2}^{m} a_j z^j + a_0$ be a polynomial of degree n, where $n - m \geq 3$ and $a_p a_m \neq 0$ for some positive integer p with $2 \leq p \leq m$ and $\gcd(p, 3) = 1$. Suppose further that $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of all distinct zeros of $P(z)$. Let k be the number of distinct zeros of the derivative $P'(z)$. If $n \geq 2k + 7$ (resp. $2k + 3$), then the following statements are equivalent*:

(i) *P is a "uniqueness polynomial" for meromorphic (resp. entire) function.*
(ii) *S is a URSM$_{3)}$ (resp. URSE$_{3)}$).*
(iii) *S is a URSM (resp. URSE).*
(iv) *P is a "uniqueness polynomial in broad sense" for meromorphic (resp. entire) function.*

Here, we also observed that to show the equivalence between the statements (ii) and (iii) in Theorem 3.2, one does not need the condition "$n - m \geq 3$". Now, we state our next result:

Theorem 3.3. *Let $P(z)$ be a polynomial of degree n such that $P(z) = a_0(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$; where $\alpha_i \neq \alpha_j$, $1 \leq i, j \leq n$. Further suppose that $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of all distinct zeros of $P(z)$. Let k be the number of distinct zeros of the derivative $P'(z)$. Let f and g be two non-constant meromorphic functions such that*

$$\Theta(\infty; f) + \Theta(\infty; g) + \frac{1}{2}\min\{\delta(0, f), \delta(0, g)\} > \frac{2k + 6 - n}{2}.$$

Then the following two statements are equivalent:

(a) *If $E_{3)}(S, f) = E_{3)}(S, g)$, then $f \equiv g$.*
(b) *If $E_f(S) = E_g(S)$, then $f \equiv g$.*

The proof of this theorem is similar to the proof of Theorem 2.2. So we omit the details.

Corollary 3.1. *Let $P(z)$ be a polynomial of degree n such that $P(z) = a_0(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$; where $\alpha_i \neq \alpha_j$, $1 \leq i, j \leq n$. Further suppose that $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of all distinct zeros of $P(z)$. Let k be the number of distinct zeros of the derivative $P'(z)$. If $n \geq 2k + 7$ (resp. $2k + 3$), then the following two statements are equivalent*:

(a) *S is a URSM$_{3)}$ (resp. URSE$_{3)}$).*
(b) *S is a URSM (resp. URSE).*

The next result is obvious in view of Theorem 2.2 and Theorem 3.3:

Corollary 3.2. *Let $P(z)$ be a polynomial of degree n such that $P(z) = a_0(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n)$; where $\alpha_i \neq \alpha_j$, $1 \leq i, j \leq n$. Further suppose that $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is the set of all distinct zeros of $P(z)$. Let k be the number of distinct zeros of the derivative $P'(z)$. Let f and g be two non-constant meromorphic functions such that*

$$\Theta(\infty; f) + \Theta(\infty; g) + \frac{1}{2}\min\{\delta(0, f), \delta(0, g)\} > \frac{2k + 6 - n}{2}.$$

Then the following statements are equivalent:

(a) *If $E_f(S) = E_g(S)$, then $f \equiv g$.*
(b) *If $E_f(S, 2) = E_g(S, 2)$, then $f \equiv g$.*
(c) *If $E_{3)}(S, f) = E_{3)}(S, g)$, then $f \equiv g$.*

4. Functions sharing two sets

In connection to the Gross's question (Question 1.1), in 1994, H. X. Yi ([23]) gave the existence of two finite sets S_1 (with 5 elements) and S_2 (with one element) such that if any two non-constant entire functions f and g satisfying the condition $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f \equiv g$.

Later in 1998, the same author ([25]) proved that there exist two finite sets S_1 (with 3 elements) and S_2 (with one element) such that any two non-constant entire functions f and g satisfying the condition $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f \equiv g$.

In the same paper ([25]), another nice observation was made.

Theorem 4.1 ([25])**.** *If S_1 and S_2 are two sets of finite distinct complex numbers such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$, must be identical, then* $\max\{\sharp(S_1), \sharp(S_2)\} \geq 3$, *where $\sharp(S)$ denotes the cardinality of the set S.*

Thus for the uniqueness of two entire functions when they share two sets, it is clear that the smallest cardinalities of S_1 and S_2 are 1 and 3 respectively.

Later, the Gross' Question for meromorphic functions was also introduced in the literature as follows:

Question 4.1 ([23])**.** Can one find two finite sets S_j $(j = 1, 2)$ such that if two non-constant meromorphic functions f and g share them, then $f \equiv g$?

In 1994, H. X. Yi ([23]) completely answered Question 4.1 by giving the existence of two finite sets S_1 (with 9 elements) and S_2 (with 2 elements) such that if any two non-constant meromorphic functions f and g satisfying the condition $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f \equiv g$.

In this direction, in 2012, B. Yi and Y. H. Li ([26]) provided a significant result. They proved that there exist two finite sets S_1 (with 5 elements) and S_2 (with 2 elements) such that if any two non-constant meromorphic functions f and g satisfying the condition $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f \equiv g$.

The motivation of writing this section is to answer Question 4.1 by giving the *existence of two generic sets* S_1 and S_2 for meromorphic functions such that if any two non-constant meromorphic functions f and g satisfying the condition $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$, then $f \equiv g$.

Suppose

$$P(z) = a_0(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n), \tag{4.1}$$

where $\alpha_i \neq \alpha_j$, $1 \leq i, j \leq n$; $a_0 \neq 0$. Further suppose that

$$P'(z) = b_0(z - \beta_1)^{q_1}(z - \beta_2)^{q_2} \cdots (z - \beta_k)^{q_k} \tag{4.2}$$

satisfying the assumption (this property was introduced by H. Fujimoto ([12])) that

$$P(\beta_{l_s}) \neq P(\beta_{l_t}) \; (1 \leq l_s < l_t \leq k). \tag{4.3}$$

Now, we state our main two theorems of this section:

Theorem 4.2. *Let $P(z)$ be a "uniqueness polynomial" of the form* (4.1) *satisfying the condition* (4.3)*. Further suppose that $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $S_2 = \{\beta_1, \beta_2, \ldots, \beta_k\}$.*

If two non-constant meromorphic (*resp. entire*) *functions f and g share the set S_1 with weight two and S_2 IM, $k \geq 3$ and $n \geq k + 7$* (*resp. $k + 3$*)*, then $f \equiv g$.*

Theorem 4.3. *Let $P(z)$ be a "uniqueness polynomial" of the form* (4.1) *satisfying the condition* (4.3). *Further suppose that $S_1 = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and $S_2 = \{\beta_1, \beta_2, \dots, \beta_k\}$.*

Moreover, assume that $k \geq 2$ and $P'(z)$ have no simple zeros. If two non-constant meromorphic (*resp. entire*) *functions f and g share the set S_1 with weight* 3 *and S_2 IM, and $n \geq \max\{10 - 2k, 5\}$* (*resp.* 5), *then $f \equiv g$.*

Example 4.1. Let

$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c,$$

where $c \in \mathbb{C} \setminus \{0, \frac{1}{2}, 1\}$ and $n \geq 6$.

Under these suppositions, $P(z)$ must be a "uniqueness polynomial" satisfying the condition (4.3) (Theorem 1.2, [9]). Suppose that

$$S_1 = \left\{ z : \frac{(n-1)(n-2)}{2} z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - c = 0 \right\}$$

and $S_2 = \{0, 1\}$. Then by Theorem 4.3, if two non-constant meromorphic functions f and g share S_1 with weight 3 and S_2 IM, then $f \equiv g$.

To prove the above two theorems, we need the following lemma:

Lemma 4.1 ([12, Proposition 7.1])**.** *Let $P(z)$ be a polynomial of degree $n \geq 5$ and of the form* (4.1) *satisfying the condition* (4.3). *Suppose that*

$$\frac{1}{P(f)} = \frac{c_0}{P(g)} + c_1,$$

where f and g are non-constant meromorphic functions and $c_0 (\neq 0)$, c_1 are constants. If $k \geq 3$, or if $k = 2$ and $P'(z)$ have no simple zeros, then $c_1 = 0$.

Proof of Theorem 4.2. Since f and g share S_2 IM, so

$$\sum_{j=1}^{k} \overline{N}(r, \beta_j; f) = \sum_{j=1}^{k} \overline{N}(r, \beta_j; g).$$

Now, we put

$$F(z) := \frac{1}{P(f(z))} \text{ and } G(z) := \frac{1}{P(g(z))}.$$

Let $S(r)$ be any function $S(r) : (0, \infty) \to \mathbb{R}$ satisfying $S(r) = o(T(r, F) + T(r, G))$ for $r \to \infty$ outside a set of finite Lebesgue measure.

Let

$$H(z) := \frac{F''(z)}{F'(z)} - \frac{G''(z)}{G'(z)}.$$

Now, we consider two cases:

Case-I. First we assume that $H \not\equiv 0$. Since $H(z)$ can be expressed as

$$H(z) = \frac{G'(z)}{F'(z)} \left(\frac{F'(z)}{G'(z)} \right)',$$

so all poles of H are simple. Also, *poles of* H *may occur* at

(1) poles of F and G.
(2) zeros of F' and G'.

But using the Laurent series expansion of H, it is clear that "simple poles" of F (hence, that of G) is a zero of H. Thus

$$N(r,\infty;F|=1)=N(r,\infty;G|=1)\le N(r,0;H). \tag{4.4}$$

Using the lemma of logarithmic derivative and the first fundamental theorem, (4.4) can be written as

$$N(r,\infty;F|=1)=N(r,\infty;G|=1)\le N(r,\infty;H)+S(r). \tag{4.5}$$

Since

$$F'(z)=-\frac{f'(z)P'(f(z))}{(P(f(z)))^2} \text{ and } G'(z)=-\frac{g'(z)P'(g(z))}{(P(g(z)))^2},$$

and f, g share $(S_1,2)$ and $(S_2,0)$, so by simple calculations, we can write

$$\begin{aligned} N(r,\infty;H)\le{}& \sum_{j=1}^{k}\overline{N}(r,\beta_j;f)+\overline{N}_0(r,0;f')+\overline{N}_0(r,0;g') \\ &+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;F,G), \end{aligned} \tag{4.6}$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function of zeros of f', which are not zeros of $\prod_{i=1}^{n}(f-\alpha_i)\prod_{j=1}^{k}(f-\beta_j)$; similarly, $\overline{N}_0(r,0;g')$ is defined. Now, using the second fundamental theorem and (4.5), (4.6), we have

$$\begin{aligned} &(n+k-1)\left(T(r,f)+T(r,g)\right) \\ &\le \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;P(f))+\overline{N}(r,0;P(g)) \\ &\quad+\sum_{j=1}^{k}\left(\overline{N}(r,\beta_j;f)+\overline{N}(r,\beta_j;g)\right)-\overline{N}_0(r,0;f')-\overline{N}_0(r,0;g')+S(r) \\ &\le 2\left(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right)+\sum_{j=1}^{k}\left(2\overline{N}(r,\beta_j;f)+\overline{N}(r,\beta_j;g)\right) \\ &\quad+\overline{N}(r,\infty;F|\ge 2)+\overline{N}(r,\infty;G)+\overline{N}_*(r,\infty;F,G)+S(r). \end{aligned} \tag{4.7}$$

Noting that

$$\overline{N}(r,\infty;F)-\frac{1}{2}N(r,\infty;F|=1)+\frac{1}{2}\overline{N}_*(r,\infty;F,G)\le\frac{1}{2}N(r,\infty;F),$$
$$\overline{N}(r,\infty;G)-\frac{1}{2}N(r,\infty;G|=1)+\frac{1}{2}\overline{N}_*(r,\infty;F,G)\le\frac{1}{2}N(r,\infty;G).$$

Thus (4.7) can be written as

$$\begin{aligned} &(n+k-1)\left(T(r,f)+T(r,g)\right) \\ &\le 2\left(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right)+\left(\frac{3}{2}k+\frac{n}{2}\right)\left(T(r,f)+T(r,g)\right)+S(r), \end{aligned} \tag{4.8}$$

which contradicts the assumption $n \geq k+7$ (resp. $k+3$). Thus $H \equiv 0$.

Case-II. Next we assume that $H \equiv 0$. Then by integration, we have

$$\frac{1}{P(f(z))} \equiv \frac{c_0}{P(g(z))} + c_1,$$

where $c_0(\neq 0)$, c_1 are constants.

Since $n \geq 5$ and $k \geq 3$, thus by applying Lemma 4.1 and noting the assumption that $P(z)$ is a "uniqueness polynomial", we have

$$f \equiv g.$$

This completes the proof. □

Proof of Theorem 4.3. Since f and g share S_2 IM, so

$$\sum_{j=1}^{k} \overline{N}(r, \beta_j; f) = \sum_{j=1}^{k} \overline{N}(r, \beta_j; g).$$

Now, we put

$$F(z) := \frac{1}{P(f(z))} \text{ and } G(z) := \frac{1}{P(g(z))}.$$

Let $S(r)$ be any function $S(r) : (0, \infty) \to \mathbb{R}$ satisfying $S(r) = o(T(r, F) + T(r, G))$ for $r \to \infty$ outside a set of finite Lebesgue measure.

Let

$$H(z) := \frac{F''(z)}{F'(z)} - \frac{G''(z)}{G'(z)}.$$

Now, we consider two cases:

Case-I. First we assume that $H \not\equiv 0$. Now, proceeding as Case-I of Theorem 4.2, we have from (4.7) that

$$\begin{aligned} (4.9) \qquad & (n+k-1)\left(T(r,f)+T(r,g)\right) \\ \leq\ & 2\left(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right) + \sum_{j=1}^{k}\left(2\overline{N}(r,\beta_j;f)+\overline{N}(r,\beta_j;g)\right) \\ & + \overline{N}(r,\infty;F| \geq 2) + \overline{N}(r,\infty;G) + \overline{N}_*(r,\infty;F,G) + S(r). \end{aligned}$$

Since f and g share the set S_1 with weight 3, so

$$\begin{aligned} & \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) - N(r,\infty;F| = 1) + \frac{5}{2}\overline{N}_*(r,\infty;F,G) \\ \leq\ & \frac{1}{2}(N(r,\infty;F) + N(r,\infty;G)). \end{aligned}$$

Thus (4.9) can be written as

$$\begin{aligned} (4.10) \qquad & (n+k-1)\left(T(r,f)+T(r,g)\right) \\ \leq\ & 2\left(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right) + 3\sum_{j=1}^{k}\overline{N}(r,\beta_j;f) \end{aligned}$$

$$+\frac{1}{2}(N(r,\infty;F)+N(r,\infty;G))-\frac{3}{2}\overline{N}_*(r,\infty;F,G)+S(r).$$

Now, let us consider the following function

$$\varphi(z):=\frac{F'(z)}{F(z)}-\frac{G'(z)}{G(z)}.$$

Next we consider two cases:

Subcase-I. Assume that $\varphi\not\equiv 0$. Thus all poles of φ are simple. Also, *poles of φ may occur* at

(1) poles of F and G,
(2) zeros of F and G.

Since $P'(z)$ have no simple zeros, thus using the first fundamental theorem and the lemma of logarithmic derivative, we have

$$\begin{aligned}2\sum_{j=1}^{k}\overline{N}(r,\beta_j;f)\leq& N(r,0;\varphi)\\ \leq& T(r,\varphi)+O(1)\\ \leq& N(r,\infty;\varphi)+S(r,F)+S(r,G)\\ \leq&\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;F,G)+S(r).\end{aligned}$$

Thus (4.10) can be written as

$$(4.11)\quad (\frac{n}{2}+k-1)\,(T(r,f)+T(r,g))\leq\frac{7}{2}\left(\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right)+S(r),$$

which is impossible if f and g both are entire functions. Also, this is impossible if f and g both are meromorphic functions and $n\geq 10-2k$. Thus $\varphi\equiv 0$.

Subcase-II. Next we assume that $\varphi\equiv 0$. Then on integration, we have

$$F\equiv AG,$$

where A is a non-zero constant, which is impossible as $H\not\equiv 0$.

Case-II. Now we consider the case $H\equiv 0$. Then by integration, we have

$$\frac{1}{P(f(z))}\equiv\frac{c_0}{P(g(z))}+c_1,$$

where $c_0(\neq 0)$, c_1 are constants.

Since $n\geq 5$, $k\geq 2$ and $P'(z)$ have no simple zeros, thus, applying Lemma 4.1 and noting the assumption that $P(z)$ is a "uniqueness polynomial", we have

$$f\equiv g.$$

This completes the proof. □

5. Some observations

The natural query would be whether there exist different classes of unique range sets, but in this direction the number of results are not sufficient. Recently, V. H. An and P. N. Hoa ([3]) exhibited a new class of unique range set for meromorphic functions. The unique range set is the zero set of the following polynomial:

$$P(z) = z^n + (az+b)^n + c, \tag{5.1}$$

where $n \geq 25$ is an integer, $a, b, c \in \mathbb{C} \setminus \{0\}$ with $c \neq \frac{b^d}{a^d}$, $a^{2d} \neq 1$, $c \neq a^d b^d$, $c \neq \frac{(-1)^d b^d}{a^{2d}}$, $c \neq (-1)^d b^d$. Also, it was assumed that $P(z)$ has only simple zeros.

This URSM has 25 elements but in literature there exist URSM with 11 elements. Also, it was proved that any URSM (resp. URSE) must contain at least six (resp. five) [see, Theorem 10.59 (resp. Theorem 10.72), ([20])] elements. So, the challenging work is to exhibit URSM (resp. URSE) with elements ≤ 11 (resp. 7).

Now, we discuss a method of P. Li and C. C. Yang (page 448, ([19])):

Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set with finite distinct elements of $\mathbb{C}$. Also, let $\alpha(\neq 0)$ and β be two complex constants. If S is a unique range set, then the set $T = \{\alpha\alpha_1 + \beta, \alpha\alpha_2 + \beta, \ldots, \alpha\alpha_n + \beta\}$ is also a unique range set.

If f and g are two meromorphic functions sharing T CM, then

$$\begin{aligned}&(f - (\alpha\alpha_1 + \beta))(f - (\alpha\alpha_2 + \beta)) \cdots (f - (\alpha\alpha_n + \beta)) \\ &= h(g - (\alpha\alpha_1 + \beta))(g - (\alpha\alpha_2 + \beta)) \cdots (g - (\alpha\alpha_n + \beta)),\end{aligned}$$

where h is a meromorphic function whose zeros come from the poles of g and the poles come from the poles of f. Thus

$$\begin{aligned}&\left(\frac{f-\beta}{\alpha} - \alpha_1\right)\left(\frac{f-\beta}{\alpha} - \alpha_2\right)\cdots\left(\frac{f-\beta}{\alpha} - \alpha_n\right) \\ &= h\left(\frac{g-\beta}{\alpha} - \alpha_1\right)\left(\frac{g-\beta}{\alpha} - \alpha_2\right)\cdots\left(\frac{g-\beta}{\alpha} - \alpha_n\right).\end{aligned}$$

Thus $\frac{f-\beta}{\alpha}$ and $\frac{g-\beta}{\alpha}$ share S CM. So, $\frac{f-\beta}{\alpha} \equiv \frac{g-\beta}{\alpha}$, i.e., $f \equiv g$. So, T is a URSM.

Remark 5.1. Since the examples of unique range sets are few in numbers, thus this method (page 448, [19]) helps us to construct new class of unique range sets. For examples,

(i) The zero set of the following polynomial gives a new class of URSM (resp. URSE) with 13 (resp. 7) elements:

$$P(z) = (z-\beta)^n + a(z-\beta)^{n-m} + b,$$

where $\beta \in \mathbb{C}$, a and b are two non-zero constants such that $z^n + az^{n-m} + b = 0$ has no multiple root. Also, $m \geq 2$ (resp. 1), $n > 2m+8$ (resp. $2m+4$) are integers with n and $n-m$ having no common factors.

(ii) The zero set of the following polynomial gives a new class of URSM with 11 elements:

$$P(z) = \frac{(n-1)(n-2)}{2}(z-\beta)^n - n(n-2)(z-\beta)^{n-1} + \frac{n(n-1)}{2}(z-\beta)^{n-2} - c,$$

where $n \geq 11$ and $c \neq 0, 1$, $\beta \in \mathbb{C}$.

We have seen from the equation (1.1) that the zero set of the polynomial

$$P(z) = z^n + z^{n-1} + 1$$

gives a URSE with $n(\geq 7)$ elements. Now,

$$z^n P\left(\frac{1}{z}\right) = z^n + z + 1.$$

Again, the zero set of the polynomial $z^n P(\frac{1}{z}) = z^n + z + 1$ gives a URSE with $n(\geq 7)$ elements (Theorem 10.57, [20]).

Next, we will see that the equation (1.3) gives that the zero set of the polynomial

$$P(z) = \frac{(n-1)(n-2)}{2}z^n - n(n-2)z^{n-1} + \frac{n(n-1)}{2}z^{n-2} - \frac{c}{2},$$

where $c \neq 0, 1, 2$ is a URSM with $n(\geq 11)$ elements. Now,

$$z^n P\left(\frac{1}{z}\right) = -\frac{1}{2}\left(cz^n - n(n-1)z^2 + 2n(n-2)z - (n-1)(n-2)\right).$$

From the equation (1.4), the zero set of the polynomial $z^n P(\frac{1}{z})$ also gives a URSM with $n(\geq 11)$ elements.

Thus the following question is obvious:

Question 5.1. Let $P(z)$ be a non-constant polynomial of degree n, having simple zeros. What are the characterizations of the polynomial $P(z)$ such that if the zero set of the polynomial $P(z)$ forms a unique range set, then the zero set of polynomial $z^n P(\frac{1}{z})$ must form a unique range set?

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