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EXISTENCE OF GLOBAL SOLUTIONS TO SOME NONLINEAR EQUATIONS ON LOCALLY FINITE GRAPHS

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Abstract. Let $G = (V, E)$ be a connected locally finite and weighted graph, Δ_p be the p-th graph Laplacian. Consider the p-th nonlinear equation

$$-\Delta_p u + h|u|^{p-2}u = f(x, u)$$

on G, where $p > 2$, h, f satisfy certain assumptions. Grigor'yan-Lin-Yang [24] proved the existence of the solution to the above nonlinear equation in a bounded domain $\Omega \subset V$. In this paper, we show that there exists a strictly positive solution on the infinite set V to the above nonlinear equation by modifying some conditions in [24]. To the m-order differential operator $\mathcal{L}_{m,p}$, we also prove the existence of the nontrivial solution to the analogous nonlinear equation.

1. Introduction

Let $G = (V, E)$ be a locally finite graph. Grigor'yan-Lin-Yang [24] firstly studied Yamabe type equations on graphs. Using the mountain pass theorem, they proved that the Yamabe type equation, $-\Delta u - \alpha u = |u|^{p-2}u$, has a strictly positive solution in a nonempty bounded domain $\Omega \subset V$ with the solution function takes a value of 0 at the boundary $\partial\Omega$. They also established local existence results about the p-th graph Laplacian Δ_p as follows

$$\begin{cases} -\Delta_p u = f(x, u) \quad \text{in} \quad \Omega^\circ, \\ u \geq 0 \ \text{ in } \ \Omega^\circ, \ \ u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\Omega \subset V$ is a bounded domain with $\Omega^\circ = \Omega \setminus \partial\Omega \neq \varnothing$ and $p > 2$. Applying the similar method, Grigor'yan-Lin-Yang [25] considered the nonlinear equation $-\Delta u + hu = f(x, u)$, they proved:

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Theorem 1.1 (Theorem 2, [25]). *Let $G = (V, E)$ be a locally finite graph. Assume that its weight satisfies $\omega_{xy} = \omega_{yx}$ for all $y \sim x \in V$, and that its measure $\mu(x) \geq \mu_{\min} > 0$ for all $x \in V$. Let $h : V \to \mathbb{R}$ be a function satisfying*

(1) *there exists a constant $h_0 > 0$ such that $h(x) \geq h_0$ for all $x \in V$;*

(2) *$h(x) \to +\infty$ as $d(x, x_0) \to +\infty$ for some fixed $x_0 \in V$.*

Suppose that $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis:

(3) *for any $s, t \in \mathbb{R}$, there exists some constant $L > 0$ such that*

$$|f(x, s) - f(x, t)| \leq L|s - t| \quad \text{for all } x \in V;$$

(4) *there exists a constant $q > 2$ such that for all $x \in V$ and $s > 0$,*

$$0 < qF(x, s) = q \int_0^s f(x, t)dt \leq sf(x, s);$$

(5) $\limsup_{s \to 0^+} \frac{2F(x,s)}{s^2} < \lambda_1 = \inf_{\int_V u^2 d\mu = 1} \int_V (|\nabla u|^2 + hu^2) d\mu$.

Then the equation $-\Delta u + hu = f(x, u)$ has a strictly positive solution.

From the above results in [24, 25], one naturally has the following question: Does the p-th nonlinear equation $-\Delta_p u + h|u|^{p-2}u = f(x, u)$ exist a positive solution on V?

The main purpose of this paper is to prove the existence of global positive solution on V to the p-th nonlinear equation

$$-\Delta_p u + h|u|^{p-2}u = f(x, u),$$

where $p > 2$ and h, f satisfy certain assumptions. However, the associated function space $\left\{u \in L^p(V) : \int_V (|\nabla u|^p + h|u|^p) \, d\mu < +\infty\right\}$ is not a Hilbert space when $p > 2$. In view of this fact, the approach in [25] is not feasible. By following the method in [20], we will use variational principles and Fatou's lemma to replace the mountain pass theorem.

Grigor'yan-Lin-Yang [24] also studied the associated equation about the m-order differential operator $\mathcal{L}_{m,p}$ on graphs. On a locally finite graph $G = (V, E)$ and $\Omega \subset V$ is a bounded domain with $\Omega^\circ \neq \varnothing$, they considered the following nonlinear equation

$$\begin{cases} \mathcal{L}_{m,p} u = f(x, u) \text{ in } \Omega^\circ, \\ |\nabla^j u| = 0 \text{ on } \partial\Omega, \ 0 \leq j \leq m - 1, \end{cases}$$

where $m \geq 2$ is an integer and $p > 1$. And they proved the existence of the nontrivial solution to the above equation with f satisfies three assumptions. Moreover, on a finite graph $G = (V, E)$ with the same three assumptions, they showed that there exists a nontrivial solution to $\mathcal{L}_{m,p} u + h|u|^{p-2}u = f(x, u)$ on V. In this paper, we will study the nonlinear equation $\mathcal{L}_{m,p} u + h|u|^{p-2}u = f(x, u)$ on a locally finite graph and prove the existence of the nontrivial global solution to this equation.

This kind of problems have been extensively studied in the Euclidean space, see for examples, Alves-Figueiredo [2], Alama-Li [1], Cao [6], Ding-Ni [13], Jeanjean [27], Kryszewski-Szulkin [29], Panda [32], and the references therein.

For the Riemannian manifold case, we refer the reader to [15, 33, 34]. Recently, the investigations of discrete weighted Laplacians and various equations on graphs have attracted much attention, see for examples Bauer-Hua-Jost [3], Chung-Lee-Chung [12], Ge [16], Ge-Hua-Jiang [19], Ge-Jiang [21, 22], Han [26], Bauer-Hua-Yau [4]. For p-Laplacian on graphs, we refer to Bühler-Hein [5], Chang [7, 8], Chang-Shao-Zhang [9, 10], Kawohl-Fridman [28], Mugnolo [31], Zhang-Chang [35], Zhang-Lin [36, 37].

The remaining part of this paper is organized as follows: In Section 2, we give some notations and main results on weighted graphs. In Section 3, we give the proof of Theorem 2.1. We prove Theorem 2.2 in Section 4. Finally, in Section 5, we consider another definition of Δ_p and prove the existence of the strictly positive global solution to the nonlinear equation (35) under the same assumptions in Theorem 2.1.

2. Settings and main results

All graphs considered in this paper are connected, undirected and weighted graphs. Now, we recall some basic notations for weighted graphs in [11, 35]. Let $G = (V, E)$ be a locally finite graph, where V, E denote the vertex set and the edge set of G, respectively. Let $\omega : V \times V \ni (x, y) \mapsto \omega_{xy} \in [0, \infty)$ be an edge weight function satisfying $\omega_{xy} = \omega_{yx}$, $\sum_{y \in V} \omega_{xy} < \infty$, for any $x \in V$, $\mu : V \ni x \mapsto \mu(x) \in (0, \infty)$ be a measure on V of full support, and for any $x, y \in V$, $\{x, y\} \in E$ if and only if $\omega_{xy} > 0$, in symbols $x \sim y$. Alternatively, ω_{xy} can be considered as a positive function on the set E, that is extended to be 0 on non-edge pairs (x, y). Note that $G = (V, E)$ possibly possesses self-loops. Any weight ω_{xy} gives rise to a function on vertices as $\mu(x) = \sum_{y \sim x} \omega_{xy}$, and $\mu(x)$ is called the weight of a vertex x. For example, if the weight ω is simple, then $\mu(x) = \deg(x)$. Throughout this paper, we denote $C_{G,h,\ldots}$ as some positive constant depending only on the information of $G, h, \ldots$. Note that the information of G contains V, E, μ and ω. Denote $C(V)$ as the set of all real functions defined on V, then $C(V)$ is an infinite dimensional linear space with the usual functions additions and scalar multiplications due to V is an infinite set.

For any function $u : V \to \mathbb{R}$, the μ-Laplacian (or Laplacian for short) of u is defined as

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} w_{xy}(u(y) - u(x)). \tag{1}$$

With respect to the vertex weight μ, the integral of u over V is defined by

$$\int_V u d\mu = \sum_{x \in V} u(x)\mu(x)$$

for any $u \in C(V)$. We consider the p-th Laplacian $\Delta_p : C(V) \to C(V)$, which is defined in distributional sense by

$$\int_V (\Delta_p u)\phi d\mu = -\int_V |\nabla u|^{p-2}\Gamma(u,\phi)d\mu,\ \forall \phi \in C_c(V), \tag{2}$$

where $C_c(V)$ denotes the set of all functions with compact support. The associated gradient form reads

$$\Gamma(u,v)(x) = \frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))(v(y)-v(x)).$$

We write $\Gamma(u) = \Gamma(u,u)$ for short. The length of its gradient $|\nabla u|$ in (2) is defined as

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}\left(u(y)-u(x)\right)^2\right)^{1/2}. \tag{3}$$

Point-wisely, Δ_p can be written as

$$\Delta_p u(x) = \frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}\Big(|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x)\Big)\big(u(y)-u(x)\big) \tag{4}$$

for $u \in C(V)$ and $x \in V$. Note that u may not be integrable generally. Denote $L^p(V)$ as the space of all p-th integrable functions on V.

We define a space of functions

$$\mathcal{H} = \left\{u \in L^p(V) : \int_V \left(|\nabla u|^p + h|u|^p\right)d\mu < +\infty\right\} \tag{5}$$

with a norm

$$\|u\|_{\mathcal{H}} = \left(\int_V \left(|\nabla u|^p + h|u|^p\right)d\mu\right)^{1/p},$$

where $|\nabla u|$ is defined as (3) and $h \in C(V)$.

Let $h : V \to \mathbb{R}$, $f : V \times \mathbb{R} \to \mathbb{R}$ be two functions. We say that $u : V \to \mathbb{R}$ is a solution of the p-th nonlinear equation

$$-\Delta_p u + h|u|^{p-2}u = f(x,u), \tag{6}$$

if (6) holds for all $x \in V$, where Δ_p is defined as (4). We shall prove the following:

Theorem 2.1. *Let $G = (V,E)$ be a connected, locally finite and weighted graph and $p > 2$. Assume its measure satisfies* $\inf\{\mu(x) : x \in V\} > 0$. *Let $h : V \to \mathbb{R}$ be a function satisfying the following assumptions*:

(H_1) $\inf_{x\in V} h(x) > 0$;

(H_2) $1/h \in L^\delta(V)$ *for some* $\delta : 0 < \delta \le \frac{1}{p-2}$.

Suppose that $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypothesis:

(H_3) *for all* $x \in V$, $f(x,0) = 0$, *and there exists a constant* $q > 0$ *such that for all* $x \in V$ *and* $s > 0$,

$$0 < q \int_0^s f(x,t)dt \leq sf(x,s);$$

(H_4) *there exists some constant* $L > 0$ *such that*

$$|f(x,t_1) - f(x,t_2)| \leq L|t_1 - t_2| \quad \text{for any } x \in V \text{ and } t_1, t_2 \in \mathbb{R}.$$

Then the equation (6) *has a strictly positive solution.*

We know that the higher order differential operators were also extensively studied on manifolds, refer to [14,30]. Inspired by Grigor'yan-Lin-Yang's work in [23, 24], we shall extend the equation (6) to nonlinear elliptic equation involving higher order derivative. The length of m-order gradient of u is defined as

$$|\nabla^m u| = \begin{cases} |\nabla \Delta^{\frac{m-1}{2}} u|, & \text{when } m \text{ is odd,} \\ |\Delta^{\frac{m}{2}} u|, & \text{when } m \text{ is even,} \end{cases} \tag{7}$$

where $|\nabla \Delta^{\frac{m-1}{2}} u|$ is defined as in (3) for the function $\Delta^{\frac{m-1}{2}} u$, and $|\Delta^{\frac{m}{2}} u|$ denotes the usual absolute of the function $\Delta^{\frac{m}{2}} u$. Then we have:

Theorem 2.2. *Let* $G = (V,E)$ *be a connected, locally finite and weighted graph and* $p > 2$. *Assume its measure satisfies* $\inf\{\mu(x) : x \in V\} > 0$. *Let* $h : V \to \mathbb{R}$ *be a function satisfying the assumptions* (H_1) *and* (H_2). *Suppose that* $f : V \times \mathbb{R} \to \mathbb{R}$ *satisfies the hypothesis* (H_4) *and*

(H_3') *for all* $x \in V$, $f(x,0) = 0$, *and there exists a constant* $q > 0$ *such that for all* $x \in V$, $|s| > 0$,

$$0 < q \int_0^s f(x,t)dt \leq sf(x,s).$$

Then there exists a nontrivial solution to

$$\mathcal{L}_{m,p} u + h|u|^{p-2}u = f(x,u), \tag{8}$$

where $\mathcal{L}_{m,p} u$ *(*m *is a positive integer) is defined in the distributional sense: for any function* $\phi \in C(V)$, *there holds*

$$\int_V (\mathcal{L}_{m,p} u)\phi d\mu = \begin{cases} \int_V |\nabla^m u|^{p-2} \Gamma(\Delta^{\frac{m-1}{2}} u, \Delta^{\frac{m-1}{2}} \phi) d\mu, & \textit{when } m \textit{ is odd}, \\ \int_V |\nabla^m u|^{p-2} (\Delta^{\frac{m}{2}} u)(\Delta^{\frac{m}{2}} \phi) d\mu, & \textit{when } m \textit{ is even}. \end{cases} \tag{9}$$

3. Proof of Theorem 2.1

For each $u \in \mathcal{H}$, we define a functional

$$J(u) = \int_V (|\nabla u|^p + h|u|^p) d\mu. \tag{10}$$

We can see that $J(u) = \|u\|_{\mathcal{H}}^p$ and J is continuously differentiable on $\mathcal{H}$ as follows:

Lemma 3.1. *Let $G = (V, E)$ be a connected, locally finite and weighted graph and $p > 2$. Assume its measure satisfies* $\inf\{\mu(x) : x \in V\} > 0$. *Suppose $h \in C(V)$ satisfies* (H_1). *Then the function J, defined as* (10)*, is continuously differentiable on $\mathcal{H}$, where $\mathcal{H}$ defined as* (5).

Proof. By direct calculation, the Fréchet derivative of $J(u)$ at a fixed $u \in \mathcal{H}$ is a $J'(u) \in \mathcal{H}^*$ with

$$\mathcal{H} \ni \xi \mapsto J'(u)(\xi) = p \int_V \left(-\Delta_p u + h|u|^{p-2}u\right) \xi d\mu,$$

which implies that $J'(u) : \mathcal{H} \to \mathcal{H}^*$ is linear. By the Hölder inequality, we know that for any vertex $x \in V$, there holds $\Gamma(u, \xi)(x) \leq |\nabla u|(x)|\nabla \xi|(x)$, then

$$\begin{aligned}|J'(u)(\xi)| &= p \int_V \left(|\nabla u|^{p-2}\Gamma(u, \xi) + h|u|^{p-2}u\xi\right) d\mu \\ &\leq p \left(\int_V |\nabla u|^{p-1}|\nabla \xi| d\mu + \int_V h|u|^{p-1}|\xi| d\mu\right) \\ &\leq C_{G,p,h}\|u\|_{\mathcal{H}}^{p-1}\|\xi\|_{\mathcal{H}}.\end{aligned}$$

Hence, we get $J' : \mathcal{H} \to \mathcal{H}^*$, the Fréchet derivative of J satisfies

$$\|J'(u)\|_{\mathcal{H}^*} \leq C_{G,p,h}\|u\|_{\mathcal{H}}^{p-1}.$$

This means J' is continuous, that is J is continuously differentiable on $\mathcal{H}$. □

For any constant $\theta > 0$, set

$$F(x, s) = \begin{cases} \int_0^{\theta s} f(x, t) dt & s \geq 0, \\ 0 & s < 0. \end{cases}$$

It is continuously differentiable with respect to s with $\partial_s F(x, s) = \theta f(x, \theta s)$ when $s \geq 0$ and $\partial_s F(x, s) = 0$ when $s < 0$. In the sequel, we write $F'(x, s) = \partial_s F(x, s)$ for short. Consider the following functional

$$K(u) = \int_V F(x, u) d\mu, \ u \in \mathcal{H}. \tag{11}$$

Lemma 3.2. *Let $G = (V, E)$ be a connected, locally finite and weighted graph and $p > 2$. Assume its measure satisfies* $\inf\{\mu(x) : x \in V\} > 0$. *Suppose that $h \in C(V)$ satisfies* (H_1), (H_2) *and $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies* (H_4). *Then the function K, defined as* (11)*, is continuously differentiable on $\mathcal{H}$, where $\mathcal{H}$ defined as* (5).

Proof. By direct calculation, the Fréchet derivative of $K(u)$ at a fixed $u \in \mathcal{H}$ is a $K'(u) \in \mathcal{H}^*$ with

$$\mathcal{H} \ni v \mapsto K'(u)(v) = \int_V F'(x, u) v d\mu.$$

In view of (H_1), there exists some constant $C_{G,h}$ such that

$$\frac{1}{h} \leq C_{G,h}, \tag{12}$$

in addition, by (H_2), we know that there exists some constant $C_{G,h,p,\delta}$ such that

$$\int_V \frac{1}{h^{1/(p-2)}} d\mu \leq C_{G,h}^{1/(p-2)-\delta} \int_V \frac{1}{h^\delta} d\mu \leq C_{G,h,p,\delta}.$$

Hence, using the Hölder inequality and (H_1), we have

$$\begin{aligned} \int_V |u_1 - u_2||v| d\mu &\leq \Big(\int_V |v|^p d\mu \Big)^{1/p} \Big(\int_V |u_1 - u_2|^{\frac{p}{p-1}} d\mu \Big)^{\frac{p-1}{p}} \\ &\leq C_{G,h,p} \|v\|_{\mathcal{H}} \Big(\int_V \frac{1}{h^{1/(p-2)}} d\mu \Big)^{\frac{p-2}{p}} \Big(\int_V h |u_1 - u_2|^p d\mu \Big)^{\frac{1}{p}} \\ &\leq C_{G,h,p,\delta} \|v\|_{\mathcal{H}} \|u_1 - u_2\|_{\mathcal{H}}. \end{aligned} \tag{13}$$

If $u_1(x) > 0$, $u_2(x) > 0$, using (H_4) and (13), we have

$$\begin{aligned} |(K'(u_1) - K'(u_2))v| &\leq L\theta^2 \int_V |u_1 - u_2||v| d\mu \\ &\leq C_{G,h,p,\delta,L,\theta} \|v\|_{\mathcal{H}} \|u_1 - u_2\|_{\mathcal{H}}. \end{aligned}$$

Noting that

$$f(x, 0) = 0 \ \text{ for all } x \in V,$$

and checking other cases for the sign of $u_1(x)$, $u_2(x)$, there also holds

$$|(K'(u_1) - K'(u_2))v| \leq C_{G,h,p,\delta,L,\theta} \|v\|_{\mathcal{H}} \|u_1 - u_2\|_{\mathcal{H}}.$$

Hence, we get $K' : \mathcal{H} \to \mathcal{H}^*$, the Fréchet derivative of K satisfies

$$\|K'(u_1) - K'(u_2)\|_{\mathcal{H}^*} \leq C_{G,h,p,\delta,L,\theta} \|u_1 - u_2\|_{\mathcal{H}}.$$

This implies that K is continuously differentiable on $\mathcal{H}$. □

Now, we consider the functional $J(u)$ under the constraint $K(u) = 1$. Since $J(u) \geq 0$,

$$\gamma = \inf\{J(u) : u \in \mathcal{H},\ K(u) = 1\}$$

is well defined. Obviously, $\gamma \geq 0$. Choose a sequence $\{u_n\}_{n\geq 1}$ in $\mathcal{H}$ with $J(u_n) \to \gamma$, $J(u_n) < \gamma + 1$ and $K(u_n) = 1$. At each vertex $x \in V$, we have

$$h(x)\mu(x)|u_n(x)|^p \leq \int_V h|u_n|^p d\mu \leq J(u_n) \leq \gamma + 1.$$

This means $|u_n(x)| \leq C_{G,h,p,\gamma}$ for all $x \in V$ and all $n \geq 1$. In other words, $\{u_n\}_{n\geq 1}$ are uniformly bounded. Noting that V is a countable set of points. Hence, there exists some $\bar{u}$ such that up to a subsequence, $u_n \to \bar{u}$ on V. We may well denote this subsequence as u_n. Because G is locally finite, $|\nabla u_n| \to |\nabla \bar{u}|$ at each vertex $x \in V$. According to Fatou's lemma, we obtain

$$\int_V (|\nabla \bar{u}|^p + h|\bar{u}|^p) d\mu \leq \gamma,$$

$$K(\bar{u}) = \int_V F(x, \bar{u}) d\mu \leq 1, \tag{14}$$

which implies $\bar{u} \in \mathcal{H}$.

Claim 1. $\bar{u}$, as above, is not identically zero on V.

Proof. Let $x_0 \in V$ be fixed. For any $\epsilon > 0$, in view of (H_2), there exists some $R > 0$ such that

$$\Big(\int_T \Big(\frac{1}{h}\Big)^\delta d\mu\Big)^{\frac{1}{\delta}} \leq \epsilon^{\frac{p}{\delta(p-2)}}, \tag{15}$$

where $T = \{x \in V : d(x, x_0) > R\}$ and $d(x, x_0)$ denotes the distance between x and x_0 on G.

If $u_n(x) \geq 0$, by (H_4), we obtain

$$f(x, \theta u_n(x)) = |f(x, \theta u_n(x)) - f(x, 0)| \leq L\theta u_n(x),$$

this leads to

$$\int_{\{x \in T: u_n(x) \geq 0\}} u_n f(x, \theta u_n) d\mu \leq C_{L,\theta} \int_{\{x \in T: u_n(x) \geq 0\}} {u_n}^2 d\mu. \tag{16}$$

Noting that $0 < \delta \leq \frac{1}{p-2}$, by the Hölder inequality and (12), (15), we get

$$\begin{aligned}
\int_T |u_n|^2 d\mu &\leq \left(\int_T \left(\frac{1}{h}\right)^{\frac{2}{p-2}} d\mu\right)^{\frac{p-2}{p}} \Big(\int_T h|u_n|^p d\mu\Big)^{\frac{2}{p}} \\
&\leq C_{G,h,p,\delta} \left(\int_T \left(\frac{1}{h}\right)^\delta d\mu\right)^{\frac{p-2}{p}} \|u_n\|^2_{\mathcal{H}} \\
&\leq C_{G,h,p,\delta} \|u_n\|^2_{\mathcal{H}}\ \epsilon. \qquad (17)
\end{aligned}$$

Combining (H_3), (16) and (17) with the definition of $F(x, s)$, we get

$$\begin{aligned}
\int_T F(x, u_n) d\mu &= \int_{\{x \in T: u_n(x) \geq 0\}} \Big(\int_0^{\theta u_n(x)} f(x, t) dt\Big) d\mu \\
&\leq \frac{1}{q} \int_{\{x \in T: u_n(x) > 0\}} \theta u_n f(x, \theta u_n) d\mu \\
&\leq C_{L,\theta,q} \int_{\{x \in T: u_n(x) > 0\}} {u_n}^2 d\mu \\
&\leq C_{G,h,L,\theta,\delta,p,q} \|u_n\|^2_{\mathcal{H}}\ \epsilon.
\end{aligned}$$

Hence, according to $K(u_n) = 1$, we have

$$\int_{\{x: d(x, x_0) \leq R\}} F(x, u_n) d\mu = 1 - \int_T F(x, u_n) d\mu \geq 1 - C_{G,h,L,\theta,\delta,p,q} \|u_n\|^2_{\mathcal{H}}\ \epsilon.$$

Let $n \to \infty$ and note that $\{x \in V : d(x, x_0) \le R\}$ is a bounded domain and $\|u_n\|_{\mathcal{H}}^p = J(u_n) \to \gamma$, we obtain

$$K(\bar{u}) = \int_V F(x, \bar{u}) d\mu \ge \int_{\{x: d(x,x_0) \le R\}} F(x, \bar{u}) d\mu \ge 1 - C_{G,h,L,\theta,\delta,p,q} \gamma^{\frac{2}{p}} \epsilon.$$

Further, let $\epsilon \to 0$, we have $K(\bar{u}) \ge 1$. By (14), we see $K(\bar{u}) = 1$, which implies that $\bar{u}$ is not identically zero. □

Claim 2. $\bar{u}$, as above, is positive everywhere on V.

Proof. We calculate the Euler-Lagrange equation at $\bar{u}$ under the constraint condition $K(\bar{u}) = 1$. By Lemma 3.2 and (2), for any $\varphi \in C_c(V)$, there holds

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \Big\{ J(\bar{u} + t\varphi) - \lambda \Big(\int_V F(x, \bar{u} + t\varphi) d\mu - 1 \Big) \Big\} \\ &= p \int_V |\nabla \bar{u}|^{p-2} \Gamma(\bar{u}, \varphi) d\mu + \int_V \Big(ph|\bar{u}|^{p-2}\bar{u} - \lambda F'(x, \bar{u}) \Big) \varphi d\mu \\ &= \int_V \Big(-p\Delta_p \bar{u} + ph|\bar{u}|^{p-2}\bar{u} - \lambda F'(x, \bar{u}) \Big) \varphi d\mu. \end{aligned}$$

Hence, we get

$$-p\Delta_p \bar{u} + ph|\bar{u}|^{p-2}\bar{u} = \lambda F'(x, \bar{u}). \tag{18}$$

Multiplying $\bar{u}$ on both sides of the equation (18), and taking integration, we get

$$\int_V (-p\bar{u}\Delta_p \bar{u} + ph|\bar{u}|^p)\, d\mu = \lambda \int_V \bar{u} F'(x, \bar{u}) d\mu.$$

By Claim 1, $\bar{u} \not\equiv 0$ on V, we know

$$LHS = p \int_V (|\nabla \bar{u}|^p + h|\bar{u}|^p) d\mu > 0,$$

and

$$RHS = \lambda \int_{\{x \in V: \bar{u}(x) > 0\}} \bar{u} F'(x, \bar{u}) d\mu = \lambda\theta \int_{\{x \in V: \bar{u}(x) > 0\}} \bar{u} f(x, \theta\bar{u}) d\mu.$$

Using (H_3), we get $\bar{u} f(x, \theta\bar{u}) > 0$ when $\bar{u} > 0$. These lead to $\lambda > 0$.

If $\bar{u}(x) < 0$, at some vertex $x \in V$, then by the equation (18), we see

$$\Delta_p \bar{u}(x) < 0.$$

However, by the definition of Δ_p, there is a $y \sim x$ with $\bar{u}(y) < \bar{u}(x) < 0$. In view of the connectedness of the graph $G = (V, E)$, by induction, we obtain a sequence $x = x_1 \sim x_2 \sim x_3 \sim \cdots$ such that

$$\cdots < \bar{u}(x_i) < \bar{u}(x_{i-1}) < \cdots < \bar{u}(x_1) < 0.$$

Then we have

$$\sum_{i=1}^n |\bar{u}(x_i)|^p \mu(x_i) \to +\infty \quad \text{as } n \to +\infty,$$

which contradicts $\bar{u} \in \mathcal{H} \subseteq L^p(V)$. Hence $\bar{u}$ is nonnegative on V. If $\bar{u}$ is not positive everywhere on V, we can always find two vertices x, y with $y \sim x$, $\bar{u}(x) = 0$, $\bar{u}(y) > 0$. Then it follows $\Delta_p \bar{u}(x) > 0$ by the definition of Δ_p, which contradicts to the equation (18). Hence $\bar{u}$ is positive everywhere on V. $\square$

Claim 3. The p-th nonlinear equation (6) has a strictly positive solution.

Proof. By Claims 1, 2, we know that $\bar{u}$ is positive everywhere on V, and it satisfies

$$-p\Delta_p \bar{u} + ph\bar{u}^{p-1} = \lambda\theta f(x, \theta\bar{u}). \tag{19}$$

Choosing $\theta = (\frac{p}{\lambda})^{1/p}$, and taking $\theta\bar{u}$ by u in (19), we know that u is positive everywhere on V, and u satisfies the following equation

$$-\Delta_p u + hu^{p-1} = f(x, u),$$

which completes the proof. $\square$

4. Proof of Theorem 2.2

The proof of Theorem 2.2 is analogous to that of Theorem 2.1. We define a space of functions

$$\mathcal{H}_1 = \left\{ u \in L^p(V) : \int_V (|\nabla^m u|^p + h|u|^p)\, d\mu < +\infty \right\} \tag{20}$$

with a norm

$$\|u\|_{\mathcal{H}_1} = \left(\int_V (|\nabla^m u|^p + h|u|^p)\, d\mu \right)^{1/p}, \tag{21}$$

where $|\nabla^m u|$ is defined as (7).

For each $u \in \mathcal{H}_1$, we define a functional

$$J_1(u) = \int_V (|\nabla^m u|^p + h|u|^p) d\mu.$$

It is easy to see that $J_1(u) = \|u\|^p_{\mathcal{H}_1}$. And similar proof of Lemma 3.1, we can see J_1 is continuously differentiable. For any constant $\theta > 0$, we set

$$K_1(u) = \int_V \int_0^{\theta u(x)} f(x, t) dt d\mu, \ u \in \mathcal{H}_1. \tag{22}$$

Now, we consider the functional $J_1(u)$ under the constraint $K_1(u) = 1$. Since $J_1(u) \geq 0$,

$$\gamma_1 = \inf\{J_1(u) : u \in \mathcal{H}_1, \ K_1(u) = 1\}$$

is well defined. Obviously, $\gamma_1 \geq 0$. Choose a sequence $\{u_n\}_{n \geq 1}$ in $\mathcal{H}_1$ with $J_1(u_n) \to \gamma_1$, $J_1(u_n) < \gamma_1 + 1$ and $K_1(u_n) = 1$. Similarly, at each vertex $x \in V$, we have

$$h(x)\mu(x)|u_n(x)|^p \leq \int_V h|u_n|^p d\mu \leq J_1(u_n) \leq \gamma_1 + 1.$$

This means $|u_n(x)| \leq C_{G,h,p,\gamma_1}$ for all $x \in V$ and all $n \geq 1$, that is, $\{u_n\}_{n\geq 1}$ are uniformly bounded. Hence, there exists some $\hat{u}$ such that up to a subsequence, $u_n \to \hat{u}$ on V. We may well denote this subsequence as u_n. Because G is locally finite, $|\nabla^m u_n| \to |\nabla^m \hat{u}|$ at each vertex x. According to Fatou's lemma, we obtain

$$\begin{aligned} &\int_V (|\nabla^m \hat{u}|^p + h|\hat{u}|^p) d\mu \leq \gamma_1, \\ &K_1(\hat{u}) = \int_V \int_0^{\theta \hat{u}(x)} f(x,t) dt d\mu \leq 1, \end{aligned} \tag{23}$$

which implies $\hat{u} \in \mathcal{H}_1$.

Lemma 4.1. *Let $G = (V, E)$ be a connected, locally finite and weighted graph and $p > 2$. Assume its measure satisfies $\inf\{\mu(x) : x \in V\} > 0$. Suppose that $h \in C(V)$ satisfies (H_1), (H_2) and $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies (H_4). Then the function K_1, defined as* (22)*, is continuously differentiable on $\mathcal{H}_1$, where $\mathcal{H}_1$ defined as* (20)*.*

Proof. By direct calculation, the Fréchet derivative of $K_1(u)$ at a fixed $u \in \mathcal{H}_1$ is a $K_1'(u) \in \mathcal{H}_1^*$ with

$$\mathcal{H} \ni v \mapsto K_1'(u)(v) = \theta \int_V f(x, \theta u(x)) v(x) d\mu.$$

Similar to the calculation of (13), we have

$$\int_V |u_1 - u_2||v| d\mu \leq C_{G,h,p,\delta} \|v\|_{\mathcal{H}_1} \|u_1 - u_2\|_{\mathcal{H}_1}. \tag{24}$$

Using (H_4) and (24), we have

$$\begin{aligned} |(K_1'(u_1) - K_1'(u_2))v| &\leq \theta \int_V |f(x, \theta u_1) - f(x, \theta u_2)||v| d\mu \\ &\leq L\theta^2 \int_V |u_1 - u_2||v| d\mu \\ &\leq C_{G,h,p,\delta,L,\theta} \|v\|_{\mathcal{H}_1} \|u_1 - u_2\|_{\mathcal{H}_1}. \end{aligned}$$

Hence, we get $K_1' : \mathcal{H}_1 \to \mathcal{H}_1^*$, the Fréchet derivative of K_1 satisfies

$$\|K_1'(u_1) - K_1'(u_2)\|_{\mathcal{H}_1^*} \leq C_{G,h,p,\delta,L,\theta} \|u_1 - u_2\|_{\mathcal{H}_1}.$$

□

Claim 4. $\hat{u}$, as above, is not identically zero on V.

Proof. Analogous proof of Claim 1, we will show $K_1(\hat{u}) = 1$.

Let $x_0 \in V$ be fixed. For any $\epsilon > 0$, in view of (H_2), there exists some $R > 0$ such that

$$\left(\int_T \left(\frac{1}{h} \right)^\delta d\mu \right)^{\frac{1}{\delta}} \leq \epsilon^{\frac{p}{\delta(p-2)}}, \tag{25}$$

where $T = \{x \in V : d(x, x_0) > R\}$.

By (H_4), we obtain

$$|f(x,\theta u_n)| = |f(x,\theta u_n) - f(x,0)| \leq L\theta |u_n|,$$

this leads to

$$\int_T u_n f(x,\theta u_n) d\mu = \int_T |u_n f(x,\theta u_n)| d\mu \leq C_{L,\theta} \int_T |u_n|^2 d\mu. \tag{26}$$

Similar to the calculation of (17), we get

$$\int_T |u_n|^2 d\mu \leq C_{G,h,p,\delta} \|u_n\|^2_{\mathcal{H}_1} \ \epsilon. \tag{27}$$

Combining (H_3'), (26) and (27), we get

$$\begin{aligned}\int_T \int_0^{\theta u_n(x)} f(x,t) dt d\mu \leq \frac{\theta}{q} \int_T u_n f(x,\theta u_n) d\mu \leq \frac{L\theta^2}{q} \int_T |u_n|^2 d\mu \\ \leq C_{G,h,L,\theta,\delta,p,q} \|u_n\|^2_{\mathcal{H}_1} \ \epsilon.\end{aligned}$$

Since $K_1(u_n) = 1$, we have

$$\int_{V\setminus T} \int_0^{\theta u_n(x)} f(x,t) dt d\mu \geq 1 - C_{G,h,L,\theta,\delta,p,q} \|u_n\|^2_{\mathcal{H}_1} \ \epsilon.$$

Let $n \to \infty$ and note that $V \setminus T = \{x \in V : d(x,x_0) \leq R\}$ is a bounded domain and $\|u_n\|^p_{\mathcal{H}_1} = J_1(u_n) \to \gamma_1$, we obtain

$$K_1(\hat{u}) \geq \int_{V\setminus T} \int_0^{\theta \hat{u}} f(x,t) dt d\mu \geq 1 - C_{G,h,L,\theta,\delta,p,q} \gamma_1{}^{\frac{2}{p}} \epsilon.$$

Further, let $\epsilon \to 0$, we have $K_1(\hat{u}) \geq 1$. By (23), we see $K_1(\hat{u}) = 1$, which implies that $\hat{u}$ is not identically zero. □

In the following, we show the equation (8) has a nontrivial solution on V. Before this, we first prove the following two lemmas:

Lemma 4.2. *Let $G = (V, E)$ be a connected, locally finite and weighted graph. For any positive integer k and any $\varphi \in \mathcal{H}_1$, there holds*

$$\frac{d}{dt}\Big|_{t=0} \Delta^k (u + t\varphi)(x) = \Delta^k \varphi(x), \ \forall x \in V,$$

where Δu is defined as (1) *and the space $\mathcal{H}_1$ is defined as* (20).

Proof. By induction on k, $\Delta^k(u + t\varphi)(x)$ is continuous when $\Delta^k(u + t\varphi)(x)$ is considered as a function of t. For $k = 1$, since the operator Δ is a linear operator, for any $x \in V$, there holds

$$\frac{d}{dt}\Big|_{t=0} \Delta(u + t\varphi)(x) = \Delta\varphi(x).$$

The inductive step from k to $k + 1$:

$$\frac{d}{dt}\Big|_{t=0} \Delta^{k+1}(u + t\varphi)(x)$$

$$
\begin{aligned}
&= \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy} \Big(\frac{d}{dt}\Big|_{t=0} \Delta^k(u+t\varphi)(y) - \frac{d}{dt}\Big|_{t=0} \Delta^k(u+t\varphi)(x) \Big) \\
&= \frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy} \Big(\Delta^k\varphi(y) - \Delta^k\varphi(x) \Big) \\
&= \Delta^{k+1}\varphi(x).
\end{aligned}
$$

□

Lemma 4.3. *Let $G=(V,E)$ be a connected, locally finite and weighted graph and $p>2$. For any positive integer m and any $\varphi\in\mathcal{H}_1$, there holds*

$$
\frac{d}{dt}\Big|_{t=0} \int_V |\nabla^m(u+t\varphi)|^p d\mu = p\int_V (\mathcal{L}_{m,p}u)\varphi d\mu,
$$

where $|\nabla^m u|$ is defined as (7)*, the operator $\mathcal{L}_{m,p}$ is defined in the distributional sense as* (9)*.*

Proof. In view of the definition of the operator $\mathcal{L}_{m,p}$, we split the proof into two cases.
Case 1. When m is even, using (7), Lemma 4.2 and (9), we obtain

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \int_V |\nabla^m(u+t\varphi)|^p d\mu &= \frac{d}{dt}\Big|_{t=0} \int_V |\Delta^{\frac{m}{2}}(u+t\varphi)|^p d\mu \\
&= p\int_V |\Delta^{\frac{m}{2}}u|^{p-2} (\Delta^{\frac{m}{2}}u)(\Delta^{\frac{m}{2}}\varphi) d\mu \\
&= p\int_V (\mathcal{L}_{m,p}u)\varphi d\mu.
\end{aligned}
$$

Case 2. When m is odd,

$$
\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \int_V |\nabla^m(u+t\varphi)|^p d\mu \\
&= \frac{d}{dt}\Big|_{t=0} \int_V |\nabla\Delta^{\frac{m-1}{2}}(u+t\varphi)|^p d\mu \\
&= \frac{d}{dt}\Big|_{t=0} \sum_{x\in V} \Big(\frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy} \big(\Delta^{\frac{m-1}{2}}(u+t\varphi)(y) - \Delta^{\frac{m-1}{2}}(u+t\varphi)(x) \big)^2 \Big)^{\frac{p}{2}} \mu(x) \\
&= \frac{p}{2} \sum_{x\in V} |\nabla\Delta^{\frac{m-1}{2}}(u+t\varphi)|^{p-2}(x) \Big(\frac{1}{\mu(x)} \sum_{y\sim x} \omega_{xy} \big(\Delta^{\frac{m-1}{2}}(u+t\varphi)(y) \\
&\quad - \Delta^{\frac{m-1}{2}}(u+t\varphi)(x) \big) \cdot \big(\Delta^{\frac{m-1}{2}}\varphi(y) - \Delta^{\frac{m-1}{2}}\varphi(x) \big) \Big) \mu(x) \Big|_{t=0} \\
&= p \sum_{x\in V} |\nabla^m u|^{p-2}(x) \Gamma(\Delta^{\frac{m-1}{2}}u, \Delta^{\frac{m-1}{2}}\varphi)(x)\mu(x) \\
&= p\int_V (\mathcal{L}_{m,p}u)\varphi d\mu.
\end{aligned}
$$

□

Claim 5. The equation (8) has a nontrivial solution on G.

Proof. We calculate the Euler-Lagrange equation at $\hat{u}$ under the constraint condition $K_1(\hat{u}) = 1$. For any $\varphi \in \mathcal{H}_1$, using Lemma 4.1 and Lemma 4.3, there holds

$$\begin{aligned} 0 = & \frac{d}{dt}\Big|_{t=0}\Big\{J_1(\hat{u}+t\varphi) - \lambda_1\Big(\int_V \int_0^{\theta(\hat{u}+t\varphi)(x)} f(x,t)dtd\mu - 1\Big)\Big\} \\ = & \frac{d}{dt}\Big|_{t=0}\Big\{\int_V (|\nabla^m(\hat{u}+t\varphi)|^p + h|(\hat{u}+t\varphi)|^p)d\mu \\ & \qquad - \lambda_1\Big(\int_V \int_0^{\theta(\hat{u}+t\varphi)(x)} f(x,t)dtd\mu - 1\Big)\Big\} \\ = & \int_V \Big(p\mathcal{L}_{m,p}\hat{u} + ph|\hat{u}|^{p-2}\hat{u} - \lambda_1\theta f(x,\theta\hat{u})\Big)\varphi d\mu. \end{aligned}$$

Hence, we get

$$p\mathcal{L}_{m,p}\hat{u} + ph|\hat{u}|^{p-2}\hat{u} = \lambda_1\theta f(x,\theta\hat{u}). \tag{28}$$

Using (3), (7) and (9), we obtain

$$\int_V (\mathcal{L}_{m,p}u)ud\mu = \int_V |\nabla^m u|^p d\mu. \tag{29}$$

Multiplying $\hat{u}$ on both sides of the equation (28), and taking integration, we get

$$\int_V (p\hat{u}\mathcal{L}_{m,p}\hat{u} + ph|\hat{u}|^p)\, d\mu = \lambda_1 \int_V \theta\hat{u}f(x,\theta\hat{u})d\mu.$$

By (29), (H_1) and Claim 4, $\hat{u} \not\equiv 0$ on V, we know

$$LHS = p\int_V (|\nabla^m\hat{u}|^p + h|\hat{u}|^p)d\mu > 0.$$

Moreover, using (H_3'), we get $\theta\hat{u}(x)f(x,\theta\hat{u}(x)) > 0$ as $\hat{u}(x) \neq 0$. These lead to $\lambda_1 > 0$. From (9), for any $\phi \in C(V)$, we have

$$\int_V (\mathcal{L}_{m,p}\frac{u}{\theta})\phi d\mu = \frac{1}{\theta^{p-1}}\int_V (\mathcal{L}_{m,p}u)\phi d\mu,$$

which implies

$$\mathcal{L}_{m,p}\frac{u}{\theta} = \frac{1}{\theta^{p-1}}\mathcal{L}_{m,p}u. \tag{30}$$

Choosing $\theta = (\frac{p}{\lambda_1})^{1/p}$, and taking $\theta\hat{u}$ by u in (28), we know that u is nontrivial on V, and u satisfies the following equation

$$\mathcal{L}_{m,p}u + h|u|^{p-2}u = f(x,u),$$

which completes the proof. $\square$

5. Extensions

In [17, 18], Ge considered another definition of the discrete p-Laplacian operator $\Delta_p : C(V) \to C(V)$, that is

$$\Delta_p u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy} |u(y) - u(x)|^{p-2} (u(y) - u(x)) \tag{31}$$

for $u \in C(V)$ and $x \in V$. The length of gradient $\nabla_p u$ is defined as

$$|\nabla_p u(x)| = \left(\frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy} |u(y) - u(x)|^p \right)^{\frac{1}{p}} \tag{32}$$

for any $u \in C(V)$ and $x \in V$. And we can see

$$\int_V |\nabla_p u|^p d\mu = \sum_{\substack{x,y \in V \\ x \sim y}} \omega_{xy} |u(y) - u(x)|^p. \tag{33}$$

We consider the following space of functions

$$\mathcal{H}_2 = \left\{ u \in L^p(V) : \int_V \left(|\nabla_p u|^p + h|u|^p \right) d\mu < +\infty \right\} \tag{34}$$

with a norm

$$\|u\|_{\mathcal{H}_2} = \left(\int_V \left(|\nabla_p u|^p + h|u|^p \right) d\mu \right)^{1/p},$$

where $|\nabla_p u|$ is defined as (32) and $h \in C(V)$.

Let $h : V \to \mathbb{R}$ and $f : V \times \mathbb{R} \to \mathbb{R}$ be two functions. Now we consider the following p-th nonlinear equation

$$-\Delta_p u + h|u|^{p-2} u = f(x, u), \tag{35}$$

where Δ_p is defined as (31). If (35) holds for all $x \in V$, we also say that $u : V \to \mathbb{R}$ is a solution to the nonlinear equation (35).

We shall prove the following:

Theorem 5.1. *Let $G = (V, E)$ be a connected, locally finite and weighted graph and $p > 2$. Assume its measure satisfies $\inf\{\mu(x) : x \in V\} > 0$. Let $h : V \to \mathbb{R}$ be a function satisfying the assumptions (H_1) and (H_2). Suppose that $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies the hypothesis (H_3) and (H_4). Then the equation (35) has a strictly positive solution.*

For each $u \in \mathcal{H}_2$, we set a functional

$$J_2(u) = \int_V \left(|\nabla_p u|^p + h|u|^p \right) d\mu. \tag{36}$$

Then we have:

Lemma 5.2. *Let $G=(V,E)$ be a connected, locally finite and weighted graph and $p>2$. Assume its measure satisfies* $\inf\{\mu(x): x\in V\}>0$. *Suppose $h\in C(V)$ satisfies* (H_1). *Then the function J_2, defined as* (36), *is continuously differentiable on $\mathcal{H}_2$, where $\mathcal{H}_2$ defined as* (34).

Proof. By direct calculation, the Fréchet derivative of $J_2(u)$ at a fixed $u\in\mathcal{H}_2$ is a $J_2'(u)\in\mathcal{H}_2^*$ with

$$\mathcal{H}_2\ni\xi\mapsto J_2'(u)(\xi)=\int_V\left(-\frac{p}{2}\Delta_p u+ph|u|^{p-2}u\right)\xi d\mu,$$

which implies that $J_2'(u):\mathcal{H}_2\to\mathcal{H}_2^*$ is linear. By the Hölder inequality, we know that for each vertex $x\in V$, there holds

$$\begin{aligned}|\Delta_p u(x)|&\le\frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}|u(y)-u(x)|^{p-1}\\&\le\frac{1}{\mu(x)}\Big(\sum_{y\sim x}\omega_{xy}\Big)^{\frac{1}{p}}\Big(\sum_{y\sim x}\omega_{xy}|u(y)-u(x)|^p\Big)^{\frac{p-1}{p}}\\&=\frac{1}{(\mu(x))^{\frac{p-1}{p}}}\Big(\sum_{y\sim x}\omega_{xy}|u(y)-u(x)|^p\Big)^{\frac{p-1}{p}}.\end{aligned}$$

In view of (H_1), then

$$\begin{aligned}|J_2'(u)(\xi)|&\le p\int_V\left(|\Delta_p u|+h|u|^{p-1}\right)|\xi|d\mu\\&\le p\sum_{x\in V}\Big(\sum_{y\sim x}\omega_{xy}|u(y)-u(x)|^p\Big)^{\frac{p-1}{p}}(\mu(x))^{\frac{1}{p}}|\xi(x)|\\&\quad+p\int_V h|u|^{p-1}|\xi|d\mu\\&\le C_{G,p,h}\|u\|_{\mathcal{H}_2}^{p-1}\|\xi\|_{\mathcal{H}_2}.\end{aligned}$$

Hence, we get $J_2':\mathcal{H}_2\to\mathcal{H}_2^*$, the Fréchet derivative of J_2 satisfies

$$\|J_2'(u)\|_{\mathcal{H}_2^*}\le C_{G,p,h}\|u\|_{\mathcal{H}_2}^{p-1}.$$

This means J_2' is continuous, that is J_2 is continuously differentiable on $\mathcal{H}$. □

For any constant $\theta>0$, we still set

$$F(x,s)=\begin{cases}\int_0^{\theta s}f(x,t)dt & s\ge 0,\\ 0 & s<0.\end{cases}$$

It is continuously differentiable with respect to s with $\partial_s F(x,s)=\theta f(x,\theta s)$ when $s\ge 0$ and $\partial_s F(x,s)=0$ when $s<0$. We still write $F'(x,s)=\partial_s F(x,s)$ for short. Consider the following functional

$$K_2(u)=\int_V F(x,u)d\mu,\ u\in\mathcal{H}_2. \tag{37}$$

Similar to the proof of Lemma 3.2, we have:

Lemma 5.3. *Let $G = (V, E)$ be a connected, locally finite and weighted graph and $p > 2$. Assume its measure satisfies $\inf\{\mu(x) : x \in V\} > 0$. Suppose that $h \in C(V)$ satisfies (H_1), (H_2) and $f : V \times \mathbb{R} \to \mathbb{R}$ satisfies (H_4). Then the function K_2, defined as (37), is continuously differentiable on $\mathcal{H}_2$, where $\mathcal{H}_2$ defined as (34).*

Now, we consider the functional $J_2(u)$ under the constraint $K_2(u) = 1$. Since $J_2(u) \geq 0$,

$$\gamma_2 = \inf\{J_2(u) : u \in \mathcal{H}_2,\ K_2(u) = 1\}$$

is well defined. Obviously, $\gamma_2 \geq 0$. Choose a sequence $\{u_n\}_{n\geq 1}$ in $\mathcal{H}_2$ with $J_2(u_n) \to \gamma_2$, $J_2(u_n) < \gamma_2 + 1$ and $K_2(u_n) = 1$. At each vertex $x \in V$, we have

$$h(x)\mu(x)|u_n(x)|^p \leq \int_V h|u_n|^p d\mu \leq J_2(u_n) \leq \gamma_2 + 1.$$

This means $|u_n(x)| \leq C_{G,h,p,\gamma_2}$ for all $x \in V$ and all $n \geq 1$. In other words, $\{u_n\}_{n\geq 1}$ are uniformly bounded. Noting that V is a countable set of points. Hence, there exists some $\tilde{u}$ such that up to a subsequence, $u_n \to \tilde{u}$ on V. We may well denote this subsequence as u_n. Because G is locally finite, $|\nabla_p u_n| \to |\nabla_p \tilde{u}|$ at each vertex x. According to Fatou's lemma, we obtain

$$\begin{aligned} &\int_V (|\nabla_p \tilde{u}|^p + h|\tilde{u}|^p) d\mu \leq \gamma_2, \\ &K_2(\tilde{u}) = \int_V F(x, \tilde{u}) d\mu \leq 1, \end{aligned} \tag{38}$$

which implies $\tilde{u} \in \mathcal{H}_2$.

Analogous proof of Claim 1, we can see $K_2(\tilde{u}) = 1$, which implies $\tilde{u}$ is not identically zero on V.

Claim 6. $\tilde{u}$, as above, is positive everywhere on V.

Proof. We calculate the Euler-Lagrange equation at $\tilde{u}$ under the constraint condition $K_2(\tilde{u}) = 1$. By Lemma 5.3 and (31), (33), for any $\varphi \in C(V)$, there holds

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \Big\{ J_2(\tilde{u} + t\varphi) - \lambda_2 \Big(\int_V F(x, \tilde{u} + t\varphi) d\mu - 1 \Big) \Big\} \\ &= p \sum_{\substack{x,y \in V \\ x \sim y}} \omega_{xy} |u(y) - u(x)|^{p-2} (u(y) - u(x))(\varphi(y) - \varphi(x)) \\ &\quad + \int_V \Big(ph|\tilde{u}|^{p-2}\tilde{u} - \lambda_2 F'(x, \tilde{u}) \Big) \varphi d\mu \\ &= \int_V \Big(-p\Delta_p \tilde{u} + ph|\tilde{u}|^{p-2}\tilde{u} - \lambda_2 F'(x, \tilde{u}) \Big) \varphi d\mu. \end{aligned}$$

Hence, we get

$$-p\Delta_p\tilde{u} + ph|\tilde{u}|^{p-2}\tilde{u} = \lambda_2 F'(x,\tilde{u}). \tag{39}$$

Since (31) and (33), we have

$$-\int_V u\Delta_p u d\mu = \int_V |\nabla_p u|^p d\mu. \tag{40}$$

Hence, by (40), (H_3) and $\tilde{u} \not\equiv 0$ on V, multiplying $\tilde{u}$ on both sides of the equation (39), and taking integration, we can see $\lambda_2 > 0$.

If $\tilde{u}(x) < 0$, at some vertex $x \in V$, then by the equation (39), we see

$$\Delta_p\tilde{u}(x) < 0.$$

However, by the definition of Δ_p in (31), there is a $y \sim x$ with $\tilde{u}(y) < \tilde{u}(x) < 0$. In view of the connectedness of the graph $G = (V, E)$, by induction, we obtain a sequence $x = x_1 \sim x_2 \sim x_3 \sim \cdots$ such that

$$\cdots < \tilde{u}(x_i) < \tilde{u}(x_{i-1}) < \cdots < \tilde{u}(x_1) < 0.$$

Then we have

$$\sum_{i=1}^{n} |\tilde{u}(x_i)|^p \mu(x_i) \to +\infty \quad \text{as } n \to +\infty,$$

which contradicts $\tilde{u} \in \mathcal{H}_2 \subseteq L^p(V)$. Hence $\tilde{u}$ is nonnegative on V. If $\tilde{u}$ is not positive everywhere on V, we can always find two vertices x, y with $y \sim x$, $\tilde{u}(x) = 0$, $\tilde{u}(y) > 0$. Then it follows $\Delta_p\tilde{u}(x) > 0$ by the definition of Δ_p, which contradicts to the equation (39). Hence $\tilde{u}$ is positive everywhere on V. □

Claim 7. The p-th nonlinear equation (35) has a strictly positive solution.

Proof. By Claim 5, we know that $\tilde{u}$ is positive everywhere on V, and it satisfies

$$-p\Delta_p\tilde{u} + ph\tilde{u}^{p-1} = \lambda_2\theta f(x, \theta\tilde{u}). \tag{41}$$

Choosing $\theta = (\frac{p}{\lambda_2})^{1/p}$, and taking $\theta\tilde{u}$ by u in (41), we know that u is positive everywhere on V, and u satisfies the following equation

$$-\Delta_p u + hu^{p-1} = f(x, u),$$

which completes the proof. □

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