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CORRIGENDUM TO "A DUAL ITERATIVE SUBSTRUCTURING METHOD WITH A SMALL PENALTY PARAMETER", [J. KOREAN MATH. SOC. 54 (2017), NO. 2, 461–477]

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Abstract. In this corrigendum, we offer a correction to [J. Korean Math. Soc. 54 (2017), No. 2, 461–477]. We construct a counterexample for the strengthened Cauchy–Schwarz inequality used in the original paper. In addition, we provide a new proof for Lemma 5 of the original paper, an estimate for the extremal eigenvalues of the standard unpreconditioned FETI-DP dual operator.

In the first and second authors' previous work [4], the strengthened Cauchy–Schwarz inequality used for [4, Eq. (3.8)] is incorrect and consequently, the statement of [4, Lemma 4] needs to be corrected. We present a new proof for [4, Lemma 5], that does not use [4, Lemma 4]. All notations are adopted from the original paper [4].

In the paragraph containing [4, Eq. (3.8)], it was claimed that by deriving a strengthened Cauchy-Schwarz inequality in a similar way to Lemma 4.3 in [3], it is shown that there exists a constant γ such that

$$2\tilde{a}(v_I + v_\Delta, v_c) \geq -\gamma(\tilde{a}(v_I + v_\Delta, v_I + v_\Delta) + \tilde{a}(v_c, v_c)),$$

where $0 < \gamma < 1$ is independent of H and h. That is, the above inequality is true when there exists a constant γ such that

$$|\tilde{a}(v_I + v_\Delta, v_c)| \leq \gamma \left(\tilde{a}(v_I + v_\Delta, v_I + v_\Delta)\right)^{1/2} \left(\tilde{a}(v_c, v_c)\right)^{1/2}, \tag{1}$$

where $0 < \gamma < 1$ is independent of h and H.

On the other hand, a specific function $w = w_I + w_c + w_\Delta$ can be constructed, for which γ approaches 1 as H decreases. In fact, it suffices to characterize such w_Δ because w_I and w_c in (1) are determined by w_Δ in terms of the discrete $\tilde{a}$-harmonic extension $\mathcal{H}^c(w_\Delta)$.

Proposition 1. *There is no γ $(0 < \gamma < 1)$, independent of h and H, satisfying* (1).

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Proof. Noting that $\mathcal{H}^c(v_\Delta)$ in X_h^c is $\tilde{a}(\cdot,\cdot)$-orthogonal to all the functions which vanish at the interface nodes except for the subdomain corners, we have that

$$\begin{aligned}\tilde{a}(v_I+v_\Delta,v_c) &= \tilde{a}\left(\mathcal{H}^c(v_\Delta)-v_c,v_c\right)\\ &= \tilde{a}\left(\mathcal{H}^c(v_\Delta),v_c\right)-\tilde{a}(v_c,v_c)\\ &= -\tilde{a}(v_c,v_c),\end{aligned}$$

which implies that for $\tilde{a}(v_I+v_\Delta,v_I+v_\Delta)\neq 0$, the estimate (1) is equivalent to

$$\frac{\tilde{a}(v_c,v_c)}{\tilde{a}(v_I+v_\Delta,v_I+v_\Delta)}\le\gamma^2, \tag{2}$$

where $\gamma<1$ is independent of h and H.

Next, let us divide $\Omega=(0,1)^2$ into $1/H\times 1/H$ square subdomains with a side length H. Each subdomain is partitioned into $2\times H/h\times H/h$ uniform right triangles. Associated with such a triangulation, we select the function w in X_h^c such that w is a conforming $\mathbb{P}_1$ element function in each subdomain, and $w_\Delta=1$ at all the nodes on the interface except for the subdomain corners. Then it is noted that w in X_h^c vanishes on $\partial\Omega$. Let us denote by $\{x_k\}$ the subdomain corners that are not on $\partial\Omega$. Hence, for w_c and w_I that are computed by the discrete harmonic extension of w_Δ, it is observed that

$$w_c=1 \text{ at all } x_k, \tag{3a}$$

$$w_I=1 \text{ in } \Omega_j \text{ for } \partial\Omega_j\cap\partial\Omega=\varnothing, \tag{3b}$$

which imply that

$$w\equiv 1 \text{ in all subdomains whose boundary does not touch } \partial\Omega. \tag{4}$$

Let us first estimate $\tilde{a}(w_c,w_c)$ in (2). Using (3a), we have that

$$\tilde{a}(w_c,w_c)=\sum_{k=1}^{(1/H-1)^2}\tilde{a}(\phi_{c,k},\phi_{c,k})=4\left(\frac{1}{H}-1\right)^2,$$

where $\phi_{c,k}$ is the nodal basis function associated with x_k. We next look over $\tilde{a}(w_I+w_\Delta,w_I+w_\Delta)$ based on the fact that, for $\partial\Omega_j\cap\partial\Omega=\varnothing$

$$\tilde{a}_{\Omega_j}(w_I+w_\Delta,w_I+w_\Delta)=\int_{\Omega_j}|\nabla(w_I+w_\Delta)|^2dx=\int_{\Omega_j}|\nabla w_c|^2dx=4, \tag{5}$$

which follows from (4). Hence it suffices to estimate $\tilde{a}_{\Omega_j}(w_I+w_\Delta,w_I+w_\Delta)$ for the following two cases:

(i) only one of the edges of the subdomain Ω_j is on $\partial\Omega$.
(ii) two edges of the subdomain Ω_j are on $\partial\Omega$.

Here, the number of subdomains corresponding to the cases (i) and (ii) is $4\left(\frac{1}{H}-2\right)$ and 4, respectively. Let us take $H/h=3$ to focus only on the dependence of γ on either H or h. By finding the discrete local harmonic extensions for the cases (i) and (ii), it is computed directly that

$$\tilde{a}_{\Omega_j}(w_I + w_\Delta, w_I + w_\Delta) = \begin{cases} \frac{17}{4} & \text{for the case (i)}, \\ \frac{14}{4} & \text{for the case (ii)}. \end{cases} \tag{6}$$

Then by using (5) and (6), it follows that

$$\begin{aligned} \tilde{a}(w_I + w_\Delta, w_I + w_\Delta) &= \left(\sum_{\substack{j \text{ for} \\ \partial\Omega_j \cap \partial\Omega = \varnothing}} + \sum_{\substack{j \text{ for} \\ \partial\Omega_j \cap \partial\Omega \neq \varnothing}} \right) \tilde{a}_{\Omega_j}(w_I + w_\Delta, w_I + w_\Delta) \\ &= 4\left(\frac{1}{H} - 2\right)^2 + 17\left(\frac{1}{H} - 2\right) + 14. \end{aligned} \tag{7}$$

Finally, from (3a) and (7), it is confirmed that for a function w given above,

$$\lim_{H \to 0} \frac{\tilde{a}(w_c, w_c)}{\tilde{a}(w_I + w_\Delta, w_I + w_\Delta)} = 1,$$

which implies that (2) does not hold. Therefore, the proof is completed. □

In [4, Lemma 5], the extremal eigenvalues of the FETI-DP dual operator $F = B_\Delta S^{-1} B_\Delta^T$ were estimated using [4, Lemma 4]. Since [4, Lemma 4] is incorrect, we provide a new estimate for F that does not utilize [4, Lemma 4]. We assume that each subdomain Ω_j is the union of elements in a conforming coarse mesh $\mathcal{T}_H$ of Ω. First, we consider the following Poincaré-type inequality that generalizes [4, Proposition 3].

Lemma 2. *For any $v_j \in X_h^j$, let $I_j^H v_j$ be the linear coarse interpolation of v_j such that $I_j^H v_j = v_j$ at vertices of a subdomain $\Omega_j \subset \mathbb{R}^d$. Then we have*

$$|v_j|_{H^1(\Omega_j)}^2 \gtrsim \begin{cases} H^{-1}\left(1 + \ln \frac{H}{h}\right)^{-1} \|v_j - I_j^H v_j\|_{L^2(\partial\Omega_j)}^2 & \text{for } d = 2, \\ h^{-1}\left(\frac{H}{h}\right)^{-2} \|v_j - I_j^H v_j\|_{L^2(\partial\Omega_j)}^2 & \text{for } d = 3. \end{cases}$$

Proof. Note that both sides of the above inequality do not change if a constant is added to v_j. Without loss of generality, we assume that v_j has the zero average, so that the following Poincaré inequality holds:

$$\|v_j\|_{H^1(\Omega_j)} \lesssim |v_j|_{H^1(\Omega_j)}, \tag{8}$$

where $\|\cdot\|_{H^1(\Omega_j)}$ is the weighted H^1-norm on Ω_j given by

$$\|v_j\|_{H^1(\Omega_j)}^2 = |v_j|_{H^1(\Omega_j)}^2 + \frac{1}{H^2}\|v_j\|_{L^2(\Omega_j)}^2.$$

Since $I_j^H v_j$ attains its extremum at vertices, we have

$$\begin{aligned} \|v_j - I_j^H v_j\|_{L^2(\partial\Omega_j)} &\lesssim H^{\frac{d-1}{2}} \|v_j - I_j^H v_j\|_{L^\infty(\partial\Omega_j)} \\ &\le H^{\frac{d-1}{2}} \left(\|v_j\|_{L^\infty(\partial\Omega_j)} + \|I_j^H v_j\|_{L^\infty(\partial\Omega_j)} \right) \\ &\lesssim H^{\frac{d-1}{2}} \|v_j\|_{L^\infty(\partial\Omega_j)}. \end{aligned} \tag{9}$$

Let $\mathcal{H}_j v_j$ be the generalized harmonic extension of $v_j|_{\partial\Omega_j}$ introduced in [7] such that $\mathcal{H}_j v_j = v_j$ on $\partial\Omega_j$ and

$$\|\mathcal{H}_j v_j\|_{H^1(\Omega_j)} = \min_{\substack{w_j \in H^1(\Omega_j) \\ w_j = v_j \text{ on } \partial\Omega_j}} \|w_j\|_{H^1(\Omega_j)}. \tag{10}$$

Then it follows that

$$\begin{aligned} H^{d-1} \|v_j\|^2_{L^\infty(\partial\Omega_j)} &\le H^{d-1} \|\mathcal{H}_j v_j\|^2_{L^\infty(\Omega_j)} \\ &\lesssim C_d(H,h) \|\mathcal{H}_j v_j\|^2_{H^1(\Omega_j)} && \text{(11a)} \\ &\le C_d(H,h) \|v_j\|^2_{H^1(\Omega_j)} && \text{(11b)} \\ &\lesssim C_d(H,h) |v_j|^2_{H^1(\Omega_j)}, && \text{(11c)} \end{aligned}$$

where

$$C_d(H,h) = \begin{cases} H\left(1 + \ln \frac{H}{h}\right) & \text{for } d = 2, \\ h \left(\frac{H}{h}\right)^2 & \text{for } d = 3, \end{cases}$$

and (11a) is due to the discrete Sobolev inequality [2, Lemma 2.3]. Also (10) and (8) are used in (11b) and (11c), respectively. Combination of (9) and (11) completes the proof. □

Note that Lemma 2 reduces to [4, Proposition 3] when v_j vanishes at vertices of Ω_j so that $I_j^H v_j = 0$. Using Lemma 2, we obtain the following estimate for F.

Proposition 3. *For $F = B_\Delta S^{-1} B_\Delta^T$, we have*

$$\underline{C}_F \lambda^T \lambda \lesssim \lambda^T F \lambda \lesssim \overline{C}_F \lambda^T \lambda \quad \forall \lambda,$$

where

$$\underline{C}_F = h^{2-d} \text{ for } d = 2, 3,$$

and

$$\overline{C}_F = \begin{cases} \left(\frac{H}{h}\right)\left(1 + \ln \frac{H}{h}\right) & \text{for } d = 2, \\ h^{-1} \left(\frac{H}{h}\right)^2 & \text{for } d = 3. \end{cases}$$

Consequently, the condition number of F satisfies the following bound:

$$\kappa(F) \lesssim \begin{cases} \left(\frac{H}{h}\right)\left(1 + \ln \frac{H}{h}\right) & \text{for } d = 2, \\ \left(\frac{H}{h}\right)^2 & \text{for } d = 3. \end{cases}$$

Proof. As the derivation of the maximum eigenvalue of S in the original paper [4] is correct, the derivation of $\underline{C}_F$ is also correct. Thus, we only estimate $\overline{C}_F$ in the following.

We first prove that

$$(B_\Delta \boldsymbol{v}_\Delta)^T (B_\Delta \boldsymbol{v}_\Delta) \lesssim \overline{C}_F \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta \quad \forall \boldsymbol{v}_\Delta. \tag{12}$$

For v_Δ, we consider the discrete $\tilde{a}$-harmonic extension $v = \mathcal{H}^c(v_\Delta)$. Let $w = v - I^H v$, where $I^H v$ is the linear coarse interpolation of v onto $\mathcal{T}^H$ such that $I^H v = v$ at the subdomain vertices. We write $w = w_I + w_\Delta$. Since $I^H v$ is continuous along Γ, we have $B_\Delta \boldsymbol{w}_\Delta = B_\Delta \boldsymbol{v}_\Delta$. Then it follows that

$$\begin{aligned}
(B_\Delta \boldsymbol{v}_\Delta)^T (B_\Delta \boldsymbol{v}_\Delta) &= (B_\Delta \boldsymbol{w}_\Delta)^T (B_\Delta \boldsymbol{w}_\Delta) \\
&= \sum_{j<k} \left(\boldsymbol{w}_\Delta^{(j)}|_{\Gamma_{jk}} - \boldsymbol{w}_\Delta^{(k)}|_{\Gamma_{jk}} \right)^T \left(\boldsymbol{w}_\Delta^{(j)}|_{\Gamma_{jk}} - \boldsymbol{w}_\Delta^{(k)}|_{\Gamma_{jk}} \right) \\
&\lesssim \sum_{j<k} \left(\left(\boldsymbol{w}_\Delta^{(j)}|_{\Gamma_{jk}} \right)^T \boldsymbol{w}_\Delta^{(j)}|_{\Gamma_{jk}} + \left(\boldsymbol{w}_\Delta^{(k)}|_{\Gamma_{jk}} \right)^T \boldsymbol{w}_\Delta^{(k)}|_{\Gamma_{jk}} \right) \\
&\lesssim \sum_{j=1}^{N_s} (\boldsymbol{w}_\Delta^{(j)})^T (\boldsymbol{w}_\Delta^{(j)}) \\
&\lesssim h^{1-d} \sum_{j=1}^{N_s} \|w\|_{L^2(\partial\Omega_j)}^2 \\
&\lesssim \overline{C}_F \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta,
\end{aligned}$$

where the last inequality is due to Lemma 2.

Then similar to [5, Theorem 4.4], we get the desired result as follows:

$$\begin{aligned}
\lambda^T F \lambda &= \max_{\boldsymbol{v}_\Delta \neq 0} \frac{\left((B_\Delta \boldsymbol{v}_\Delta)^T \lambda \right)^2}{\boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta} \\
&\lesssim \overline{C}_F \max_{B_\Delta \boldsymbol{v}_\Delta \neq 0} \frac{\left((B_\Delta \boldsymbol{v}_\Delta)^T \lambda \right)^2}{(B_\Delta \boldsymbol{v}_\Delta)^T B_\Delta \boldsymbol{v}_\Delta} \\
&\leq \overline{C}_F \max_{\mu \neq 0} \frac{(\mu^T \lambda)^2}{\mu^T \mu} \\
&= \overline{C}_F \lambda^T \lambda,
\end{aligned}$$

where we used [5, Lemma 4.3] in the first equality. Consequently, this completes the proof. □

It must be mentioned that the conclusion of Proposition 3 agrees with Lemma 5 of the original paper [4]. Since the conclusion of [4, Lemma 5] is true, it requires no additional correction in the remaining part of that paper.

For the sake of completeness, we present a correct estimate for the extremal eigenvalues of S that replaces [4, Lemma 4].

Proposition 4. *For $S = A_{\Delta\Delta} - A_{I\Delta}^T A_{II}^{-1} A_{I\Delta}$, we have*

$$\underline{C}_S \boldsymbol{v}_\Delta^T \boldsymbol{v}_\Delta \lesssim \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta \lesssim \overline{C}_S \boldsymbol{v}_\Delta^T \boldsymbol{v}_\Delta \quad \forall \boldsymbol{v}_\Delta,$$

where

$$\underline{C}_S = \begin{cases} Hh\left(1 + \ln \frac{H}{h}\right)^{-1} & \text{for } d = 2, \\ h^3 & \text{for } d = 3, \end{cases}$$

and

$$\overline{C}_S = h^{d-2} \text{ for } d = 2, 3.$$

Proof. Since the derivation of $\overline{C}_S$ in the original paper [4] is correct, we only consider an estimate for $\underline{C}_S$. Take any $\boldsymbol{v}_\Delta$ and its corresponding finite element function v_Δ. Let $v = \mathcal{H}^c(v_\Delta)$ be the discrete $\tilde{a}$-harmonic extension of v_Δ. Proceeding as in [6, Lemma 4.11], we get

$$\begin{aligned} \boldsymbol{v}_\Delta^T \boldsymbol{v}_\Delta &\lesssim h^{1-d} \sum_{j=1}^{N_s} \|v_\Delta\|_{L^2(\partial\Omega_j)}^2 \\ &\lesssim Hh^{1-d} \sum_{j=1}^{N_s} \left(|v|_{H^1(\Omega_j)}^2 + H^{-2} \|v\|_{L^2(\Omega_j)}^2 \right) \\ &= Hh^{1-d} \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta + H^{-1} h^{1-d} \|v\|_{L^2(\Omega)}^2. \end{aligned}$$

Note that we cannot apply the discrete Poincaré inequality [1, Lemma 5.1] in each subdomain Ω_j since $\mathcal{H}v_\Delta$ does not vanish at the subdomain vertices in general.

It remains to show that

$$\|v\|_{L^2(\Omega)}^2 \lesssim \begin{cases} \left(1 + \ln \frac{H}{h}\right) \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta & \text{for } d = 2, \\ \frac{H}{h} \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta & \text{for } d = 3. \end{cases} \tag{13}$$

Let $I^H v$ be the linear nodal interpolation of v onto the coarse mesh $\mathcal{T}_H$. Since $I^H v$ is continuous along the subdomain interfaces Γ, we can apply the Poincaré inequality to obtain

$$\|I^H v\|_{L^2(\Omega)} \lesssim |I^H v|_{H^1(\Omega)}.$$

Then it follows that

$$\begin{aligned} \|v\|_{L^2(\Omega)}^2 &\lesssim \|v - I^H v\|_{L^2(\Omega)}^2 + \|I^H v\|_{L^2(\Omega)}^2 \\ &\lesssim \|v - I^H v\|_{L^2(\Omega)}^2 + |I^H v|_{H^1(\Omega)}^2 \\ &\lesssim \begin{cases} \left(1 + \ln \frac{H}{h}\right) \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta & \text{for } d = 2, \\ \frac{H}{h} \boldsymbol{v}_\Delta^T S \boldsymbol{v}_\Delta & \text{for } d = 3, \end{cases} \end{aligned}$$

where the last inequality is due to [6, Remark 4.13] for $d=2$ and [6, Lemma 4.12] for $d = 3$, respectively. This completes the proof. □

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