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ON THE STRUCTURE OF ZERO-DIVISOR ELEMENTS IN A NEAR-RING OF SKEW FORMAL POWER SERIES

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Abstract. The main purpose of this paper is to study the zero-divisor properties of the zero-symmetric near-ring of skew formal power series $R_0[[x;\alpha]]$, where R is a symmetric, α-compatible and right Noetherian ring. It is shown that if R is reduced, then the set of all zero-divisor elements of $R_0[[x;\alpha]]$ forms an ideal of $R_0[[x;\alpha]]$ if and only if $Z(R)$ is an ideal of R. Also, if R is a non-reduced ring and $ann_R(a-b)\cap Nil(R)\neq 0$ for each $a,b\in Z(R)$, then $Z\big(R_0[[x;\alpha]]\big)$ is an ideal of $R_0[[x;\alpha]]$. Moreover, if R is a non-reduced right Noetherian ring and $Z\big(R_0[[x;\alpha]]\big)$ forms an ideal, then $ann_R(a-b)\cap Nil(R)\neq 0$ for each $a,b\in Z(R)$. Also, it is proved that the only possible diameters of the zero-divisor graph of $R_0[[x;\alpha]]$ is 2 and 3.

1. Introduction and preliminary definitions

Throughout this paper R always denotes an associative ring with unity. Recall that a ring R is said to be *symmetric* if $abc=0$, then $bac=0$ for each $a,b,c\in R$. Also, R is called *reversible* if $ab=0$ implies $ba=0$ for each $a,b\in R$. Moreover, R is said to be *semicommutative* if $ab=0$ implies $aRb=0$ for each $a,b\in R$.

Recall that a ring R is said to be *right* (*left*) *uniserial* if its right (left) ideals are linearly ordered by inclusion.

We denote the set of all nilpotent elements of a ring R by $Nil(R)$. Recall that a ring R is called *reduced* if $Nil(R)=\{0\}$. Also, if $X\subseteq R$, then $\langle X\rangle_\ell$, $\langle X\rangle_r$ and $\langle X\rangle$ denote the left ideal generated by X, the right ideal generated by X and the ideal generated by X, respectively. Moreover, for a given ring or near-ring R, we write $Z(R)=Z_\ell(R)\cup Z_r(R)$, where $Z_\ell(R)$ and $Z_r(R)$ are the set of all left zero-divisors of R and the set of all right zero-divisors of R, respectively.

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Let α be an endomorphism of a ring R. Then the *skew power series* ring $R[[x;\alpha]]$ is the ring of power series over R in the variable x, with term-wise addition and with coefficient written on the left of x, subject to the skew-multiplication rule $xr = \alpha(r)x$ for $r \in R$. Following [20], an endomorphism α of ring R is called *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. Also, R is called α-rigid if there exists a rigid endomorphism α of R. In [16], authors proved that any rigid endomorphism of a ring is a monomorphism and α-rigid rings are reduced.

According to [13], a ring R is called α-*compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\alpha(b) = 0$. Hence α-compatible rings are a generalization of rigid rings. In fact, R is an α-rigid ring if and only if R is reduced and α-compatible, by [13, Lemma 2.2].

The collection of all skew power series with positive orders using the operations of addition and substitution is denoted by $R_0[[x;\alpha]]$. Notice that the system $(R_0[[x;\alpha]], +, \circ)$ is a zero-symmetric left near-ring, since the operation "$\circ$", left distributes but does not right distribute over addition. For example, let $f = \sum_{i=1}^{\infty} a_i x^i$ and $g = \sum_{j=1}^{\infty} b_j x^j \in R_0[[x;\alpha]]$, then

$$\begin{aligned} g \circ f &= a_1 g + a_2 g^2 + a_3 g^3 + \cdots \\ &= a_1 b_1 x + \big[a_1 b_2 + a_2 b_1 \alpha(b_1)\big] x^2 \\ &= \big[a_1 b_3 + a_2 b_1 \alpha(b_2) + a_2 b_2 \alpha^2(b_1) + a_3 b_1 \alpha(b_1) \alpha^2(b_1)\big] x^3 + \cdots , \end{aligned}$$

where g^i is the product of i copies of g in the ring $R[[x;\alpha]]$ for each i.

Recall that a graph G is *connected* if there is a path between any two distinct vertices of G. Also, the *diameter* of G is

$$diam(G) = \sup\{d(a,b) \,|\, a, b \text{ are vertices of } G\},$$

where $d(a,b)$ is the length of the shortest path from a to b.

In [7], Beck introduced and studied the zero-divisor graph of a commutative ring. Since then, the concept of zero-divisor graphs has been studied extensively by many authors, (cf. [3–5, 7, 18, 22]). Now, we are interested to study the undirected zero-divisor graph of a near-ring $R_0[[x;\alpha]]$ which is denoted by $\Gamma\big(R_0[[x;\alpha]]\big)$ and defined as follows: the set of vertices of $\Gamma\big(R_0[[x;\alpha]]\big)$ is the non-zero zero-divisor elements of $R_0[[x;\alpha]]$ and two distinct vertices f and g are adjacent if and only if $f \circ g = 0$ or $g \circ f = 0$.

In this work, we first characterize the zero-divisor elements of a near-ring $R_0[[x;\alpha]]$, where R is a symmetric, α-compatible and right Noetherian ring. Then we study the zero-divisor graph of $R_0[[x;\alpha]]$, and show that $diam\big(\Gamma(R_0[[x;\alpha]])\big)$ is 2 or 3, where R is a symmetric and α-compatible ring. Moreover, we prove that if R is an α-rigid right Noetherian ring, then $Z\big(R_0[[x;\alpha]]\big)$ forms an ideal of $R_0[[x;\alpha]]$ if and only if $Z(R)$ is an ideal of R. Also, giving some examples, we will show that the assumption being right Noetherian for R is not redundant. Finally, for symmetric non-reduced rings, it is proved that (1) if $ann_R(a-b) \cap Nil(R) \neq 0$ for each $a, b \in Z(R)$, then

$Z\big(R_0[[x;\alpha]]\big)$ is an ideal of $R_0[[x;\alpha]]$, and (2) if R is right Noetherian and $Z\big(R_0[[x;\alpha]]\big)$ forms an ideal of $R_0[[x;\alpha]]$, then $ann_R(a-b)\cap Nil(R)\neq 0$ for each $a,b\in Z(R)$.

2. Zero-divisor elements in a near-ring of skew formal power series

We start by summarizing some useful lemmas, which will become building blocks of the main results. The following lemma can be found in [14].

Lemma 2.1 ([14, Lemma 2.3])**.** *Let R be an α-compatible ring. Then we have the following*:

(1) *If $ab=0$, then $a\alpha^n(b)=\alpha^n(a)b=0$ for any positive integer n.*
(2) *If $\alpha^k(a)b=0$ for some positive integer k, then $ab=0$.*
(3) *If $f=\sum_{i=0}^{\infty}a_ix^i\in R[[x;\alpha]]$ and $r\in R$, then $fr=0$ if and only if $a_ir=0$ for each i.*
(4) *If $f\in R[[x;\alpha]]$ and $r\in R$, then $rf=0$ if and only if $rxf=0$.*

Let f be an element of a ring $R[[x;\alpha]]$ or a near-ring $R_0[[x;\alpha]]$. Then we use C_f to denote the set of all coefficients of f.

Lemma 2.2. *Let R be an α-rigid ring. Then we have the following*:

(1) [12, Proposition 2.3] *If f and g are elements of a ring $R[[x;\alpha]]$, then $fg=0$ if and only if $a_ib_j=0$ for all $a_i\in C_f$ and all $b_j\in C_g$.*
(2) [10, Lemma 2.4] *If f and g are elements of a near-ring $R_0[[x;\alpha]]$, then $f\circ g=0$ if and only if $a_ib_j=0$ for all $a_i\in C_f$ and all $b_j\in C_g$.*

As an immediate consequence of Lemma 2.2, we get the following lemma.

Lemma 2.3. *Let R be an α-rigid ring. Then*

$$Z_\ell\big(R_0[[x;\alpha]]\big)=Z_r\big(R_0[[x;\alpha]]\big)=Z_r\big(R[[x;\alpha]]\big)x=Z_\ell\big(R[[x;\alpha]]\big)x.$$

According to [17], a ring R has (*right*) *left Property* (A), if every finitely generated ideal consisting entirely of (left) right zero-divisor has a left (right) non-zero annihilator. Also, a ring R is said to have *Property* (A) if R has both right and left Property (A).

Since every symmetric ring is semicommutative by [8], then we get the following result from [11, Theorem 2.6].

Lemma 2.4. *Let R be a symmetric and right Noetherian ring. Then R has left Property (A).*

Motivated by [2], the authors in [14] calls a ring R with an endomorphism α to be *right α-power-serieswise McCoy*, whenever power series $f,g\in R[[x;\alpha]]\setminus\{0\}$ satisfy $fg=0$, then there exists a non-zero element $c\in R$ such that $fc=0$. *Left α-power-serieswise McCoy* is defined similarly. If a ring R is both right and left α-power-serieswise McCoy, then R is called *α-power-serieswise McCoy*.

Lemma 2.5 ([14, Corollary 2.7])**.** *If R is a reversible, α-compatible and right Noetherian ring, then R is α-power-serieswise McCoy.*

Remark 2.6. Let R be a reversible, α-compatible and right Noetherian ring. Notice that $Z\big(R[[x;\alpha]]\big) \subseteq Z(R)[[x;\alpha]]$, by Lemma 2.5. Now, let $Z(R)$ be an ideal of R and $f \in Z(R)[[x;\alpha]]$. Since R is right Noetherian, then $Z(R)$ is finitely generated as right ideal. Hence there exists $0 \neq r \in R$ such that $rZ(R) = 0$, by Lemma 2.4. Since $C_f \subseteq Z(R)$, then $rC_f = 0$. It means that $rf = 0$, and thus $Z(R)[[x;\alpha]] \subseteq Z\big(R[[x;\alpha]]\big)$. Therefore $Z\big(R[[x;\alpha]]\big) = Z(R)[[x;\alpha]]$.

Combining Lemma 2.3 and Remark 2.6, we obtain the following corollary.

Corollary 2.7. *Let R be an α-rigid and right Noetherian ring. If $Z(R)$ is an ideal of R, then $Z\big(R_0[[x;\alpha]]\big) = Z\big(R[[x;\alpha]]\big)x = Z(R)_0[[x;\alpha]]$.*

Lemma 2.8. *Let R be a symmetric and α-compatible ring. If $f = \sum_{i=1}^{\infty} a_i x^i$ is a zero-divisor of $R_0[[x;\alpha]]$, then $a_1 \in Z(R)$.*

Proof. Let $a_1 \neq 0$. Since $f \in Z\big(R_0[[x;\alpha]]\big)$, then there exists a non-zero $g = \sum_{j=1}^{\infty} b_j x^j \in R_0[[x;\alpha]]$ such that $g \circ f = 0$ or $f \circ g = 0$. Let k be the smallest integer such that $b_k \neq 0$. If $g \circ f = 0$, then $a_1 b_k = 0$, and so the result follows. Now suppose that $f \circ g = 0$. Then $b_k a_1 \alpha(a_1) \cdots \alpha^{k-1}(a_1) = 0$, since it is the coefficient of x^k in $f \circ g$. Hence $b_k a_1^k = 0$, by Lemma 2.1. If $b_k a_1 = 0$, then $a_1 \in Z(R)$. Now, assume that $b_k a_1 \neq 0$. Then there exists $1 \leq s \leq k-1$ such that $b_k a_1^s \neq 0$ but $(b_k a_1^s) a_1 = 0$, as desired. □

In [1], the authors studied the skew generalized power series rings over nil rings and provided some conditions under which the skew generalized power series ring is nil. Following [19], a ring R is *α-nil-Armendariz* whenever $f = \sum_{i=0}^{\infty} a_i x^i$ and $g = \sum_{j=0}^{\infty} b_j x^j$ be elements of $R[[x;\alpha]]$ with $fg \in Nil(R)[[x;\alpha]]$, then $a_i \alpha^i(b_j) \in Nil(R)$ for each i, j.

Recall that an ideal I of R is an *α-ideal* if $\alpha(I) \subseteq I$. For example, if R is an α-compatible ring and $Nil(R)$ is an ideal of R, then it is also an α-ideal. Therefore, by a similar way as used in the proof of [15, Proposition 1], one can prove the following result.

Proposition 2.9. *Let R be an α-compatible ring and $Nil(R)$ be an ideal of R. Then R is an α-nil-Armendariz ring.*

We will make use of the following lemma which appears in [19, Theorem 3.14].

Proposition 2.10. *Let R be an α-compatible and α-nil-Armendariz ring. If $Nil(R)$ is a nilpotent ideal of R, then $Nil\big(R[[x;\alpha]]\big) = Nil(R)[[x;\alpha]]$.*

Corollary 2.11. *Let R be a symmetric, α-compatible and right Noetherian ring. Then $Nil\big(R[[x;\alpha]]\big) = Nil(R)[[x;\alpha]]$.*

Proof. Since R is symmetric and right Noetherian, then $Nil(R)$ is a nilpotent ideal of R. Hence the assertion follows from Propositions 2.9 and 2.10. □

Example 2.12. (1) Let D be an integral domain and $R = \{\left[\begin{smallmatrix} a & d \\ 0 & a \end{smallmatrix}\right] \mid a, d \in D\}$. Suppose that $u \in U(D)$. Consider $\alpha : R \longrightarrow R$ by $\alpha\left(\left[\begin{smallmatrix} a & d \\ 0 & a \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} a & ud \\ 0 & a \end{smallmatrix}\right]$. Thus R is a commutative and α-compatible ring. Also, $Nil(R) = \{\left[\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right] \mid a \in D\}$ is a nilpotent ideal of R. Therefore $Nil\big(R[[x;\alpha]]\big) = Nil(R)[[x;\alpha]]$, by Propositions 2.9 and 2.10.

(2) Let R be a right Artinian and right uniserial ring, and $S = R[y]$. Then R is right Noetherian, and so S is right Noetherian. Moreover, by [21, Proposition 3.5], S is symmetric. Hence $Nil\big(R[[x]]\big) = Nil(R)[[x]]$, by Corollary 2.11.

Now we bring the following theorem, which has a key rule in our results.

Theorem 2.13. *Let R be a symmetric, α-compatible and right Noetherian ring. Let $f = \sum_{i=1}^{\infty} a_i x^i$ and $g = \sum_{j=1}^{\infty} b_j x^j$ be non-zero elements of a near-ring $R_0[[x;\alpha]]$. If $f \circ g = 0$, then*

(1) $a_1 b_1 = 0$,
(2) $rf = 0$ *for some non-zero* $r \in R$,
(3) f *is nilpotent or* $sg = 0$ *for some non-zero* $s \in R$.

Proof. (1) It is clear, since $b_1 a_1$ is the coefficient of x in $f \circ g$.

(2) Since $f \circ g = 0$, it follows that $b_1 f + b_2 f^2 + b_3 f^3 + \cdots = 0$. Hence

$$\big(b_1 + b_2 f + b_3 f^2 + \cdots\big) f = 0.$$

Since $b_1 + b_2 f + b_3 f^2 + \cdots$ is non-zero, $rf = 0$ for some non-zero $r \in R$, by Lemma 2.5.

(3) Notice that $\langle C_g \rangle_r = \langle b_1, \ldots, b_n \rangle_r$ for some $n \geq 1$, since R is right Noetherian. Suppose that f is not nilpotent. It follows that there exists $a = a_i$ such that $a \notin Nil(R)$, by Corollary 2.11. Let $\overline{R} = R/Nil(R)$. Since $f \circ g = 0$, then $\overline{f} \circ \overline{g} = \overline{0}$ in a near-ring $\overline{R}_0[[x;\overline{\alpha}]]$. Since $\overline{R}$ is a reduced and $\overline{\alpha}$-compatible ring, it follows that $\overline{R}$ is an $\overline{\alpha}$-rigid ring, by [13, Lemma 2.2]. Thus $\overline{a}_i \overline{b}_j = \overline{0}$, by Lemma 2.2. Since R is right Noetherian, then $Nil(R)$ is nilpotent, and so $Nil(R)^k = 0$ for some positive integer k. Thus $a^k b_j^k = 0$ for each $j \geq 1$. Hence there exist integers $0 \leq t_j \leq k$ such that $a^k b_j^{t_j} \neq 0$ but $a^k b_j^{t_j+1} = 0$ for each $j \geq 1$. Therefore there exist integers $0 \leq s_j \leq t_j$ such that $a^k b_1^{s_1} b_2^{s_2} \cdots b_n^{s_n} \neq 0$ but $a^k b_1^{s_1} b_2^{s_2} \cdots b_n^{s_n} b_j = 0$ for each $1 \leq j \leq n$. Let $s = a^k b_1^{s_1} b_2^{s_2} \cdots b_n^{s_n}$. Thus $sg = 0$, since $\langle C_g \rangle_r = \langle b_1, \ldots, b_n \rangle_r$. □

Now, we determine the structure of the set of all zero-divisor elements of $R_0[[x;\alpha]]$, where R is α-rigid.

Proposition 2.14. *Let R be an α-rigid and right Noetherian ring. Then*

$$Z\big(R_0[[x;\alpha]]\big) = \big\{f \in R_0[[x;\alpha]] \,|\, rf = 0 \text{ for some non-zero } r \in R\big\}.$$

Proof. We have $Z\big(R_0[[x;\alpha]]\big) \subseteq \big\{f \in R_0[[x;\alpha]] \,|\, rf = 0 \text{ for some non-zero } r \in R\big\}$, by Lemma 2.3 and Theorem 2.13. Now, suppose that $f \in R_0[[x;\alpha]]$ and $rf = 0$ for some non-zero $r \in R$. Thus $f \circ rx = 0$, and so $f \in Z\big(R_0[[x;\alpha]]\big)$. This completes the proof. □

Lemma 2.15. *Let R be a symmetric, α-compatible and right Noetherian ring. Then $Z_\ell\big(R_0[[x;\alpha]]\big) = \big\{f \in R_0[[x;\alpha]] \,|\, rf = 0$ for some non-zero $r \in R\big\}$, when R is not reduced. In particular, $Z_\ell\big(R_0[[x;\alpha]]\big) \subseteq Z_r\big(R_0[[x;\alpha]]\big)$.*

Proof. Let $0 \neq f \in R_0[[x;\alpha]]$. Notice that if $rf = 0$ for some $0 \neq r \in R$, then $f \circ rx = 0$, and so $\big\{f \in R_0[[x;\alpha]] \,|\, rf = 0$ for some non-zero $r \in R\big\} \subseteq Z_\ell\big(R_0[[x;\alpha]]\big)$. Hence $Z_\ell\big(R_0[[x;\alpha]]\big) = \big\{f \in R_0[[x;\alpha]] \,|\, rf = 0$ for some non-zero $r \in R\big\}$, by Theorem 2.13.

For proving the last statement, suppose that $f = \sum_{i=1}^{\infty} a_i x^i \in Z_\ell\big(R_0[[x;\alpha]]\big)$. Then $rf = 0$ for some $0 \neq r \in R$, and thus $ra_i = 0$ for each i, which implies that $rx \circ f = 0$. Hence $f \in Z_r\big(R_0[[x;\alpha]]\big)$, as wanted. □

Next, we want to characterize the zero-divisor elements of the near-ring $R_0[[x;\alpha]]$, where R is not reduced.

Theorem 2.16. *Let R be a symmetric, α-compatible and right Noetherian ring which is not reduced. Then $Z\big(R_0[[x;\alpha]]\big) = Z_\ell\big(R_0[[x;\alpha]]\big) \cup B$, where $B = \big\{\sum_{i=1}^{\infty} a_i x^i \,|\, ann_R(a_1) \cap Nil(R) \neq 0$ and $a_i \in R$ for each $i \geq 2\big\}$.*

Proof. Let $f = \sum_{i=1}^{\infty} a_i x^i$ be a non-zero element of $R_0[[x;\alpha]]$. If $ann_R(a_1) \cap Nil(R) \neq 0$, then $ba_1 = 0$ for some $0 \neq b \in Nil(R)$. Hence there exists a positive integer t such that $b^t = 0$ but $b^{t-1} \neq 0$. Therefore $b^{t-1}x \circ f = \sum_{i=1}^{\infty} a_i (b^{t-1}x)^i = 0$, by Lemma 2.1, which implies that $f \in Z\big(R_0[[x;\alpha]]\big)$.

Now assume that $f \in Z\big(R_0[[x;\alpha]]\big)$. Then $g \circ f = 0$ for some non-zero $g = \sum_{j=1}^{\infty} b_j x^j \in R_0[[x;\alpha]]$, since $Z\big(R_0[[x;\alpha]]\big) = Z_r\big(R_0[[x;\alpha]]\big)$, by Lemma 2.15. If g is nilpotent, then $b_i \in Nil(R)$ for each i, by Corollary 2.11. Suppose that s is the smallest integer such that $b_s \neq 0$. Then $b_s a_1 = 0$, which implies that $ann_R(a_1) \cap Nil(R) \neq 0$. On the other hand, if g is not nilpotent, then $rf = 0$ for some non-zero $r \in R$, by Theorem 2.13. This shows that $f \in Z_\ell\big(R_0[[x;\alpha]]\big)$, by Lemma 2.15. □

3. The diameter of the zero-divisor graph $\Gamma(R_0[[x;\alpha]])$

According to [9, Theorem 2.2], we have $diam\big(\Gamma(N)\big) \leq 3$, for every zero-symmetric near-ring N. Since $R_0[[x;\alpha]]$ is a zero-symmetric near-ring with respect to "$\circ$", then $diam\big(\Gamma(R_0[[x;\alpha]])\big) \leq 3$. In the following theorem, we determine the lower bound of $diam\big(\Gamma(R_0[[x;\alpha]])\big)$, where R is a symmetric and α-compatible ring.

Theorem 3.1. *Let R be a symmetric and α-compatible ring with $Z(R) \neq 0$. Then $diam\big(\Gamma(R_0[[x;\alpha]])\big) \geq 2$.*

Proof. First suppose that R is a reduced ring and $0 \neq a \in Z(R)$. Then $ax, ax^2 \in Z\big(R_0[[x;\alpha]]\big)$. Since $ax \circ ax^2 \neq 0 \neq ax^2 \circ ax$, it follows that $d(ax, ax^2) \geq 2$. Now, assume that R is not reduced. It means that there exists $0 \neq c \in R$ such that $c^2 = 0$. Thus $cx \circ x^2 = c\alpha(c)x^2 = 0$ and $cx \circ x^3 = c\alpha(c)\alpha^2(c)x^3 = 0$, by Lemma 2.1, which implies that $x^2, x^3 \in$

$Z\big(R_0[[x;\alpha]]\big)$. Since $x^2 \circ x^3 \neq 0 \neq x^3 \circ x^2$, then $d(x^2, x^3) \geq 2$. Therefore $diam\big(\Gamma(R_0[[x;\alpha]])\big) \geq 2$. □

Proposition 3.2. *Let R be a symmetric, α-compatible and right Noetherian ring which is not reduced. If $diam\big(\Gamma(R_0[[x;\alpha]])\big) = 2$, then*

$$Z\big(R_0[[x;\alpha]]\big) = Z(R)x + R_0[[x;\alpha]]x.$$

Proof. Let $0 \neq f = \sum_{i=1}^{\infty} a_i x^i \in R_0[[x;\alpha]]$. If $f \in Z\big(R_0[[x;\alpha]]\big)$, then $a_1 \in Z(R)$, by Lemma 2.8. Hence $Z\big(R_0[[x;\alpha]]\big) \subseteq Z(R)x + R_0[[x;\alpha]]x$. For the reverse inclusion, if $a_1 \in Nil(R)$, then we are done, by Theorem 2.16. Hence suppose that $a_1 \in Z(R) \setminus Nil(R)$. Since R is not reduced, it follows that a_1x, x^2 are zero-divisor elements of $R_0[[x;\alpha]]$ with $a_1x \circ x^2 \neq 0 \neq x^2 \circ a_1x$. Since $diam\big(\Gamma(R_0[[x;\alpha]])\big) = 2$, there exists a non-zero nilpotent element $g = \sum_{j=1}^{\infty} b_j x^j$ such that $a_1x - g - x^2$ is a path. Thus $b_j \in Nil(R)$ for each j, by Corollary 2.11. Let s be the smallest integer such that $b_s \neq 0$. If $g \circ (a_1x) = 0$, then $a_1b_s = 0$. It means that $ann_R(a_1) \cap Nil(R) \neq 0$, and so $f \in Z\big(R_0[[x;\alpha]]\big)$, by Theorem 2.16. Now assume that $(a_1x) \circ g = 0$. Thus $\sum_{j=s}^{\infty} b_j(a_1x)^j = 0$, which implies that $b_s a_1 \alpha(a_1)\alpha^2(a_1)\cdots\alpha^{s-1}(a_1) = 0$, since it is the coefficient of x^s in this equation. Hence $b_s a_1^s = 0$, by Lemma 2.1. If $b_s a_1 = 0$, then we are done. Now suppose that $b_s a_1 \neq 0$. Then there exists a positive integer $1 \leq k \leq s-1$ such that $b_s a_1^k \neq 0$ but $(b_s a_1^k)a_1 = 0$. Therefore $b_s a_1^k \in ann_R(a_1) \cap Nil(R)$, and so the result follows from Theorem 2.16. □

Theorem 3.3. *Let R be a symmetric, α-compatible ring which is not reduced. Then we have the following*:

(1) *If $ann_R\big(\{a,b\}\big) \cap Nil(R) \neq 0$ for each $a, b \in Z(R)$, then*

$$diam\big(\Gamma(R_0[[x;\alpha]])\big) = 2.$$

(2) *If R is right Noetherian and $diam\big(\Gamma(R_0[[x;\alpha]])\big) = 2$, then for each $a, b \in Z(R)$, $ann_R\big(\{a,b\}\big) \cap Nil(R) \neq 0$.*

Proof. (1) Assume that $f = \sum_{i=1}^{\infty} a_i x^i$ and $g = \sum_{j=1}^{\infty} b_j x^j$ are non-zero zero-divisor elements of $R_0[[x;\alpha]]$. By Lemma 2.8, $a_1, b_1 \in Z(R)$, which implies that there exists $c \in Nil(R)$ such that $ca_1 = 0 = cb_1$. Hence there exists a positive integer k such that $c^k = 0$ but $c^{k-1} \neq 0$. It follows that $f - c^{k-1}x - g$ is a path, by Lemma 2.1. This shows that $d(f, g) \leq 2$, and so the result follows from Theorem 3.1.

(2) Let $a, b \in Z(R)$. Then by Proposition 3.2,

$$\{ax + x^2, bx + x^2\} \in Z\big(R_0[[x;\alpha]]\big).$$

Since $diam\big(\Gamma(R_0[[x;\alpha]])\big) = 2$, there exists a non-zero nilpotent f such that $f \circ (ax + x^2) = 0$ and $f \circ (bx + x^2) = 0$, by Theorem 2.13. Let $f = \sum_{i=k}^{\infty} c_i x^i$ and $c_k \neq 0$. Thus $ac_k = 0 = bc_k$. Also, by Corollary 2.11, we have $c_k \in Nil(R)$. Hence $ann_R\big(\{a,b\}\big) \cap Nil(R) \neq 0$. □

In [14, Theorems 2.21 and 2.23], the authors characterized the diameter of the zero-divisor graph $\Gamma\big(R[[x;\alpha]]\big)$, where R is a reversible and α-compatible ring with $Z(R) \neq 0$. They proved that $diam\big(\Gamma(R[[x;\alpha]])\big) = 1$ if and only if R is a non-reduced ring with $Z(R)^2 = 0$. Also, if R is right Noetherian, then (1) $diam\big(\Gamma(R[[x;\alpha]])\big) = 2$ if and only if $|Z(R)| > 3$ and either (i) R is a reduced ring with exactly two minimal primes, or (ii) $Z(R)$ is an ideal of R with $Z(R)^2 \neq 0$. (2) $diam\big(\Gamma(R[[x;\alpha]])\big) = 3$ if and only if R is not a reduced ring with exactly two minimal primes and $Z(R)$ is not an ideal of R.

Applying Lemma 2.2, one can get the next interesting result.

Proposition 3.4. *Let R be an α-rigid ring. Then*

(1) $diam\big(\Gamma(R[[x;\alpha]])\big) = 2$ *if and only if* $diam\big(\Gamma(R_0[[x;\alpha]])\big) = 2$.

(2) $diam\big(\Gamma(R[[x;\alpha]])\big) = 3$ *if and only if* $diam\big(\Gamma(R_0[[x;\alpha]])\big) = 3$.

The next example shows that the assumption "R is α-rigid" in Proposition 3.4 is not superfluous.

Example 3.5. (1) Let p be a prime integer number and $S = \mathbb{Z}(+)\mathbb{Z}(p^\infty)$ be the idealization of $\mathbb{Z}(p^\infty)$. Clearly, S is neither reduced nor Noetherian. Consider $\alpha : S \to S$ by $\alpha(n, \overline{m}) = (n, -\overline{m})$ for each $n \in \mathbb{Z}$ and $\overline{m} \in \mathbb{Z}(p^\infty)$. Clearly, S is an α-compatible ring. Let $g = (0, (\overline{1/p})) + (0, (\overline{1/p^2}))x + (0, (\overline{1/p^3}))x^2 + \cdots$ and $f = (p, 0) + (1, 0)x$. Then $fg = 0$, and so $f \in Z\big(S[[x;\alpha]]\big)$. Now, let $h = (p, 0)$. Obviously, $h \in Z\big(S[[x;\alpha]]\big)$ but $hf \neq 0 \neq fh$. Notice that $ann_{S[[x;\alpha]]}(h) = \big\{ \sum_{i=0}^{\infty} (0, a_i)x^i \,|\, a_i \in \{0, (\overline{1/p})\} \text{ for each } i \geq 0 \big\}$. This shows that f and h have no common non-zero annihilator, and hence $diam\big(\Gamma(S[[x;\alpha]])\big) = 3$, by [22, Theorem 3.2]. On the other hand, we have $Z(S) = p\mathbb{Z}(+)\mathbb{Z}(p^\infty)$, by [18, Example 5.6]. Since $(0, (\overline{1/p}))Z(S) = 0$ and $(0, (\overline{1/p})) \in Nil(S)$, then $diam\big(\Gamma(S_0[[x;\alpha]])\big) = 2$, by Theorem 3.3.

(2) Let K be a field and $D = K[w, y, z]_M$, where w, y and z are algebraically independent indeterminates and $M = \langle w, y, z\rangle K[w, y, z]$. Clearly, D is a domain. Let $\mathcal{P}$ denote the height two primes of D and Q be the maximal ideal of D. Also, let $B = \sum F_\gamma$ where $F_\gamma = qf(D/P_\gamma)$ for each $P_\gamma \in \mathcal{P}$. Let $R = D(+)B$ be the idealization of B over D. Clearly, R is neither reduced nor Noetherian. Lucas [18, Example 5.2] showed that $diam\big(\Gamma(R[[x]])\big) = 3$ and R is a local ring with maximal ideal $Q(+)B = Z(R)$. He also proved that each two elements of $Z(R)$ has a non-zero nilpotent annihilator. This shows that $diam\big(\Gamma(R_0[[x]])\big) = 2$, by Theorem 3.3.

(3) Let R be a commutative non-reduced Noetherian ring with $Z(R)^2 = 0$. Then $diam\big(\Gamma(R[[x]])\big) = 1$, by [6, Theorem 3]. But $diam\big(\Gamma(R_0[[x]])\big) = 2$, by Theorem 3.3.

The following interesting result gives conditions under which $Z\big(R_0[[x;\alpha]]\big)$ forms an ideal of $R_0[[x;\alpha]]$.

Proposition 3.6. *Let R be an α-rigid and right Noetherian ring. Then $Z\big(R_0[[x;\alpha]]\big)$ is an ideal of $R_0[[x;\alpha]]$ if and only if $Z(R)$ is an ideal of R.*

Proof. First suppose that $Z\big(R_0[[x;\alpha]]\big)$ is an ideal of $R_0[[x;\alpha]]$ and $a, b \in Z(R)$. Then $ax, bx \in Z\big(R_0[[x;\alpha]]\big)$, and so $(a+b)x = ax + bx \in Z\big(R_0[[x;\alpha]]\big)$. By Proposition 2.14, there exists $0 \neq r \in R$ such that $r(a+b)x = 0$, which implies that $a + b \in Z(R)$, and thus $Z(R)$ is an ideal of R.

Conversely, let $f = \sum_{i=1}^{\infty} a_i x^i$ and $g = \sum_{j=1}^{\infty} b_j x^j$ be elements of $Z\big(R_0[[x;\alpha]]\big)$. By Proposition 2.14, there exist non-zero $r, s \in R$ such that $rf = 0 = sg$, which implies that $a_i, b_j \in Z(R)$ for each i, j. Let $\beta = \{a_i + b_i \mid a_i \in C_f \text{ and } b_i \in C_g \text{ for each } i \geq 1\}$. Since R is right Noetherian, then there exists a positive integer n such that $\beta R = \langle a_1 + b_1, \ldots, a_n + b_n\rangle_r$. Since $Z(R)$ is an ideal, then $\langle a_1 + b_1, \ldots, a_n + b_n\rangle \subseteq Z(R)$. Also, by Lemma 2.4, R has left Property (A), and thus $t\langle a_1 + b_1, \ldots, a_n + b_n\rangle = 0$ for some $0 \neq t \in R$. Thus $t(a_i + b_i) = 0$ for each $1 \leq i \leq n$, which implies that $t(f + g) = 0$, and so $f + g \in Z\big(R_0[[x;\alpha]]\big)$, by Proposition 2.14.

Let $f = \sum_{i=1}^{\infty} a_i x^i$ and $g = \sum_{j=1}^{\infty} b_j x^j$ be elements of $R_0[[x;\alpha]]$ and $z = \sum_{k=1}^{\infty} c_k x^k \in Z\big(R_0[[x;\alpha]]\big)$. Note that $f \circ z = \sum_{k=1}^{\infty} c_k f^k$ and

$$\begin{aligned}
&(z+f)\circ g - f\circ g\\
&= \sum_{j=1}^{\infty} b_j (z+f)^j - \sum_{j=1}^{\infty} b_j f^j\\
&= b_1c_1x + [b_1c_2 + b_2c_1\alpha(c_1) + b_2c_1\alpha(a_1) + b_2a_1\alpha(c_1)]x^2 + \cdots .
\end{aligned}$$

Since $c_k \in Z(R)$ for each $k \geq 1$ and $Z(R)$ is an ideal of R, then $(z+f)\circ g - f\circ g$ and $f \circ z \in Z\big(R_0[[x;\alpha]]\big)$, by Corollary 2.7. Hence $Z\big(R_0[[x;\alpha]]\big)$ is an ideal of $R_0[[x;\alpha]]$. $\square$

The next example shows that the condition "R is right Noetherian" in Proposition 3.6 can not be dropped.

Example 3.7. Let R be the commutative ring introduced in [18, Example 5.3] and α be the identity endomorphism on R. Thus R is an α-rigid ring which is not Noetherian. Lucas proved that $Z(R)$ is an ideal of R and there exist a countably generated ideal $A = \langle a_1, a_2, \ldots\rangle$ and an element $b \in R$ such that the ideal $A + bR$ is a countably generated ideal contained in $Z(R)$ that has no non-zero annihilator, but both A and bR have non-zero annihilators. Consider $f = a_1x^2 + a_2x^3 + \cdots$ and $g = bx$. Thus $f, g \in Z\big(R_0[[x;\alpha]]\big)$. If $f + g \in Z\big(R_0[[x;\alpha]]\big)$, then $h \circ (f+g) = 0$ for some $0 \neq h = \sum_{j=1}^{\infty} c_j x^j \in R_0[[x;\alpha]]$. Let k be the smallest integer such that $c_k \neq 0$. Thus $c_k a_i = 0 = c_k b$ for each $i \geq 1$, by Lemma 2.2. It means that $0 \neq c_k \in ann_R(A + bR)$, which is a contradiction. This shows that $Z\big(R_0[[x;\alpha]]\big)$ is not an ideal of $R_0[[x;\alpha]]$.

Proposition 3.8. *Let R be a symmetric and α-compatible ring which is not reduced. Then we have the following*:

(1) *If $ann_R(a-b) \cap Nil(R) \neq 0$ for each $a, b \in Z(R)$, then $Z\big(R_0[[x;\alpha]]\big)$ forms an ideal of $R_0[[x;\alpha]]$.*

(2) *If R is right Noetherian ring and $Z\big(R_0[[x;\alpha]]\big)$ forms an ideal of $R_0[[x;\alpha]]$, then $ann_R(a-b)\cap Nil(R)\neq 0$ for each $a,b\in Z(R)$.*

Proof. (1) Let $f=\sum_{i=1}^{\infty} a_i x^i$ and $g=\sum_{j=1}^{\infty} b_j x^j$ be non-zero elements of $Z\big(R_0[[x;\alpha]]\big)$. Then $a_1,b_1\in Z(R)$, by Lemma 2.8. Also, by hypothesis, we have $c(a_1-b_1)=0$ for some $0\neq c\in Nil(R)$. Hence there exists a positive integer k such that $c^k=0$ but $c^{k-1}\neq 0$. Thus $c^{k-1}x\circ(f-g)=0$, by Lemma 2.1. Therefore $f-g\in Z\big(R_0[[x;\alpha]]\big)$. Now, assume that $h=\sum_{i=1}^{\infty} c_i x^i$ and $k=\sum_{j=1}^{\infty} d_j x^j\in R_0[[x;\alpha]]$. Hence c_1a_1 and d_1a_1 are the coefficients of x respectively in $h\circ f$ and $(f+h)\circ k-h\circ k$. Since $ann_R(a_1)\cap Nil(R)\neq 0$, then by a similar argument as used above, we have $h\circ f$ and $(f+h)\circ k-h\circ k\in Z\big(R_0[[x;\alpha]]\big)$. Therefore $Z\big(R_0[[x;\alpha]]\big)$ forms an ideal of $R_0[[x;\alpha]]$.

(2) Suppose that $a,b\in Z(R)$. Then $ax,bx,x^2\in Z\big(R_0[[x;\alpha]]\big)$, which implies that $(a-b)x+x^2\in Z\big(R_0[[x;\alpha]]\big)$, since $Z\big(R_0[[x;\alpha]]\big)$ is an ideal of $R_0[[x;\alpha]]$. By Theorem 2.13, there exists a nilpotent element $f=\sum_{i=1}^{\infty} c_i x^i$ such that $f\circ(a-b)x+x^2=0$. Thus $c_i\in Nil(R)$ for each $i\geq 1$, by Corollary 2.11. Let k be the smallest integer such that $c_k\neq 0$. Then $(a-b)c_k=0$, and so $ann_R(a-b)\cap Nil(R)\neq 0$. □

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