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SOME FREDHOLM THEORY RESULTS AROUND RELATIVE DEMICOMPACTNESS CONCEPT

Wajdi Chaker, Aref Jeribi, and Bilel Krichen

Abstract. In this paper, we provide a characterization of upper semi-Fredholm operators via the relative demicompactness concept. The obtained results are used to investigate the stability of various essential spectra of closed linear operators under perturbations belonging to classes involving demicompact, as well as, relative demicompact operators.

1. Introduction and preliminaries

In 1966, W. V. Petryshyn [15] introduced the concept of demicompactness for nonlinear operators acting on Hilbert spaces in order to study an iterative method for a construction of fixed points. This notion was also used by the same author to investigate the structure of fixed point sets for nonlinear operators defined on a closed subset of a Banach space. This definition asserts that a nonlinear operator $T : \mathcal{D}(T) \subset X \longrightarrow X$, where X is a Banach space is demicompact if every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $\{x_n - Tx_n\}$ converges in X, have a convergent subsequence. The demicompactness of an operator is not as restrictive as it seems, for instance each of the following conditions imply that T is demicompact, (i) T is compact. (ii) The range of $I-T$ is closed, the inverse $(I-T)^{-1}$ exists and is continuous, where I denotes the identity operator of X. For more informations, see [15, 16].

In the theory of Fredholm operator, W. V. Petryshyn [16] and Y. Akashi [1] used the class of demicompact, 1-set contraction linear operators to obtain some results of Fredholm perturbations. Recently, W. Chaker, A. Jeribi and B. Krichen [3] continued this study in order to investigate the essential spectra of closed linear operators. In 2014, B. Krichen [11], introduced the relative demicompactness class with respect to a given closed linear operator as a generalization of the demicompactness notion. This definition asserts that if X is a Banach space, $T : \mathcal{D}(T) \subset X \longrightarrow X$, and $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ are two linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$, then T is said to be S_0-demicompact (or relative demicompact with respect to S_0), if every bounded sequence $\{x_n\}$

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in $\mathcal{D}(T)$ such that $\{S_0x_n - Tx_n\}$ converges in X, have a convergent subsequence. It was shown in [11] that when $\mathcal{D}(T)$ lies in a finite dimensional subspace of X, the condition of relative demicompactness is automatically satisfied. As examples of S_0-demicompact operators, we cite operators T such that $(S_0 - T)^{-1}$ exists and is continuous on the range of $S_0 - T$. Recently, B. Krichen and D. O'Regan [12] discussed some topological properties of the set $\mathcal{F}(S_0, T, z) := \{x \in X : S_0x \in Tx + z\}$, where T is a nonlinear multi-valued mapping and S_0 is a single-valued mapping acting on a Banach space X. Their study was based on a new concept, the so called weakly relative demicompactness for nonlinear operators.

Now let us recall some standard definitions and results from Fredholm theory (see [7]). Let X and Y be two Banach spaces. By an operator T from X into Y, we mean a linear operator with domain $\mathcal{D}(T) \subset X$ and range $\mathcal{R}(T) \subset Y$. By $\mathcal{C}(X, Y)$ we denote the set of all closed linear operators from X into Y, by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators from X into Y, and by $\mathcal{K}(X, Y)$ the subspace of all compact operators of $\mathcal{L}(X, Y)$. The sets $\mathcal{C}(X, X)$, $\mathcal{L}(X, X)$ and $\mathcal{K}(X, X)$ are simply denoted by $\mathcal{C}(X)$, $\mathcal{L}(X)$ and $\mathcal{K}(X)$, respectively. Given an operator $T \in \mathcal{C}(X)$, then $\rho(T)$ denotes the resolvent set of T, and $\sigma(T) := \mathbb{C} \setminus \rho(T)$ the spectrum of T. If $T \in \mathcal{C}(X, Y)$, then we denotes by $\alpha(T)$ the dimension of the kernel $\mathcal{N}(T)$, and by $\beta(T)$ the codimension of $\mathcal{R}(T)$ in Y. The classes of upper semi-Fredholm and lower semi-Fredholm from X into Y are respectively defined by:

$$\Phi_+(X, Y) := \{T \in \mathcal{C}(X, Y) : \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\} \text{ and}$$
$$\Phi_-(X, Y) := \{T \in \mathcal{C}(X, Y) : \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed in } Y\}.$$

$\Phi(X, Y) := \Phi_+(X, Y) \cap \Phi_-(X, Y)$ is the set of Fredholm operators from X into Y, and $\Phi_\pm(X, Y) := \Phi_+(X, Y) \cup \Phi_-(X, Y)$ is the set of semi-Fredholm operators from X into Y. If $X = Y$, the sets $\Phi(X, Y)$, $\Phi_+(X, Y)$, $\Phi_-(X, Y)$ and $\Phi_\pm(X, Y)$ are replaced by $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_\pm(X)$, respectively. The index of an operator $T \in \Phi_\pm(X)$ is $i(T) := \alpha(T) - \beta(T)$.

Let $T \in \mathcal{C}(X, Y)$, T is said to have a left (resp. right) inverse modulo compact operators if there are $T_l \in \mathcal{L}(Y, X)$ (resp. $T_r \in \mathcal{L}(Y, X)$) and $K \in \mathcal{K}(X)$ (resp. $K \in \mathcal{K}(Y)$), such that $T_lT = I - K$ on $\mathcal{D}(T)$ (resp. $TT_r = I - K$ on Y). These operators T_l and T_r are called left and right inverse modulo compact operators of T, respectively. A bounded linear operator which is both a left and right inverse modulo compact operator of T is said an inverse modulo compact operator of T. It is well known that T belongs to $\Phi_+(X, Y)$, $\Phi_-(X, Y)$, and $\Phi(X, Y)$ if it possesses a left, right, and two-sided inverse modulo compact operator, respectively.

For $T \in \mathcal{C}(X)$, we can define the iterates of T by: $T^0 = I$ and for $n \geq 1$

$$\begin{cases} \mathcal{D}(T^n) := \left\{x : x, Tx, \ldots, T^{n-1}x \in \mathcal{D}(T)\right\}, \\ T^nx = T(T^{n-1}x) \quad \text{for every } x \in \mathcal{D}(T^n). \end{cases}$$

Definition ([10])**.** Let X and Y be two Banach spaces and let S and T be two linear operators from X into Y such that $\mathcal{D}(T) \subset \mathcal{D}(S)$. Then S is called relatively bounded with respect to T (or T-bounded) if there exist two positives constants a_S and b_S such that

$$\|Sx\| \leq a_S\|x\| + b_S\|Tx\|,\ x \in \mathcal{D}(T). \tag{1}$$

The infimum, denoted δ, of all b_S which satisfy (1) for some a_S is called the T-bound.

It follows from the closedness of an operator $T \in \mathcal{C}(X)$ that $\mathcal{D}(T)$ endowed with the graph norm $\| \ \|_T$, that is, $\| x \|_T := \| x \| + \| Tx \|$, and denoted by X_T, is a Banach space. In this space the operator T satisfies $\|Tx\| \leq \|x\|_T$, which implies that T is a bounded operator from X_T into X. Given a linear operator J, if $\mathcal{D}(T) \subset \mathcal{D}(J)$, then J is called T-defined and we denote by $\hat{J}$ its restriction to X_T. Moreover, if $\hat{J} \in \mathcal{L}(X_T, X)$, we see that J is T-bounded. One easily can checks that if J is closed (or closable), then J is T-bounded (see [10, Remark 1.5, p. 191]). Furthermore, we have the obvious relations:

$$\begin{cases} \alpha(\hat{T}) = \alpha(T),\ \beta(\hat{J}) = \beta(J),\ \mathcal{R}(\hat{J}) = \mathcal{R}(J), \\ \alpha(\hat{T} + \hat{J}) = \alpha(T + J),\ \beta(\hat{T} + \hat{J}) = \beta(T + J), \\ \text{and } \mathcal{R}(\hat{T} + \hat{J}) = \mathcal{R}(T + J). \end{cases}$$

Hence, $T \in \Phi(X)$ (resp. $\Phi_+(X)$) if, and only if, $\hat{T} \in \Phi(X_T, X)$ (resp. $\Phi_+(X_T, X)$).

Having further applications in various physical domains, as magnetohydrodynamics or transport operators [6, 8], the spectral studies of the operator pencils $\lambda S - T$ attract an increasing importance for many mathematicians, in particular the invariance of the relative essential spectrum [6, 8] under some classes of perturbations. We recall that when X and Y are two Banach spaces, $S \in \mathcal{L}(X, Y)$ such that $S \neq 0$, and $T \in \mathcal{C}(X, Y)$, then the S-resolvent set of T and the S-spectrum of T are defined, respectively, by:

$$\rho_S(T) := \{\lambda \in \mathbb{C} : \lambda S - T \text{ has a bounded inverse}\} \text{ and}$$
$$\sigma_S(T) := \mathbb{C} \setminus \rho_S(T).$$

Furthermore, as the notions of essential spectra (see for instance [8]), the notion of S-essential spectrum was introduced in [6] as a generalization of the usual notion of Wolf essential spectrum [18]. Recently, some investigations of this spectrum involving the class of Fredholm perturbations were established in [8]. In this work, we are concerned with the following S-essential spectra:

$$\sigma_{e_G,S}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda S - T \notin \Phi_+(X, Y)\} := \mathbb{C} \setminus \Phi^+_{T,S},$$
$$\sigma_{e_W,S}(T) := \{\lambda \in \mathbb{C} \ : \ \lambda S - T \notin \Phi(X, Y)\} := \mathbb{C} \setminus \Phi_{T,S}.$$

The paper is organized in the following way. In Section 2, we prove that an upper semi-Fredholm operator can be characterized by means of demicompactness concept. Furthermore, some sufficient conditions leading to the demicompactness and relative demicompactness are given. Section 3 is devoted to investigate the stability of various relative essential spectra of closed linear operators under perturbations belonging to classes involving relative demicompact operators. In Section 4, we give an example in transport theory to illustrate the results of the previous section.

2. Characterization of upper semi-Fredholm operators

We begin this section by introducing the following definition which generalizes the relative demicompactness concept introduced in [11].

Definition. Let $(Y, \|\cdot\|_Y)$ be a Banach space and let X be a subspace of Y endowed with a norm $\|\cdot\|_X$ such that $(X, \|\cdot\|_X)$ is a Banach space. Let $T : \mathcal{D}(T) \subset X \longrightarrow Y$ and $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow Y$ be two linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$. T is called S_0-demicompact (or relative demicompact with respect to S_0), if every bounded sequence $\{x_n\}$ in $\mathcal{D}(T)$ such that $\{S_0x_n - Tx_n\}$ converges in $(Y, \|\cdot\|_Y)$ have a convergent subsequence in $(X, \|\cdot\|_X)$.

We will denote by $\mathcal{DC}_{S_0}(X, Y)$ the class of S_0-demicompact linear operators from X into Y, and $\mathcal{DC}_{S_0}(X)$ if $(X, \|\cdot\|_X) = (Y, \|\cdot\|_Y)$. When $X = Y$ and S_0 is the identity operator of X, we find the usual demicompactness concept. In this case, the class $\mathcal{DC}_{S_0}(X)$ is simply denoted by $\mathcal{DC}(X)$.

Remark 2.1. (i) We should mention that if S_0 is invertible and T is bounded such that $S_0^{-1}T$ is compact or nilpotent, i.e., there exists $n \in \mathbb{N} \setminus \{0\}$ such that $(S_0^{-1}T)^n = 0$, then T is an S_0-demicompact operator.

(ii) Let T be a bounded linear operator and let $p \in \mathbb{N} \setminus \{0\}$. Obviously, if T^p is demicompact, then T is demicompact.

(iii) The converse of (ii) is false. Indeed, take X an infinite dimensional Banach space and T a bounded demicompact linear operator such that $T^2 = I$. Clearly, T^2 is not demicompact.

Proposition 2.2. *Let X be a Banach space and let $T : \mathcal{D}(T) \subset X \longrightarrow X$ be a closed linear operator. If $S_0 : X \longrightarrow X$ is a bounded linear operator, then*

$$T \in \mathcal{DC}_{S_0}(X) \quad \textit{if and only if} \quad \widehat{T} \in \mathcal{DC}_{S_0}(X_T, X).$$

Proof. Let $T \in \mathcal{DC}_{S_0}(X)$ and let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that $S_0x_n - Tx_n \to y$, in $(X, \|\cdot\|)$. Since T is S_0-demicompact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in X$ such that $\|x_{n_k} - x\| \to 0$. It follows that $\|Tx_{n_k} - S_0x + y\| \to 0$. Taking into account that T is closed, we deduce that $x \in \mathcal{D}(T)$ and $y = S_0x - Tx$. Thus, $\|Tx_{n_k} - Tx\| \to 0$. Hence, $\|x_{n_k} - x\|_T \to 0$ and consequently, $\widehat{T}$ is an S_0-demicompact operator from X_T into X. Conversely, let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that $S_0x_n - Tx_n \to y$ in $(X, \|\,.\,\|)$. Since $\widehat{T} \in \mathcal{DC}_{S_0}(X_T, X)$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in X$

such that $\|x_{n_k} - x\|_T \to 0$. The S_0-demicompactness of T follows immediately from the inequality $\|x_{n_k} - x\| \leq \|x_{n_k} - x\|_T$. □

Proposition 2.3. *Let X be a Banach space, and let $T : \mathcal{D}(T) \subset X \longrightarrow X$ and $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be two closed linear operators such that $\mathcal{D}(T) \subset \mathcal{D}(S_0)$. If $S_0 - T$ is a closed operator having a left inverse modulo $\mathcal{DC}(X)$, then T is S_0-demicompact.*

Proof. Let $T : \mathcal{D}(T) \subset X \longrightarrow X$ and $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be two closed linear operators such that $\mathcal{D}(T) \subset \mathcal{D}(S_0)$. Since $S_0 - T$ is a closed operator which has a left inverse modulo $\mathcal{DC}(X)$, there exist $K \in \mathcal{DC}(X)$ and $T_l \in \mathcal{L}(X)$ such that

$$T_l(S_0 - T) = I - K \quad \text{on} \quad \mathcal{D}(T).$$

Let $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T)$ such that $(S_0 - T)x_n \to x \in X$, it follows that $x_n - Kx_n \to T_l x \in X$. Taking into account that K is demicompact, we deduce that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges in X. □

It was shown in [11, Theorem 2.3] that if $T : \mathcal{D}(T) \subset X \longrightarrow X$ and $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ are two densely defined closed linear operators with $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ such that $S_0 - T$ is closed, and T is S_0-demicompact, then $S_0 - T$ is an upper semi-Fredholm operator. In particular, its range is a closed subset of X.

Next, and for any closed linear operator $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$, we will use the following notation:

$$\mathcal{C}_{S_0}(X) := \{T \in \mathcal{C}(X) : \mathcal{D}(T) \subset \mathcal{D}(S_0) \text{ and } S_0 - T \text{ is closed}\}.$$

The following Lemma provides another proof for the closedness of $\mathcal{R}(S_0 - T)$.

Lemma 2.4. *Let $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be a closed linear operator and $T \in \mathcal{C}_{S_0}(X)$. If T is S_0-demicompact, then $\mathcal{R}(S_0 - T)$ is a closed subset of X.*

Proof. By using [4, Theorem 5, p. 489], it is sufficient to show that $S_0 - T$ takes closed bounded sets of X into closed set of X. For this purpose, let $\{x_n\}$ be a sequence of a closed bounded set D such that $y_n = (S_0 - T)x_n \to y \in X$. Since T is S_0-demicompact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to $x \in X$. As D is closed, it follows that $x \in D$. Taking into account that $S_0 - T$ is a closed operator, we deduce that $y = (S_0 - T)x$ and consequently, $(S_0 - T)(D)$ is closed. □

Lemma 2.5. *Let $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be a closed linear operator and let $T \in \mathcal{C}_{S_0}(X)$. If $\mathcal{N}(S_0 - T)$ is finite dimensional and $\mathcal{R}(S_0 - T)$ is a closed subset of X, then T is S_0-demicompact.*

Proof. Since $\mathcal{N}(S_0 - T)$ is finite dimensional, there exists a closed subspace C of X such that

$$\mathcal{N}(S_0 - T) \oplus C = X.$$

It follows that

$$\mathcal{N}(S_0 - T) \oplus (C \cap \mathcal{D}(T)) = \mathcal{D}(T).$$

Taking into account that $\mathcal{R}(S_0 - T) = \mathcal{R}(\widehat{S_0} - \widehat{T})$ is a closed subset of X, we deduce that the restriction $(\widehat{S_0} - \widehat{T})_{|_{C\cap\mathcal{D}(T)}}$ of $S_0 - T$ on $C \cap \mathcal{D}(T)$ is an isomorphism between the Banach spaces $(C\cap\mathcal{D}(T), \|\cdot\|_T)$ and $(\mathcal{R}(S_0-T), \|\cdot\|)$. Hence, $((\widehat{S_0} - \widehat{T})_{|_{C\cap\mathcal{D}(T)}})^{-1}$ is bounded from $(\mathcal{R}(S_0 - T), \|\cdot\|)$ into $(C \cap \mathcal{D}(T), \|\cdot\|_T)$. Consequently, $\widehat{T}$ is $\widehat{S_0}$-demicompact. Now it suffices to apply Proposition 2.2 to deduce that T is S_0-demicompact. □

Theorem 2.6. *Let $S_0 : \mathcal{D}(S_0) \subset X \longrightarrow X$ be a closed linear operator and let $T \in \mathcal{C}_{S_0}(X)$. Then, T is S_0-demicompact if and only if $S_0 - T$ is upper semi-Fredholm.*

Proof. The proof follows immediately from Lemmas 2.4 and 2.5. □

An important consequence of Theorem 2.6 is the following theorem which shows that upper semi-Fredholm operators can be characterized by the demicompactness concept.

Theorem 2.7. *Let X be a Banach space and let $T : \mathcal{D}(T) \subset X \longrightarrow X$ be a closed linear operator. Then, T is demicompact if and only if $I - T$ is an upper semi-Fredholm operator.*

As an immediate consequence of Theorem 2.7, we have the following corollary.

Corollary 2.8. *Let X be a Banach space and let $T : X \longrightarrow X$ be a bounded linear operator. Then,*

$$T^2 \text{ is demicompact if and only if } T \text{ and } -T \text{ are demicompact.}$$

It was shown in [2], that if P and Q are two bounded linear operators and $\lambda \neq 0$, the operators $\lambda - PQ$ and $\lambda - QP$ have many common basic properties. In particular, we have:

Corollary 2.9. *Let X be a Banach space and let $P : X \longrightarrow X$, and $Q : X \longrightarrow X$ be two bounded linear operators. Then we have*:

$$PQ \in \mathcal{DC}(X) \Longleftrightarrow QP \in \mathcal{DC}(X).$$

Moreover, if $PQP = P^2$ and $QPQ = Q^2$, then

$$P \in \mathcal{DC}(X) \Longleftrightarrow PQ \in \mathcal{DC}(X) \Longleftrightarrow QP \in \mathcal{DC}(X) \Longleftrightarrow Q \in \mathcal{DC}(X).$$

Proof. The proof of the first part is an immediate consequence of both Theorem 2.7 and Theorem 6 in [2]. The second part follows from Theorem 2.7 combined with Theorem 1.2 in [17]. □

Remark 2.10. Let X be a Banach space. It follows from Corollary 2.9 that if P and Q are two idempotent operators, i.e., $P^2 = P$ and $Q^2 = Q$, then

$$PQ \in \mathcal{DC}(X) \Longleftrightarrow PQP \in \mathcal{DC}(X) \Longleftrightarrow QPQ \in \mathcal{DC}(X).$$

Theorem 2.11. *Let X be a Banach space and let $T \in \mathcal{C}(X)$ such that $\mathcal{R}(T) \subset \mathcal{D}(T)$. Then, for all $p \in \mathbb{N} \setminus \{0\}$, we have the following*:

$$I - T \in \mathcal{DC}(X) \text{ implies that } I - T^p \in \mathcal{DC}(X).$$

Proof. We first show by induction that for all $p \in \mathbb{N} \setminus \{0\}$, T^p is a closed operator. The case $p = 1$ is obvious. Assume that T^p is closed. Since $I - T \in \mathcal{DC}(X)$, by using Theorem 2.7, it follows that $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$. Applying Proposition XVII 3.2 in [7], we deduce that $T^{p+1} = TT^p$ is closed. Now, we will again prove by induction that for all $p \in \mathbb{N}\setminus\{0\}$, $I-T^p \in \mathcal{DC}(X)$. The case $p = 1$ follows from the hypothesis. Assume that $I - T^p \in \mathcal{DC}(X)$ and take $\{x_n\}$ be a bounded sequence of $\mathcal{D}(T^{p+1})$ such that $T^{p+1}x_n \to y$, $y \in X$. Put $z_n := T^p x_n$. Firstly, the sequence $\{z_n\}$ is bounded. Indeed, since $\alpha(T) < \infty$, there exists a closed subspace X_0 of X such that

$$X_T = \mathcal{N}(T) \oplus X_0 \cap \mathcal{D}(T),$$

and then the mapping $\widehat{T} : X_0 \cap \mathcal{D}(T) \to \mathcal{R}(T)$ is bijective. As $\mathcal{R}(T)$ is a closed subspace of X, it follows $\widehat{T}^{-1} : (\mathcal{R}(T), \|\cdot\|) \to (X_0 \cap \mathcal{D}(T), \|\cdot\|_T)$ is bounded. Hence

$$\|z_n - \widehat{T}^{-1}(y)\|_T = \|\widehat{T}^{-1}(T^{p+1}x_n) - \widehat{T}^{-1}(y)\|_T \to 0,$$

then $\{z_n\}$ is bounded. Next, since $Tz_n \to y$ and $I - T \in \mathcal{DC}(X)$, there exists a subsequence $\{x_{\varphi(n)}\}$ of $\{x_n\}$ such that $\{T^p x_{\varphi(n)}\}$ converges. Now, the result follows from the demicompactness of $I - T^p$. □

Corollary 2.12. *Let X be a Banach space and let T be a closed linear operator such that $\mathcal{R}(T) \subset \mathcal{D}(T)$. If $T \in \Phi_+(X)$, then for all $n \in \mathbb{N}$, $T^n \in \Phi_+(X)$.*

Proof. The proof follows from Theorems 2.11 and 2.7. □

Now recall that the Kuratowski's measure of noncompactness definition [13] is stated as follows: let X be a Banach space and Ω_X be the collection of all nonempty bounded subsets of X. For any $D \in \Omega_X$, the Kuratowski's measure of noncompactness of D, denoted by $\gamma(D)$, is the infimum of the set of real $\varepsilon > 0$ such that D can be covered by a finite number of sets of diameter less than or equal to ε. The following proposition gives some properties of the Kuratowski's measure of noncompactness which are frequently used.

Proposition 2.13. *Let D, $D' \in \Omega_X$. Then we have the following properties*:
(i) $\gamma(D) = 0$ *if, and only if, D is relatively compact.*
(ii) *If $D \subset D'$, then $\gamma(D) \leq \gamma(D')$.*
(iii) $\gamma(D + D') \leq \gamma(D) + \gamma(D')$.
(iv) *For every $\alpha \in \mathbb{C}$, $\gamma(\alpha D) = |\alpha|\gamma(D)$.*

Clearly, for any subset $D \in \Omega_X$, we have the following:

$\gamma(D) = 0$ if, and only if, $\gamma(\{x_n, n \in \mathbb{N}\}) = 0$ for every sequence $\{x_n\}$ of D.

For $T \in \mathcal{L}(X)$, consider the measure of noncompactness $\bar{\gamma}$ of T with respect to γ as follows:

$$\bar{\gamma}(T) := \sup \left\{ \frac{\gamma(T(D))}{\gamma(D)} : D \in \Omega_X \text{ and } \gamma(D) > 0 \right\}.$$

The following proposition gives some properties of $\bar{\gamma}$ that we will need later.

Proposition 2.14 ([5])**.** *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then we have the following properties*:

(i) $\bar{\gamma}(T) = 0$ *if, and only if, T is compact.*
(ii) *If $S \in \mathcal{L}(X)$, then $\bar{\gamma}(ST) \leq \bar{\gamma}(S)\bar{\gamma}(T)$.*
(iii) *If $K \in \mathcal{K}(X)$, then $\bar{\gamma}(T + K) = \bar{\gamma}(T)$.*
(iv) *If B is a bounded subset of X, then $\gamma(T(B)) \leq \bar{\gamma}(T)\gamma(B)$.*

A sufficient condition which asserts that a closed linear operator T verifies μT is S_0-demicompact for all $\mu \in [0, 1)$ is given in the following proposition.

Proposition 2.15. *Let X be a Banach space and let S_0 and T be two closed linear operators such that $\mathcal{D}(T) \subset \mathcal{D}(S_0)$ and $S_0 - T$ is closed. If $S_0 - T$ has a left inverse T_l modulo compact operators such that $\overline{\gamma}(T_l\widehat{T}) < 1$, then μT is S_0-demicompact operator for all $\mu \in [0, 1)$.*

Proof. Let $\mu \in [0, 1)$ and let $\{x_n\}$ be a sequence of $\mathcal{D}(T)$ such that

$$y_n = (S_0 - \mu T)x_n \to y \in X.$$

Suppose that $\gamma(\{x_n,\, n \in \mathbb{N}\}) \neq 0$. Since T has a left inverse modulo compact operator, denoted by T_l, there exists an operator $K \in \mathcal{K}(X)$ such that

$$T_l(S_0 - T) = I - K \quad \text{on } \mathcal{D}(T).$$

It follows that

$$T_lTx_n + x_n - Kx_n - \mu T_lTx_n \to T_ly.$$

Since K is a compact operator, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{Kx_{n_i}\}$ converges to an element of X. By using Proposition 2.14(iv), combined with the facts that $\widehat{T} = T$ on $\mathcal{D}(T)$ and $\overline{\gamma}(T_l\widehat{T}) < 1$, we deduce that

$$\begin{aligned} \gamma\{x_{n_i}\} &\leq (1 - \mu)\gamma\{T_lTx_{n_i}\} \\ &\leq (1 - \mu)\gamma\{T_l\widehat{T}x_{n_i}\} \\ &\leq (1 - \mu)\overline{\gamma}\{T_l\widehat{T}\}\gamma\{x_{n_i}\} \end{aligned}$$

which contradict the assertion $\gamma(\{x_n,\, n \in \mathbb{N}\}) \neq 0$. We conclude that $\gamma(\{x_n,\, n \in \mathbb{N}\}) = 0$ and the proof is achieved. □

3. Some perturbation results

In his study of the relative demicompactness and its interaction with the relative essential spectra, B. Krichen proved in [11] some stability theorems. We recall these results using the following notations:

Let $T \in \mathcal{C}(X)$, $S_0 \in \mathcal{L}(X) \setminus \{0\}$, and $n \in \mathbb{N} \setminus \{0\}$. We denote by:

$$\mathcal{L}_T(X) := \{J \in \mathcal{C}(X) : J \text{ is } T\text{-bounded}\},$$

$$\mathcal{F}_T^l(X) := \{T_l \in \mathcal{L}(X, X_T) : T_l \text{ is a left Fredholm inverse of } T\},$$

$$\mathcal{M}_{+,S,T}^n := \{J \in \mathcal{L}_T(X) : \forall \lambda \in \Phi_{+T,S}\ \exists T_{\lambda l} \in \mathcal{F}_{\lambda S-T}^l(X),\ (JT_{\lambda l})^n \in \mathcal{DC}(X)\},$$

$$\mathcal{M}_{S,T}^n := \{J \in \mathcal{L}_T(X) : \forall \lambda \in \Phi_{T,S}\ \exists T_{\lambda l} \in \mathcal{F}_{\lambda S-T}^l(X), \quad \forall \mu \in [0,1]\ \mu(JT_{\lambda l})^n \in \mathcal{DC}(X)\}.$$

Theorem 3.1 ([11]). *Let $T \in \mathcal{C}(X)$ and let $S \in \mathcal{L}(X) \setminus \{0\}$. Then, for every $J \in \mathcal{M}_{+,S,T}^1$, we have*:

$$\sigma_{e_G,S}(T+J) \subset \sigma_{e_G,S}(T). \tag{2}$$

Theorem 3.2 ([11]). *Let $T \in \mathcal{C}(X)$ and let $S \in \mathcal{L}(X) \setminus \{0\}$. Then, for every $J \in \mathcal{M}_{S,T}^1$, we have*:

$$\sigma_{e_W,S}(T+J) \subset \sigma_{e_W,S}(T). \tag{3}$$

Note that if T_1 and T_2 are bounded self-adjoint operators in a Hilbert space, the classical theorem of Weyl states that the essential spectra of T_1 and T_2 coincide if $T_1 - T_2$ is compact. Known generalizations of this result replace the compactness requirement of $T_1 - T_2$ by the condition that $(\lambda - T_1)^{-1} - (\lambda - T_2)^{-1}$ is compact for $\lambda \in \rho(T_1) \cap \rho(T_2)$ and relax to various degrees the self-adjointness restriction on T_1 and T_2. In what follows, we will give a refinement of the Weyl's theorem valid for a somewhat large variety of subsets of operators.

Theorem 3.3. *Let X be a Banach space. Let T_1, T_2 be two closed densely defined linear operators on X and $S \in \mathcal{L}(X)$ be an invertible operator on X.*

(i) *If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, we have $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{M}_{+,S^{-1},-(\lambda S-T_1)^{-1}}^1$, then*

$$\sigma_{e_G,S}(T_1) \subset \sigma_{e_G,S}(T_2).$$

(ii) *If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, we have $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{M}_{S^{-1},-(\lambda S-T_1)^{-1}}^1$, then*

$$\sigma_{e_W,S}(T_1) \subset \sigma_{e_W,S}(T_2).$$

Proof. First assume that $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$. Without loss of generality assume that $\lambda = 0$, it follows that $0 \in \rho_S(T_1) \cap \rho_S(T_2)$. For every $\mu \neq 0$ and $i \in \{1,2\}$, we have

$$\mu S - T_i = -\mu S(\mu^{-1}S^{-1} - T_i^{-1})T_i.$$

Since $0 \in \rho_S(T_i)$, it follows that $\alpha(\mu S - T_i) = \alpha(\mu^{-1}S^{-1} - T_i^{-1})$, $\beta(\mu S - T_i) = \beta(\mu^{-1}S^{-1} - T_i^{-1})$ and $\mathcal{R}(\mu S - T_i)$ is closed if, and only if $\mathcal{R}(\mu^{-1}S^{-1} - T_i^{-1})$ is closed. This shows that $\mu \in \Phi^+_{T_i,S}$ (resp. $\mu \in \Phi^-_{T_i,S}$) if, and only if, $\mu^{-1} \in \Phi^+_{T_i^{-1},S^{-1}}$ (resp. $\mu^{-1} \in \Phi^-_{T_i^{-1},S^{-1}}$). Similarly, we have $\mu \in \Phi_{T_i,S}$ if, and only if, $\mu^{-1} \in \Phi_{T_i^{-1},S^{-1}}$.

(i) Since $(-T_1)^{-1} - (-T_2)^{-1} \in \mathcal{M}^1_{+,S^{-1},T_1^{-1}}$, we infer, by Theorem 3.1, that $\Phi^+_{T_2,S} \subset \Phi^+_{T_1,S}$. We conclude that

$$\sigma_{e_G,S}(T_1) \subset \sigma_{e_G,S}(T_2).$$

(ii) Since $(-T_1)^{-1} - (-T_2)^{-1} \in \mathcal{M}^1_{S^{-1},T_1^{-1}}$, we infer, by Theorem 3.2, that $\Phi_{T_2,S} \subset \Phi_{T_1,S}$ and so,

$$\sigma_{e_W,S}(T_1) \subset \sigma_{e_W,S}(T_2). \qquad \square$$

Corollary 3.4. *Let X be a Banach space. Let T_1, T_2 be two closed densely defined linear operators on X and $S \in \mathcal{L}(X)$ be an invertible operator on X.*

(i) *If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, we have $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{M}^2_{+,S^{-1},-(\lambda S - T_2)^{-1}} \cap \mathcal{M}^2_{+,S^{-1},-(\lambda S - T_1)^{-1}}$, then*

$$\sigma_{e_G,S}(T_2) = \sigma_{e_G,S}(T_1).$$

(ii) *If for some $\lambda \in \rho_S(T_1) \cap \rho_S(T_2)$, we have $(\lambda S - T_1)^{-1} - (\lambda S - T_2)^{-1} \in \mathcal{M}^2_{S^{-1},-(\lambda S - T_2)^{-1}} \cap \mathcal{M}^2_{S^{-1},-(\lambda S - T_1)^{-1}}$, then*

$$\sigma_{e_W,S}(T_1) = \sigma_{e_W,S}(T_2).$$

Proof. The proof is immediate from Theorem 3.3 and Remark 2.1. $\square$

4. An example in transport theory

In this section we will apply the results obtained in Section 3 to describe the S-essential spectrum of the integro-differential operator with abstract boundary conditions, in the Banach space $X_1 := L_1((-a,a)\times(-1,1); dxdv)$, $a > 0$,

$$A_H = T_H + K.$$

Here, T_H is defined by:

$$\begin{cases} T_H : \mathcal{D}(T_H) \subset X_1 \longrightarrow X_1, \\ \qquad\qquad \varphi \qquad\quad \mapsto (T_H\varphi)(x,v) = -v\frac{\partial \varphi}{\partial x}(x,v) - \sigma(v)\varphi(x,v), \\ \\ \mathcal{D}(T_H) = \{\varphi \in W \text{ such that } \varphi^i = H\varphi^o\}, \end{cases}$$

where W is the space defined by $W = \{\varphi \in X_1 \text{ such that } v\frac{\partial \varphi}{\partial x} \in X_1\}$ and $\sigma(.) \in L^\infty(-1,1)$. H is the boundary operator connecting the outgoing and the incoming fluxes. It describes the transport of particles (neutrons, photons, molecules of gas, etc.) in a slab with thickness $2a$. The function $\varphi(\cdot,\cdot)$ represents the number density of gas particles having the position x and the direction cosine of propagation v. φ^o, φ^i represent respectively the outgoing and

the incoming fluxes related by the boundary operator H ("o" for the outgoing and "i" for the incoming) and given by:

$$\begin{cases} \varphi^i(v) = \varphi(-a, v), & v \in (0,1), \\ \varphi^i(v) = \varphi(a, v), & v \in (-1,0), \\ \varphi^o(v) = \varphi(-a, v), & v \in (-1,0), \\ \varphi^o(v) = \varphi(a, v), & v \in (0,1). \end{cases}$$

The operator K is defined by:

$$\begin{cases} K: & X_1 & \longrightarrow & X_1, \\ & u & \mapsto & Ku(x,v) = \displaystyle\int_{-1}^{1} \kappa(x,v,v')u(x,v')dv', \end{cases} \tag{4}$$

where the kernel $\kappa : (-a,a) \times (-1,1) \times (-1,1) \longrightarrow \mathbb{R}$ is assumed to be measurable such that $Ku \in X_1$ for all $u \in X_1$. Notice that the operator K is a bounded linear operator on X_1 and acts only on the velocity v', so x may be seen, simply, as a parameter in $[-a,a]$. Then, we will consider K as a function

$$K(.) : x \in [-a,a] \mapsto K(x) \in \mathcal{L}(L_1([-1,1]; dv)).$$

Definition ([14])**.** A collision operator in the form (4) is said to be regular if it satisfies the assumptions:

$$\begin{cases} - \text{ the function } K(\cdot) \text{ is mesurable, i.e., if } \mathcal{O} \text{ is an open subset of} \\ \quad \mathcal{L}(L_1([-1,1]; dv)), \text{ then } \{x \in [-a,a] \text{ such that } K(x) \in \mathcal{O}\} \text{ is mesurable,} \\ - \text{ there exists a compact subset } \mathcal{C} \subseteq \mathcal{L}(L_1([-1,1]; dv)) \text{ such that} \\ \quad K(x) \in \mathcal{C} \text{ a.e. on } [-a,a], \\ - K(x) \in \mathcal{K}(L_1([-1,1]; dv)) \text{ a.e. on } [-a,a]. \end{cases}$$

The object of this example is to determine the S-essential spectra of the transport operator A_H where S is the operator defined by:

$$\begin{cases} S: & X_1 & \longrightarrow & X_1 \\ & \varphi & \mapsto & (S\varphi)(x,v) = \eta(v)\varphi(x,v), \end{cases}$$

where $\eta(\cdot) \in L^\infty(-1,1)$.

In the sequel, we will consider the following hypothesis.

$(H_1) : \forall \varepsilon > 0,$ there exists $\alpha \in (0,1)$ such that $\sigma(v) \leq \varepsilon$ for all $v \in [-\alpha, \alpha]$.

The following lemma can be found in [8].

Lemma 4.1. *Assume that (H_1) holds. If the kernel $\kappa(x,v,v')$ of the operator (4) defines a regular operator, then the operator $(\lambda S - T_H)^{-1}K$ is weakly compact on X_1.*

Theorem 4.2. *Assume that (H_1) holds. If the operator H is weakly compact on X_1 and the operator K is regular on X_1, then*

$$\sigma_{e_G,S}(A_H) = \{\lambda \in \mathbb{C} : Re(\lambda) \leq 0\}.$$

Proof. By using Remark 3.1 in [9], we can consider $\lambda \in \rho_S(T_H)$ such that $r_\sigma((\lambda S - T_H)^{-1}K) := \lim_{n\to\infty} \| \left((\lambda S - T_H)^{-1}K\right)^n \|^{\frac{1}{n}} < 1$, ($r_\sigma$ denotes the spectral radius). For a such λ, the equation $(\lambda S - T_H - K)\varphi = \psi$ may be transformed into

$$(\lambda S - T_H)^{-1}\psi = (I - (\lambda S - T_H)^{-1}K)\varphi.$$

The fact that $r_\sigma((\lambda S - T_H)^{-1}K) < 1$ implies

$$(\lambda S - A_H)^{-1} = \left[\sum_{n\geq 0}((\lambda S - T_H)^{-1}K)^n\right](\lambda S - T_H)^{-1},$$

and so

$$(\lambda S - A_H)^{-1} - (\lambda S - T_H)^{-1} = \left[\sum_{n\geq 1}((\lambda S - T_H)^{-1}K)^n\right](\lambda S - T_H)^{-1}.$$

Since K is regular, then it follows, from Lemma 4.1, that the operator $(\lambda S - A_H)^{-1} - (\lambda S - T_H)^{-1}$ is weakly compact on X_1. Since X_1 has the Dunford-Pettis property [8], it follows, by using Corollary 2.8, that $\mu[(\lambda S - A_H)^{-1} - (\lambda S - T_H)^{-1}]^2$ is a demicompact operator for every $\mu \in [0, 1]$. The use of Corollary 3.4 leads to

$$\sigma_{e_i,S}(A_H) = \sigma_{e_i,S}(T_H), \quad i \in \{G, W\}.$$

Now, apply Theorem 3.1 in [9], we get

$$\sigma_{e_i,S}(T_H) = \sigma_{e_i,S}(T_0), \quad i \in \{G, W\}.$$

Hence,

$$\sigma_{e_i,S}(A_H) = \{\lambda \in \mathbb{C} : Re(\lambda) \leq 0\}, \quad i \in \{G, W\}. \qquad \square$$

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Wajdi Chaker
Department of New Economy
University of Sfax
Higher Institute of Comercial Studies of Sfax
3061, Sfax, Tunisia
Email address: wajdi_chaker@yahoo.fr

Aref Jeribi
Department of Mathematics
University of Sfax
Faculty of Sciences of Sfax
Road Soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia
Email address: aref.jeribi@fss.rnu.tn

Bilel Krichen
Department of Mathematics
University of Sfax
Faculty of Sciences of Sfax
Road Soukra Km 3.5, B.P. 1171, 3000, Sfax, Tunisia
Email address: krichen_bilel@yahoo.fr