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A NEW EXTENSION OF BESSEL FUNCTION

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Abstract. In this paper, we propose an extension of the classical Bessel function by means of our ℓ-hypergeometric function [2]. As the main results, the infinite order differential equation, the generating function relation, and contour integral representations including Schläfli's integral analogue are derived. With the aid of these, other results including some inequalities are also obtained. At the end, the graphs of these functions are plotted using the Maple software.

1. Introduction

We defined and studied ℓ-hypergeometric function [2]:

$$H\left[\begin{array}{lll} a; & & z \\ b; & (c:\ell); & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n\ (c)_n^{\ell n}} \frac{z^n}{n!}, \tag{1.1}$$

in which $a, \ell, z \in \mathbb{C}$ with $\Re(\ell) \geq 0$, and $b, c \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

This function can evidently be considered as the extension of the generalized hypergeometric function ${}_1F_{1+s}$ ([4, Ch. 4]), which reduces to the so-called hyper-Bessel function ${}_0F_s$ if $a = b$. Here s in the second index goes to infinity together with the summation index n in the power series.

When $\ell = 0$, this function reduces to the *confluent* hypergeometric function ${}_1F_1[z]$.

As a particular case: $a = b$, $c = 1$ of this ℓ-H function, we defined the ℓ-H exponential function as follows.

Definition 1. The ℓ-H exponential function is denoted and defined by [2]

$$e_H^{\ell}(z) = H\left[\begin{array}{lll} -; & & z \\ -; & (1:\ell); & \end{array}\right] = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\ell n+1}} \tag{1.2}$$

for all $z \in \mathbb{C}$ and $\Re(\ell) \geq 0$.

Remark 1.1. Obviously, $e_H^0(z) = e^z$ and $e_H^{\ell}(0) = 1$.

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In (1.2), replacing z by iz, we are led to the ℓ-H trigonometric functions [2]. That is,

$$e_H^\ell(iz) = \sum_{n=0}^{\infty} \frac{(-1)^n\ z^{2n}}{((2n)!)^{2\ell n+1}} + i \sum_{n=0}^{\infty} \frac{(-1)^n\ z^{2n+1}}{((2n+1)!)^{2\ell n+\ell+1}} \tag{1.3}$$

$$= \cos_H^\ell(z) + i \sin_H^\ell(z). \tag{1.4}$$

Remark 1.2. It is easy to see that

$$\cos_H^\ell(-z) = \cos_H^\ell(z),\ \sin_H^\ell(-z) = -\sin_H^\ell(z), \tag{1.5}$$

and

$$\cos_H^0(z) = \cos z,\ \sin_H^0(z) = \sin z. \tag{1.6}$$

Also,

$$\frac{1}{2}\left[e_H^\ell(iz) + e_H^\ell(-iz)\right] = \cos_H^\ell(z), \tag{1.7}$$

$$\frac{1}{2i}\left[e_H^\ell(iz) - e_H^\ell(-iz)\right] = \sin_H^\ell(z). \tag{1.8}$$

It is noteworthy that in parallel to the Kiryakova's generalized sine and cosine functions [5] termed as r-even functions in [8]; these ℓ-H trigonometric functions may be regarded as belonging to the scheme of the new ℓ-Hypergeometric functions (1.1).

Interestingly, the Laurent's series expansion of the product of two ℓ-H exponential functions enables us to define an ℓ-extension of the Bessel function $J_n(z)$. The ℓ-H trigonometric functions then lead us to derive ℓ-analogues of certain properties of the Bessel function occurring in the literature hitherto. The proposed function is defined as follows.

Definition 2. Let $\Re(\ell) \geq 0, n \in \mathbb{N}\cup\{0\}$. Then the new class of Bessel functions is denoted and defined by

$$J_{n,H}^\ell(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\ \left(z/2\right)^{n+2k}}{(k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}}. \tag{1.9}$$

We shall call this function as the ℓ-H Bessel function or briefly, ℓ-HBF.

Remark 1.3. For $\ell = 0$, the ℓ-HBF (1.9) reduces to

$$J_{n,H}^0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k\ \left(z/2\right)^{n+2k}}{k!\ (n+k)!} = J_n(z). \tag{1.10}$$

We first show the convergence of the series in (1.9).

Theorem 1.4. *If $\Re(\ell) \geq 0$ and $\Re(2\ell n + \ell + 2) > 0$, then ℓ-HBF is an entire function of z.*

Proof. Consider

$$J^{\ell}_{n,H}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \ (z/2)^{n+2k}}{(k!)^{\ell k+1} \ ((n+k)!)^{\ell n+\ell k+1}} = \sum_{k=0}^{\infty} \varphi_{n,k} \ z^{n+2k}$$

with

$$\varphi_{n,k} = \frac{(-1)^k}{(k!)^{\ell k+1} \ ((n+k)!)^{\ell n+\ell k+1} \ 4^k}.$$

Applying here the Stirling's formula [4]:

$$\Gamma(\alpha+k) \sim \sqrt{2\pi} \ e^{-(\alpha+k)} \ (\alpha+k)^{(\alpha+k-1/2)} \tag{1.11}$$

for large k and taking $\alpha = 1, n+1$ in turn, we get

$$\begin{aligned} |\varphi_{n,k}|^{\frac{1}{k}} &= \left| \frac{1}{\Gamma^{\ell k+1}(k+1) \ \Gamma^{\ell n+\ell k+1}(n+k+1) \ 4^k} \right|^{\frac{1}{k}} \\ &\sim \frac{1}{4} \ \frac{\left| \sqrt{2\pi} \ e^{-(k+1)} \ (k+1)^{k+1-\frac{1}{2}} \right|^{\frac{1}{k}-\ell}}{\left| \sqrt{2\pi} \ e^{-(n+k+1)} \ (n+k+1)^{n+k+1-\frac{1}{2}} \right|^{\frac{\ell n}{k}+\ell+\frac{1}{k}}}. \end{aligned}$$

Hence using the Cauchy-Hadamard formula, we further have

$$\begin{aligned} \frac{1}{R} &= \limsup_{k\to\infty} \sqrt[k]{|\varphi_{n,k}|} \\ &= \frac{1}{4 \ (2\pi)^{\ell}} \ \limsup_{k\to\infty} \frac{\left| e^{2\ell n+2\ell+2} \right|}{\left| e^{\ell n+2\ell} \ k^{2\ell n+\ell+2} \right|} \ \left| \frac{e}{k} \right|^{2\ell k} \\ &= \frac{e^2}{4} \left| \frac{e^n}{2\pi} \right|^{\ell} \ \limsup_{k\to\infty} \frac{1}{k^{2\ell n+\ell+2}} \ \left| \frac{e}{k} \right|^{2\ell k} \\ &= 0, \end{aligned}$$

provided $\Re(\ell) \geq 0$ and $\Re(2\ell n + \ell + 2) > 0$. □

In order to obtain the properties of the ℓ-HBF (1.9), we need to extend the binomial coefficient and thereby the binomial theorem.

Definition 3. For $0 \leq k \leq n$, the ℓ-binomial coefficient may be denoted and defined by

$$\binom{n}{k}^{(\ell)} = \frac{(n!)^{\ell n+1}}{((n-k)!)^{\ell n-\ell k+1} \ (k!)^{\ell k+1}}.$$

For $z_1, z_2 \in \mathbb{C}$, let us denote by $(z_1 +_{\ell} z_2)^n$ the ℓ-analogue of $(z_1+z_2)^n$. Then the binomial theorem admits the extension in the form:

$$(z_1 +_{\ell} z_2)^n = \sum_{k=0}^{n} \binom{n}{k}^{(\ell)} \ z_1^{n-k} \ z_2^k \tag{1.12}$$

which we call the *ℓ-binomial theorem.* Here we make a convention that $z_1 +_{\ell} z_2$ denotes the usual sum z_1+z_2 only and $z_1 +_{\ell} (-z_2)$ denotes the usual subtraction

$z_1 - z_2$, while $(z_1 +_\ell z_2)^n$ indicates that we have to consider this expansion through ℓ-binomial coefficient.

Remark 1.5. For $\ell = 0$, the ℓ-binomial theorem reduces to the binomial theorem.

In view of Definition 1, we have

$$e_H^\ell(z +_\ell w) = \sum_{n=0}^{\infty} \frac{(z +_\ell w)^n}{(n!)^{\ell n+1}} \tag{1.13}$$

for $z, w \in \mathbb{C}$ and $\Re(\ell) \geq 0$.

Using the expansion (1.12), we prove the ℓ-analogue of the identity:

$$\exp(z_1 + z_2) = \exp(z_1)\ \exp(z_2)$$

as:

Lemma 1.6. *For the ℓ-H exponential function, the following identity holds*:

$$e_H^\ell(z_1 +_\ell z_2) = e_H^\ell(z_1)\ e_H^\ell(z_2). \tag{1.14}$$

Proof. In view of (1.12) and (1.13), we have

$$\begin{aligned}
e_H^\ell(z_1 +_\ell z_2) &= \sum_{n=0}^{\infty} \frac{1}{(n!)^{\ell n+1}} \ \sum_{k=0}^{n} \frac{(n!)^{\ell n+1}}{((n-k)!)^{\ell n-\ell k+1}\ (k!)^{\ell k+1}}\ z_1^{n-k}\ z_2^k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z_1^{n-k}\ z_2^k}{((n-k)!)^{\ell n-\ell k+1}\ (k!)^{\ell k+1}} \\
&= \sum_{n=0}^{\infty} \frac{z_1^n}{(n!)^{\ell n+1}} \ \sum_{k=0}^{\infty} \frac{z_2^k}{(k!)^{\ell k+1}} \\
&= e_H^\ell(z_1)\ e_H^\ell(z_2).
\end{aligned}$$

□

Remark 1.7. Since $e_H^\ell(0) = 1$, it follows from (1.14) with $z_2 = -z_1$ that

$$e_H^\ell(z_1)\ e_H^\ell(-z_1) = e_H^\ell(z_1 +_\ell (-z_1)) = e_H^\ell(z_1 - z_1) = e_H^\ell(0) = 1. \tag{1.15}$$

Along with this, the relation: $J_{-n}(z) = (-1)^n J_n(z)$ is also put in the ℓ-form which will be used later in obtaining certain properties.

Lemma 1.8. *For $n \in \mathbb{Z}$,*

$$(-1)^n\ J_{n,H}^\ell(z) = J_{-n,H}^\ell(z). \tag{1.16}$$

Proof. We begin with

$$\begin{aligned}
(-1)^n\ J_{n,H}^\ell(z) &= (-1)^n \sum_{s=0}^{\infty} \frac{(-1)^s\ (z/2)^{n+2s}}{(s!)^{\ell s+1}\ ((n+s)!)^{\ell n+\ell s+1}} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^{s+n}\ (z/2)^{2s+2n-n}}{(s!)^{\ell s+1}\ ((s+n)!)^{\ell s+\ell n+1}}.
\end{aligned}$$

Taking $s+n=k$ we have

$$\begin{aligned}(-1)^n\ J^{\ell}_{n,H}(z) &= \sum_{k=n}^{\infty}\frac{(-1)^k\ (z/2)^{2k-n}}{(k!)^{\ell k+1}\ ((k-n)!)^{\ell k-\ell n+1}}\\ &= J^{\ell}_{-n,H}(z).\end{aligned}$$

□

2. Main results

For the ℓ-HBF, we first derive the generating function relation (GFR) and then derive the differential equation and the integral representations.

2.1. Generating function relation

The derivation of the GFR of ℓ-HBF uses the finite summation identity [9, Lemma 12, p. 112] which is referred to here as:

Lemma 2.1. *For $n\geq 1$,*

$$\sum_{k=0}^{n}A(k,n)=\sum_{k=0}^{\left[\frac{n}{2}\right]}A(n-k,n)+\sum_{k=0}^{\left[\frac{n-1}{2}\right]}A(k,n). \tag{2.1}$$

Proof. First note that for $n\geq 1$,

$$n=1+\left[\frac{n}{2}\right]+\left[\frac{n-1}{2}\right],$$

in which $[*]$ is the usual greatest integer symbol.

Hence

$$\sum_{k=0}^{n}A(k,n)=\sum_{k=0}^{\left[\frac{n}{2}\right]}A(k,n)+\sum_{k=0}^{1+\left[\frac{n}{2}\right]+\left[\frac{n-1}{2}\right]}A(k,n). \tag{2.2}$$

Now replacing k by $n-k$ that is, k by $1+\left[\frac{n}{2}\right]+\left[\frac{n-1}{2}\right]-k$ in the last summation in (2.2), leads to (2.1). □

We now derive the GFR in:

Theorem 2.2. *For $t\neq 0$ and for all finite $|z|$,*

$$\sum_{n=-\infty}^{\infty}J^{\ell}_{n,H}(z)\ t^n=e^{\ell}_{H}\left(\frac{z}{2}\left(t+_{\ell}(-t^{-1})\right)\right), \tag{2.3}$$

where $e^{\ell}_{H}(z)$ is as defined in (1.2).

Proof. The left hand side

$$\begin{aligned}\sum_{n=-\infty}^{\infty}J^{\ell}_{n,H}(z)\ t^n &= \sum_{n=-\infty}^{-1}J^{\ell}_{n,H}(z)\ t^n+\sum_{n=0}^{\infty}J^{\ell}_{n,H}(z)\ t^n\\ &= \sum_{n=0}^{\infty}J^{\ell}_{-n-1,H}(z)\ t^{-n-1}+\sum_{n=0}^{\infty}J^{\ell}_{n,H}(z)\ t^n.\end{aligned}$$

In view of Lemma 1.8 and defining series (1.9), we further have

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty} J^{\ell}_{n,H}(z)\ t^{n} \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}\ J^{\ell}_{n+1,H}(z)\ t^{-n-1} + \sum_{n=0}^{\infty} J^{\ell}_{n,H}(z)\ t^{n} \\
&= \sum_{n,k=0}^{\infty} \frac{(-1)^{n+k+1}\ (z/2)^{n+2k+1}\ t^{-n-1}}{(k!)^{\ell k+1}\ ((n+k+1)!)^{\ell n+\ell k+\ell+1}} + \sum_{n,k=0}^{\infty} \frac{(-1)^{k}\ (z/2)^{n+2k}\ t^{n}}{(k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{n-2k+k+1}\ (z/2)^{n-2k+2k+1}\ t^{-n+2k-1}}{(k!)^{\ell k+1}\ ((n-2k+k+1)!)^{\ell n-2\ell k+\ell k+\ell+1}} \\
&\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{k}\ (z/2)^{n-2k+2k}\ t^{n-2k}}{(k!)^{\ell k+1}\ ((n-2k+k)!)^{\ell n-2\ell k+\ell k+1}} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^{n-k}\ (z/2)^{n}\ t^{-n+2k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}} + 1 \\
&\quad + \sum_{n=1}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{k}\ (z/2)^{n}\ t^{n-k-k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}} \\
&= 1 + \sum_{n=1}^{\infty}\left[\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{(-1)^{n-k}\ (z/2)^{n}\ t^{-n+2k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}} \right.\\
&\qquad\qquad \left. + \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^{k}\ (z/2)^{n}\ t^{n-k-k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}}\right].
\end{aligned}
$$

From the result stated in Lemma 2.1, we have

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} J^{\ell}_{n,H}(z)\ t^{n} &= 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{(-1)^{k}\ (z/2)^{n}\ t^{n-2k}}{(k!)^{\ell k+1}\ ((n-k)!)^{\ell n-\ell k+1}} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{k}\ (z/2)^{n+k}\ t^{n-k}}{(k!)^{\ell k+1}\ (n!)^{\ell n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(z/2)^{n}\ t^{n}}{(n!)^{\ell n+1}} \sum_{k=0}^{\infty} \frac{(-1)^{k}\ (z/2)^{k}\ t^{-k}}{(k!)^{\ell k+1}} \\
&= e^{\ell}_{H}\left(\frac{zt}{2}\right) e^{\ell}_{H}\left(\frac{-z}{2t}\right) = e^{\ell}_{H}\left(\frac{z}{2}\left(t +_{\ell} (-t^{-1})\right)\right).
\end{aligned}
$$

Thus, the GFR follows from Lemma 1.6. □

2.2. Differential equation

The differential equation of ℓ-HBF is obtained by means of the infinite order hyper-Bessel type differential operator as defined below [2,3].

Definition 4. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $0 \neq z \in \mathbb{C}$, $p \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{C}$. Define

$$
{}_p\Delta_\alpha^\Theta(f(z)) = \begin{cases} \sum\limits_{n=1}^{\infty} a_n (\alpha)_{n-1}^p (\Theta + \alpha - 1)^{pn} z^n, & \text{if} \quad p \in \mathbb{N}, \\ f(z), & \text{if} \quad p = 0, \end{cases} \tag{2.4}
$$

where Θ is either differential operator $\theta = z\frac{d}{dz}$ or hyper-Bessel type differential operators (see for instance [5–7])

$$
(\mathbb{D}_z)^n = \underbrace{\frac{d}{dz} z \frac{d}{dz} \cdots \frac{d}{dz} z \frac{d}{dz}}_{n \text{ derivatives}}
$$

with $(\Theta + \alpha)^r = \underbrace{(\Theta + \alpha)(\Theta + \alpha) \cdots (\Theta + \alpha)}_{r \text{ times}}$.

Also, we need the following operators.

Definition 5. For $f(z) = \sum_{k=0}^{\infty} a_k \, z^{\alpha k}$, $\alpha \in \mathbb{R}$, define the lowering operator:

$$
\mathcal{O}_{\alpha-} f(z) = \sum_{k=0}^{\infty} a_k \; z^{(\alpha-1)k}, \tag{2.5}
$$

the raising operator:

$$
\mathcal{O}_{\alpha+} f(z) = \sum_{k=0}^{\infty} a_k \; z^{(\alpha+1)k} \tag{2.6}
$$

and as suggested by (2.4), the operator:

$$
{}_\ell\Lambda_M(f(z)) = {}_\ell\Delta_1^\theta \; (\theta(f(z))). \tag{2.7}
$$

We put

$$
{}_\ell\Omega_n^{(z)} \equiv \left(\mathcal{O}_{1+} \; {}_\ell\Lambda_M z^{-n} \; {}_\ell\Lambda_M \mathcal{O}_{2-} \right). \tag{2.8}
$$

With these operators, the differential equation of ℓ-HBF is derived in:

Theorem 2.3. *For $\ell, n \in \mathbb{N} \cup \{0\}$ and $z \neq 0$, $w = J_{n,H}^\ell(z)$ satisfies the differential equation*

$$
\left[{}_\ell\Omega_n^{(z)} + \frac{z^{-n+1}}{4} \right] w = 0, \tag{2.9}
$$

where ${}_\ell\Omega_n^{(z)}$ is as defined in (2.8).

Proof. We start with

$$
\begin{aligned}
&{}_{\ell}\Omega_n^{(z)}\ J_{n,H}^{\ell}(z)\\
&= \left(\mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ {}_{\ell}\Lambda_M \mathcal{O}_{2-}\right) \left(\sum_{k=0}^{\infty} \frac{(-1)^k\ (z/2)^{n+2k}}{(k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ {}_{\ell}\Lambda_M \left(\sum_{k=0}^{\infty} \frac{(-1)^k\ z^{n+k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ {}_{\ell}\Delta_M \left(\sum_{k=0}^{\infty} \frac{(-1)^k\ z^{n+k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k}\ (n+k-1)!}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ \left(\sum_{k=0}^{\infty} \frac{(-1)^k\ (1)_{n+k-1}^{\ell}\ \theta^{\ell n+\ell k}\ z^{n+k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k}\ (n+k-1)!}\right).
\end{aligned}
$$

Since $\theta^{\ell n+\ell k}\ z^{n+k} = (n+k)^{\ell n+\ell k}\ z^{n+k}$, we have

$$
\begin{aligned}
&{}_{\ell}\Omega_n^{(z)}\ J_{n,H}^{\ell}(z)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M z^{-n}\ \left(\sum_{k=0}^{\infty} \frac{(-1)^k\ z^{n+k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Lambda_M\ \left(\sum_{k=0}^{\infty} \frac{(-1)^k\ z^{k}}{2^{n+2k}\ (k!)^{\ell k+1}\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+}\ {}_{\ell}\Delta_M\ \left(\sum_{k=1}^{\infty} \frac{(-1)^k\ z^{k}}{2^{n+2k}\ (k!)^{\ell k}\ (k-1)!\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+} \left(\sum_{k=1}^{\infty} \frac{(-1)^k\ (1)_{k-1}^{\ell}\ \theta^{\ell k}\ z^{k}}{2^{n+2k}\ (k!)^{\ell k}\ (k-1)!\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+} \left(\sum_{k=1}^{\infty} \frac{(-1)^k\ z^{k}}{2^{n+2k}\ ((k-1)!)^{\ell k-\ell+1}\ ((n+k-1)!)^{\ell n+\ell k-\ell+1}}\right)\\
&= \mathcal{O}_{1+} \left(\sum_{k=0}^{\infty} \frac{(-1)^{k+1}\ z^{k+1}}{2^{n+2k+2}\ (k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}}\right)\\
&= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}\ z^{2k+1}}{2^{n+2k+2}\ (k!)^{\ell k+1}\ ((n+k)!)^{\ell n+\ell k+1}}\\
&= -\ \frac{z^{-n+1}}{4}\ J_{n,H}^{\ell}(z).
\end{aligned}
$$

□

Remark 2.4. The zero order ℓ-HBF, that is $w = J_{0,H}^{\ell}(z)$ satisfies the differential equation:

$$\left[{}_{\ell}\Omega_0^{(z)} + \frac{z}{4}\right] w = 0.$$

2.3. ℓ-HBF integral

By using the ℓ-H trigonometric functions (1.4), the ℓ-analogue of the Bessel's integral is obtained in:

Theorem 2.5. *For $n \in \mathbb{Z}$,*

$$J^{\ell}_{n,H}(z) = \frac{1}{\pi}\int_0^{\pi} \left[\cos n\theta \;\; \cos^{\ell}_{H}(z\sin\theta) + \sin n\theta \;\; \sin^{\ell}_{H}(z\sin\theta)\right] \mathrm{d}\theta. \tag{2.10}$$

Proof. The generating function relation of ℓ-HBF (2.3) may be regarded as the Laurent series expansion of the function $e^{\ell}_{H}\left(\frac{z}{2}(t - t^{-1})\right)$; valid near the essential singularity $t = 0$. We then have the coefficient

$$J^{\ell}_{n,H}(z) = \frac{1}{2\pi i}\int^{(0+)} u^{-n-1} \; e^{\ell}_{H}\left(\frac{z}{2}(u +_{\ell} (-u^{-1}))\right) \mathrm{du} \tag{2.11}$$

in which the contour (0+) is a simple closed path encircling the origin $u = 0$ once in the positive direction.

In (2.11), let us choose the particular path

$$u = e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ runs from $-\pi$ to π. Then $u^{-1} = \cos\theta - i\sin\theta$, hence (2.11) yields

$$\begin{aligned} J^{\ell}_{n,H}(z) &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{(-n-1)i\theta} \; e^{\ell}_{H}\left(\frac{z}{2}(2i\sin\theta)\right) ie^{i\theta}\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cos n\theta - i\sin n\theta)\left[\cos^{\ell}_{H}(z\sin\theta) + i\sin^{\ell}_{H}(z\sin\theta)\right]\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos n\theta \;\; \cos^{\ell}_{H}(z\sin\theta)\;\mathrm{d}\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin \mathrm{n}\theta \;\; \sin^{\ell}_{\mathrm{H}}(\mathrm{z}\sin\theta)\;\mathrm{d}\theta \qquad (2.12) \\ &\quad - \frac{i}{2\pi}\left[\int_{-\pi}^{\pi}\sin n\theta \;\; \cos^{\ell}_{H}(z\sin\theta)\;\mathrm{d}\theta - \int_{-\pi}^{\pi}\cos \mathrm{n}\theta \;\; \sin^{\ell}_{\mathrm{H}}(\mathrm{z}\sin\theta)\;\mathrm{d}\theta\right] \\ &= I_1 + I_2 + I_3 + I_4 \quad \text{(say)}. \end{aligned}$$

From (1.5), we note that the integrands in I_1 and I_2 are even functions of θ whereas the integrands in I_3 and I_4 are odd functions of θ, hence $I_3 = I_4 = 0$. Thus the theorem. □

Remark 2.6. When ℓ=0, then in view of (1.6) and (1.10), (2.10) reduces to the Bessel's integral [9, Theorem 40, p. 114]:

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - z\sin\theta)\,\mathrm{d}\theta$$

for integral n.

2.4. Schläfli's integral analogue

Here the integral of Theorem 2.5 is modified so as to include the non integral values of n. For that considering the alternative form:

$$J^{\ell}_{n,H}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \ (z/2)^{n+2k} \ t^n}{(k!)^{\ell k+1} \ \Gamma^{\ell n+\ell k+1}(n+k+1)}, \tag{2.13}$$

we have:

Theorem 2.7. *If $\Re(z) > 0$, then for general values of n,*

$$J^{\ell}_{n,H}(z) = \frac{1}{\pi}\Bigg\{ \int_0^{\pi} \cos n\theta \ \cos^{\ell}_{H}(z\sin\theta) + \sin n\theta \ \sin^{\ell}_{H}(z\sin\theta) \mathrm{d}\theta$$
$$- \frac{\sin n\pi}{\pi} \int_0^{\infty} e^{-n\theta} \ e^{\ell}_{H}\left(-z\sinh\theta\right) \mathrm{d}\theta \Bigg\}. \tag{2.14}$$

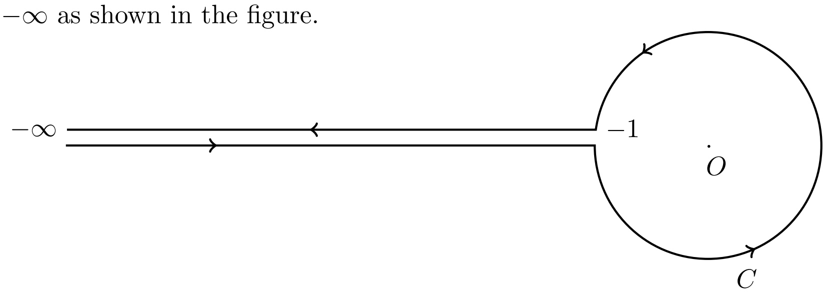
Proof. Let us consider the integral in (2.11), that is

$$J^{\ell}_{n,H}(z) = \frac{1}{2\pi i} \int^{(0+)} u^{-n-1} \ e^{\ell}_{H}\left(\frac{z}{2}(u +_{\ell} (-u^{-1}))\right) \mathrm{du}.$$

We integrate the branch

$$u^{-n-1} \ e^{\ell}_{H}\left(\frac{z}{2}(u +_{\ell} (u^{-1}))\right) \quad (|u| > 0, \ -\pi < \arg u < \pi)$$

with branch cut $\arg u = \pi$ around the contour C which is traced out by a point moving (i) along the lower edge of the cut from $-\infty$ to -1, then (ii) around the circle $|u| = 1$, and finally (iii) along the upper edge of the cut from -1 to $-\infty$ as shown in the figure.



Hence for $\Re(z) > 0$, we have

$$\begin{aligned} J^{\ell}_{n,H}(z) &= \frac{1}{2\pi i} \int_C u^{-n-1} \ e^{\ell}_{H}\left(\frac{z}{2}(u +_{\ell} (u^{-1}))\right) \mathrm{du} \\ &= \frac{1}{2\pi i} \left\{ \int_{-\infty}^{-1} + \int_{|u|=1} + \int_{-1}^{-\infty} \right\} u^{-n-1} \ e^{\ell}_{H}\left(\frac{z}{2}(u +_{\ell} (u^{-1}))\right) \mathrm{du}. \end{aligned}$$

Writing $u = e^{\mp i\pi}\ t$ in the first and third integrals respectively and $u = e^{i\theta}$, $-\pi < \theta < \pi$ in the second, we further have

$$
\begin{aligned}
J^{\ell}_{n,H}(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} e^{\ell}_{H}(iz\sin\theta)\mathrm{d}\theta \\
&\quad - \frac{1}{2\pi i}\int_{-\infty}^{-1} e^{(-n-1)(-i\pi)} t^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}\left(e^{-i\pi}t +_{\ell}\left(-\frac{e^{i\pi}}{t}\right)\right)\right)\mathrm{dt} \\
&\quad - \frac{1}{2\pi i}\int_{1}^{\infty} e^{(-n-1)(i\pi)} t^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}\left(e^{i\pi}t +_{\ell}\left(-\frac{e^{-i\pi}}{t}\right)\right)\right)\mathrm{dt} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} e^{\ell}_{H}(iz\sin\theta)\mathrm{d}\theta \\
&\quad + \int_{1}^{\infty} t^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}\left(-t +_{\ell} t^{-1}\right)\right)\left[\frac{e^{(n+1)\pi i} - e^{-(n+1)\pi i}}{2\pi i}\right]\mathrm{dt} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} e^{\ell}_{H}(iz\sin\theta)\mathrm{d}\theta \\
&\quad + \frac{\sin(n+1)\pi}{\pi}\int_{1}^{\infty} t^{-n-1}\ e^{\ell}_{H}\left(\frac{z}{2}\left(t^{-1} +_{\ell} (-t)\right)\right)\mathrm{dt}.
\end{aligned}
$$

Now evaluating the first integral by following the procedure of obtaining (2.10) from (2.12), and evaluating the second integral by putting $t = e^{\theta}$, we finally find

$$
\begin{aligned}
J^{\ell}_{n,H}(z) &= \frac{1}{\pi}\int_{0}^{\pi}\left[\cos n\theta\ \cos^{\ell}_{H}(z\sin\theta) + \sin n\theta\ \sin^{\ell}_{H}(z\sin\theta)\right]\mathrm{d}\theta \\
&\quad + \frac{\sin(n+1)\pi}{\pi}\int_{0}^{\infty} e^{-n\theta}\ \ e^{\ell}_{H}\left(\frac{z}{2}\left(e^{-\theta} +_{\ell} (-e^{\theta})\right)\right)\mathrm{d}\theta \\
&= \frac{1}{\pi}\int_{0}^{\pi}\left[\cos n\theta\ \cos^{\ell}_{H}(z\sin\theta) + \sin n\theta\ \sin^{\ell}_{H}(z\sin\theta)\right]\mathrm{d}\theta \\
&\quad - \frac{\sin n\pi}{\pi}\int_{0}^{\infty} e^{-n\theta}\ e^{\ell}_{H}\left(-z\sinh\theta\right)\mathrm{d}\theta. \qquad \square
\end{aligned}
$$

Remark 2.8. (1) We call the integral (2.14) as the *ℓ-Schläfli's integral.* If $\ell = 0$, then (2.14) yields the Schläfli's integral for Bessel function $J_n(z)$ given by [11, Sec. 17.231, p. 362]

$$
J_n(z) = \frac{1}{\pi}\int_{0}^{\pi}\cos n(\theta - z\sin\theta)\ \mathrm{d}\theta - \frac{\sin n\pi}{\pi}\int_{0}^{\infty} e^{-n\theta - z\sinh\theta}\mathrm{d}\theta.
$$

(2) If n is an integer, then the integral (2.14) reduces to the ℓ-HBF integral (2.10).

3. Other properties

Here a differential recurrence relation, summation formula and two inequalities will be derived and then with the help of the GFR and ℓ-HBF integral representation, some other properties will be deduced.

3.1. Differential recurrence relation

Let ${}_\ell\Delta_1^\theta$ be as defined in (2.4). Then by using the operator [2,3]:

$$ {}_\ell\mathcal{D}_M^{(z)}\,(f(z)) = \left(z^{-1}\ {}_\ell\Delta_1^\theta\ \theta\right)(f(z)) \quad \text{or} \quad \left(\equiv\ {}_\ell\Delta_1^{\mathbb{D}_z}\ \theta(f(z))\right), \tag{3.1}$$

we derive the differential recurrence relation in:

Theorem 3.1. *For $\ell \in \mathbb{N}\cup\{0\}$,*

$$ 2\ {}_\ell\mathcal{D}_M^{(z)} J_{n,H}^\ell(z) = J_{n-1,H}^\ell(z) - J_{n+1,H}^\ell(z). \tag{3.2}$$

The theorem is proved by using the following lemma which describes the eigen function property of ℓ-H exponential function.

Lemma 3.2. *The ℓ-H exponential function $e_H^\ell\left(\frac{z}{2}\left(t-t^{-1}\right)\right)$ with t fixed, is an eigen function with respect to the operator ${}_\ell\mathcal{D}_M^{(z)}$ as defined in* (3.1). *That is,*

$$ {}_\ell\mathcal{D}_M^{(z)}\left[e_H^\ell\left(\frac{z}{2}\left(t +_\ell (-t^{-1})\right)\right)\right] = \frac{1}{2}\left(t-t^{-1}\right)\ e_H^\ell\left(\frac{z}{2}\left(t +_\ell (-t^{-1})\right)\right) \tag{3.3}$$

for fixed t.

Proof. We begin with

$$\begin{aligned} {}_\ell\mathcal{D}_M^{(z)}\left[e_H^\ell\left(\frac{z}{2}\left(t +_\ell (-t^{-1})\right)\right)\right] &= {}_\ell\Delta_1^{\mathbb{D}_z}\left(\theta\ \sum_{n=0}^{\infty}\frac{\left(t +_\ell (-t^{-1})\right)^n\ z^n}{2^n\ (n!)^{\ell n+1}}\right)\\ &= {}_\ell\Delta_1^{\mathbb{D}_z}\left(\sum_{n=1}^{\infty}\frac{\left(t +_\ell (-t^{-1})\right)^n\ z^n}{2^n\ (n!)^{\ell n}\ (n-1)!}\right)\\ &= \sum_{n=1}^{\infty}\frac{\left(t +_\ell (-t^{-1})\right)^n\ (1)_{n-1}^\ell\ (\mathbb{D}^z)^n\ z^n}{2^n\ (n!)^{\ell n}\ (n-1)!}. \end{aligned}$$

Now since

$$ (\mathbb{D}_z)^{\ell n}\ z^n = \underbrace{\frac{d}{dz}z\frac{d}{dz}\cdots\frac{d}{dz}z\frac{d}{dz}}_{\ell n\ \text{derivatives}}\ z^n = n^{\ell n}\ z^{n-1},$$

we obtain

$$\begin{aligned} {}_\ell\mathcal{D}_M^{(z)}\left[e_H^\ell\left(\frac{z}{2}\left(t-t^{-1}\right)\right)\right] &= \sum_{n=1}^{\infty}\frac{\left(t +_\ell (-t^{-1})\right)^n\ (1)_{n-1}^\ell\ n^{\ell n}\ z^{n-1}}{2^n\ (n!)^{\ell n}\ (n-1)!}\\ &= \sum_{n=1}^{\infty}\frac{\left(t +_\ell (-t^{-1})\right)^n\ z^{n-1}}{2^n\ ((n-1)!)^{\ell n-\ell+1}}\\ &= \frac{1}{2}\left(t-t^{-1}\right)\ e_H^\ell\left(\frac{z}{2}\left(t +_\ell (-t^{-1})\right)\right). \end{aligned}$$

□

Proof of Theorem 3.1. From Theorem 2.2 we have

$$ e_H^\ell\left(\frac{z}{2}\left(t +_\ell (-t^{-1})\right)\right) = \sum_{n=-\infty}^{\infty} J_{n,H}^\ell(z)\ t^n.$$

Applying the operator ${}_{\ell}\mathcal{D}_{M}^{(z)}$ both the sides, we obtain

$$ {}_{\ell}\mathcal{D}_{M}^{(z)}\left[e_{H}^{\ell}\left(\frac{z}{2}\left(t +_{\ell}\left(-t^{-1}\right)\right)\right)\right] = \sum_{n=-\infty}^{\infty} {}_{\ell}\mathcal{D}_{M}^{(z)}\left(J_{n,H}^{\ell}(z)\right)\ t^{n}. $$

Then from Lemma 3.2,

$$ \frac{1}{2}\left(t - t^{-1}\right)\ e_{H}^{\ell}\left(\frac{z}{2}\left(t +_{\ell}\left(-t^{-1}\right)\right)\right) = \sum_{n=-\infty}^{\infty} {}_{\ell}\mathcal{D}_{M}^{(z)}\left(J_{n,H}^{\ell}(z)\right)\ t^{n}. $$

Once again using Theorem 2.2, we find

$$ \begin{aligned} &\sum_{n=-\infty}^{\infty} {}_{\ell}\mathcal{D}_{M}^{(z)}\left(J_{n,H}^{\ell}(z)\right)\ t^{n} \\ &= \frac{t}{2}\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^{n} + \left(\frac{-1}{2t}\right)\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^{n} \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^{n+1} - \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n,H}^{\ell}(z)\ t^{n-1} \\ &= \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n-1,H}^{\ell}(z)\ t^{n} - \frac{1}{2}\sum_{n=-\infty}^{\infty} J_{n+1,H}^{\ell}(z)\ t^{n}. \end{aligned} $$

On comparing the coefficients of t^{n} both sides, we get (3.2). □

The iteration of the relation (3.2) yields the following general formula.

Theorem 3.3. *If* $n \in \mathbb{N} \cup \{0\}$ *and* $k \in \mathbb{N}$, *then* (*cf.* [9, Ex. 7, p. 121])

$$ 2^{k}\left({}_{\ell}\mathcal{D}_{M}^{(z)}\right)^{k}\ J_{n,H}^{\ell}(z) = \sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}J_{n+k-2m,H}^{\ell}(z). \tag{3.4} $$

Proof. For $k = 1$, this theorem holds true from Theorem 3.1. That is,

$$ 2\ {}_{\ell}\mathcal{D}_{M}^{(z)} J_{n,H}^{\ell}(z) = J_{n-1,H}^{\ell}(z) - J_{n+1,H}^{\ell}(z). $$

Here, applying the operator $2\ {}_{\ell}\mathcal{D}_{M}^{(z)}$ both the sides, we find that

$$ \begin{aligned} 2^{2}\left({}_{\ell}\mathcal{D}_{M}^{(z)}\right)^{2}\ J_{n,H}^{\ell}(z) &= 2\ {}_{\ell}\mathcal{D}_{M}^{(z)} J_{n-1,H}^{\ell}(z) - 2\ {}_{\ell}\mathcal{D}_{M}^{(z)} J_{n+1,H}^{\ell}(z) \\ &= J_{n-2,H}^{\ell}(z) - 2J_{n,H}^{\ell}(z) + J_{n+2,H}^{\ell}(z) \\ &= \sum_{m=0}^{2}(-1)^{2-m}\binom{2}{m}J_{n+2-2m,H}^{\ell}(z). \end{aligned} \tag{3.5} $$

Similarly,

$$ \begin{aligned} &2^{3}\left({}_{\ell}\mathcal{D}_{M}^{(z)}\right)^{3}\ J_{n,H}^{\ell}(z) \\ &= 2\ {}_{\ell}\mathcal{D}_{M}^{(z)} J_{n-2,H}^{\ell}(z) - 2\ {}_{\ell}\mathcal{D}_{M}^{(z)}\left[2J_{n,H}^{\ell}(z)\right] + 2\ {}_{\ell}\mathcal{D}_{M}^{(z)} J_{n+2,H}^{\ell}(z) \end{aligned} $$

$$= J^{\ell}_{n-3,H}(z) - 3J^{\ell}_{n-1,H}(z) + 3J^{\ell}_{n+1,H}(z) - J^{\ell}_{n+3,H}(z)$$
$$= \sum_{m=0}^{3} (-1)^{3-m} \binom{3}{m} J^{\ell}_{n+3-2m,H}(z).$$

The recursive procedure k-times, leads to the theorem. □

Remark 3.4. Alternatively, this theorem can also be proved by using the principle of mathematical induction on k.

3.2. Summation formula and inequalities

In the following theorems, the properties involving series and definite integrals are extended.

Theorem 3.5. *For an integer n,* (*cf.* [9, Ex. 2, p. 120])

$$\cos^{\ell}_{H}(z) = J^{\ell}_{0,H}(z) + 2\sum_{n=1}^{\infty} (-1)^n \; J^{\ell}_{2n,H}(z), \tag{3.6}$$

$$\sin^{\ell}_{H}(z) = 2\sum_{n=0}^{\infty} (-1)^n \; J^{\ell}_{2n+1,H}(z). \tag{3.7}$$

Proof. From the GFR of ℓ-HBF (2.3) and the identity (1.16),

$$e^{\ell}_{H}\left(\frac{z}{2}\left(t +_{\ell} (-t^{-1})\right)\right) = \sum_{n=-\infty}^{\infty} J^{\ell}_{n,H}(z)\; t^n$$
$$= J^{\ell}_{0,H}(z) + \sum_{n=1}^{\infty} J^{\ell}_{n,H}(z)\; t^n + \sum_{n=1}^{\infty} J^{\ell}_{-n,H}(z)\; t^{-n}$$
$$= J^{\ell}_{0,H}(z) + \sum_{n=1}^{\infty} J^{\ell}_{n,H}(z)\; [t^n + (-1)^n\; t^{-n}]. \tag{3.8}$$

Taking $t = i$, we have $t^{-1} = -i$ hence

$$e^{\ell}_{H}(iz) = J^{\ell}_{0,H}(z) + \sum_{n=1}^{\infty} J^{\ell}_{n,H}(z)\; [i^n + (-1)^n\; i^{-n}]$$
$$= J^{\ell}_{0,H}(z) + \sum_{n=1}^{\infty} J^{\ell}_{2n,H}(z)\; [i^{2n} + (-1)^{2n}\; i^{-2n}]$$
$$+ \sum_{n=0}^{\infty} J^{\ell}_{2n+1,H}(z)\; [i^{2n+1} + (-1)^{2n+1}\; i^{-(2n+1)}].$$

In view of (1.4), this gives

$$\cos^{\ell}_{H}(z) + i\sin^{\ell}_{H}(z) = J^{\ell}_{0,H}(z) + 2\sum_{n=1}^{\infty} (-1)^n\; J^{\ell}_{2n,H}(z) + 2i\sum_{n=0}^{\infty} (-1)^n\; J^{\ell}_{2n+1,H}(z).$$

On comparing the real and imaginary parts, we obtain the required result. □

Theorem 3.6. *If $n \in \mathbb{Z}$, then* (*cf.* [9, Ex. 3, p. 120])

$$\cos_H^\ell(z \sin_H^\ell \theta) = J_{0,H}^\ell(z) + 2\sum_{n=1}^{\infty} J_{2n,H}^\ell(z)\ \cos_H^\ell 2n\theta, \tag{3.9}$$

$$\sin_H^\ell(z \sin_H^\ell \theta) = 2\sum_{n=0}^{\infty} J_{2n+1,H}^\ell(z)\ \sin_H^\ell(2n+1)\theta. \tag{3.10}$$

Proof. In (3.8), substituting $t = e_H^\ell(i\theta)$ the ℓ-H exponential function, we find

$$\begin{aligned}
&e_H^\ell\left(\frac{z}{2}\left(e_H^\ell(i\theta) +_\ell (-e_H^\ell(-i\theta))\right)\right)\\
&= J_{0,H}^\ell(z) + \sum_{n=1}^{\infty} J_{n,H}^\ell(z)\ [e_H^\ell(in\theta) + (-1)^n\ e_H^\ell(-in\theta)]\\
&= J_{0,H}^\ell(z) + \sum_{n=1}^{\infty} J_{2n,H}^\ell(z)\ [e_H^\ell(2in\theta) + (-1)^{2n}\ e_H^\ell(-2in\theta)]\\
&\quad + \sum_{n=0}^{\infty} J_{2n+1,H}^\ell(z)\ [e_H^\ell((2n+1)i\theta) + (-1)^{2n+1}\ e_H^\ell(-(2n+1)i\theta)]\\
&= J_{0,H}^\ell(z) + 2\sum_{n=1}^{\infty} J_{2n,H}^\ell(z) \cos_H^\ell 2n\theta + 2i\ \sum_{n=0}^{\infty} J_{2n+1,H}^\ell(z)\ \sin_H^\ell(2n+1)\theta.
\end{aligned}$$

From (1.8),

$$\begin{aligned}
e_H^\ell\left(\frac{z}{2}\left(e_H^\ell(i\theta) +_\ell (-e_H^\ell(-i\theta))\right)\right) &= e_H^\ell(iz \sin_H^\ell \theta)\\
&= \cos_H^\ell(z \sin_H^\ell \theta) + i \sin_H^\ell(z \sin_H^\ell \theta),
\end{aligned}$$

hence the result follows by comparison of the real and imaginary parts. $\square$

Theorem 3.7. *For $n \in \mathbb{Z}$, the following equalities hold* (*cf.* [9, Ex. 3, p. 120]):

$$[1 + (-1)^n]\ J_{n,H}^\ell(z) = \frac{2}{\pi}\int_0^\pi \cos n\theta\ \cos_H^\ell(z \sin\theta)\mathrm{d}\theta, \tag{3.11}$$

$$[1 - (-1)^n]\ J_{n,H}^\ell(z) = \frac{2}{\pi}\int_0^\pi \sin n\theta\ \sin_H^\ell(z \sin\theta)\mathrm{d}\theta. \tag{3.12}$$

Further,

$$J_{2n,H}^\ell(z) = \frac{1}{\pi}\int_0^\pi \cos 2n\theta\ \cos_H^\ell(z \sin\theta)\mathrm{d}\theta, \tag{3.13}$$

$$J_{2n+1,H}^\ell(z) = \frac{1}{\pi}\int_0^\pi \sin(2n+1)\theta\ \sin_H^\ell(z \sin\theta)\mathrm{d}\theta \tag{3.14}$$

and

$$\int_0^\pi \cos(2n+1)\theta\ \cos_H^\ell(z \sin\theta)\mathrm{d}\theta = 0, \tag{3.15}$$

$$\int_0^\pi \sin 2n\theta \ \sin_H^\ell(z\sin\theta)\mathrm{d}\theta = 0. \tag{3.16}$$

Proof. From the ℓ-HBF integral, we have

$$J_{n,H}^\ell(z) = \frac{1}{\pi}\int_0^\pi \left[\cos n\theta \ \cos_H^\ell(z\sin\theta) + \sin n\theta \ \sin_H^\ell(z\sin\theta)\right] \mathrm{d}\theta \tag{3.17}$$

$$\Rightarrow J_{-n,H}^\ell(z) = \frac{1}{\pi}\int_0^\pi \left[\cos(-n\theta) \ \cos_H^\ell(z\sin\theta) + \sin(-n\theta) \ \sin_H^\ell(z\sin\theta)\right] \mathrm{d}\theta$$

(3.18)

$$\Rightarrow (-1)^n J_{n,H}^\ell(z) = \frac{1}{\pi}\int_0^\pi \left[\cos n\theta \ \cos_H^\ell(z\sin\theta) - \sin n\theta \ \sin_H^\ell(z\sin\theta)\right] \mathrm{d}\theta.$$

Hence adding (subtracting) (3.17) and (3.18) we obtain (3.11) ((3.12)).

If n is an even (odd) integer, then (3.11) ((3.12)) yields (3.13) ((3.14)).

By considering even ordered ℓ-HBF in (3.11) and odd ordered ℓ-HBF in (3.12) yield (3.13) and (3.14) respectively.

Similarly, if the order n is an odd (even) integer, then (3.11) ((3.12)) furnishes (3.15) ((3.16)). □

3.3. ℓ-Analogue of Bessel's inequality due to Cauchy

Here we first obtain the ℓ-analogue of the inequality [10, p. 16]

$$\frac{(n+r)!}{n!} \geq (n+1)^r.$$

Lemma 3.8. *If $\Re(\ell) \geq 0$ and $r \in \mathbb{N} \cup \{0\}$, then*

$$\frac{((n+r)!)^{\ell n+1}}{(n!)^{\ell n+1}} \ \geq \ (n+1)^{r(\ell n+1)}. \tag{3.19}$$

Proof. For $r = 0$, (3.19) is evident. For the remaining values of r, we use the principle of mathematical induction.

For $r = 1$,

$$\text{L.H.S.} = \frac{((n+1)!)^{\ell n+1}}{(n!)^{\ell n+1}} = (n+1)^{\ell n+1} = \text{R.H.S.}$$

Suppose that (3.19) is true for $r =$ some positive integer k. That is

$$\frac{((n+k)!)^{\ell n+1}}{(n!)^{\ell n+1}} \ \geq \ (n+1)^{k(\ell n+1)} \tag{3.20}$$

holds true. Then for $r = k+1$, it suffice to prove

$$\frac{((n+k+1)!)^{\ell n+1}}{(n!)^{\ell n+1}} \ \geq \ (n+1)^{(k+1)(\ell n+1)}.$$

We begin with the left hand side of this inequality and make use of the assumption in (3.20) to get

$$\begin{aligned}\frac{((n+k+1)!)^{\ell n+1}}{(n!)^{\ell n+1}} &= (n+k+1)^{\ell n+1}\ \frac{((n+k)!)^{\ell n+1}}{(n!)^{\ell n+1}}\\ &\geq (n+k+1)^{\ell n+1}\ (n+1)^{k(\ell n+1)}\\ &\geq (n+1)^{\ell n+1}\ (n+1)^{k(\ell n+1)}\\ &= (n+1)^{(k+1)(\ell n+1)},\end{aligned}$$

whenever $\Re(\ell) \geq 0$. □

Using this, we establish below the ℓ-Bessel's inequality due to Cauchy.

Theorem 3.9. *For $\ell, n \in \mathbb{N} \cup \{0\}$,*

$$\left|J^{\ell}_{n,H}(z)\right| \leq \frac{\left|\frac{z}{2}\right|^n}{(n!)^{\ell n+1}}\ e^{\ell}_{H}\left(\left|\frac{z^2}{4}\right|\right). \tag{3.21}$$

Proof. From Definition 2 (of ℓ-HBF) we have

$$\left|J^{\ell}_{n,H}(z)\right| \leq \left|\frac{z}{2}\right|^n\ \sum_{k=0}^{\infty} \frac{\left|\frac{z}{2}\right|^{2k}}{((n+k)!)^{\ell n+\ell k+1}\ (k!)^{\ell k+1}} \tag{3.22}$$

in which

$$\frac{1}{((n+k)!)^{\ell n+\ell k+1}} \leq \frac{1}{((n+k)!)^{\ell k}\ (n!)^{\ell n+1}\ (n+1)^{k(\ell n+1)}}$$

in view of Lemma 3.8. Thus (3.22) leads us to

$$\begin{aligned}\left|J^{\ell}_{n,H}(z)\right| &\leq \frac{\left|\frac{z}{2}\right|^n}{(n!)^{\ell n+1}} \sum_{k=0}^{\infty} \frac{\left|\frac{z}{2}\right|^{2k}}{((n+k)!)^{\ell k}\ (n!)^{\ell n+1}\ (n+1)^{k(\ell n+1)}\ (k!)^{\ell k+1}}\\ &\leq \frac{\left|\frac{z}{2}\right|^n}{(n!)^{\ell n+1}} \sum_{k=0}^{\infty} \frac{\left|\frac{z}{2}\right|^{2k}}{(k!)^{\ell k+1}}\\ &= \frac{\left|\frac{z}{2}\right|^n}{(n!)^{\ell n+1}}\ e^{\ell}_{H}\left(\left|\frac{z^2}{4}\right|\right),\end{aligned}$$

when $\ell, n \in \mathbb{N} \cup \{0\}$. □

Remark 3.10. The special case $\ell = 0$ yields the inequality [10, Eq. (14), p. 16]:

$$|J_n(z)| \leq \frac{\left|\frac{z}{2}\right|^n}{n!}\ \exp\left(\left|\frac{z^2}{4}\right|\right).$$

3.4. Addition formula

Theorem 3.11. *For the ℓ-HBF, the following equality holds* (*cf.* [10, Sec. 2.4, 2.5, p. 30])

$$J^{\ell}_{n,H}(z_1+z_2) = \sum_{m=-\infty}^{\infty} J^{\ell}_{m,H}(z_1)\ J^{\ell}_{n-m,H}(z_2). \tag{3.23}$$

Proof. On substituting $z = z_1+z_2$ in the contour integral (2.11) and then using the property (1.14), we obtain

$$\begin{aligned}
&J^{\ell}_{n,H}(z_1+z_2)\\
&= \frac{1}{2\pi i}\int^{(0+)} u^{-n-1}\ e^{\ell}_{H}\left(\frac{(z_1+z_2)\ (u+_{\ell}(-u^{-1}))}{2}\right)\mathrm{du}\\
&= \frac{1}{2\pi i}\int^{(0+)} u^{-n-1}\ e^{\ell}_{H}\left(\frac{z_1}{2}(u+_{\ell}(-u^{-1}))\right)\ e^{\ell}_{H}\left(\frac{z_2}{2}(u+_{\ell}(-u^{-1}))\right)\mathrm{du}\\
&= \frac{1}{2\pi i}\int^{(0+)} u^{-n-1}\ \sum_{m=-\infty}^{\infty} J^{\ell}_{m,H}(z_1)\ u^m\ e^{\ell}_{H}\left(\frac{z_2}{2}(u+_{\ell}(-u^{-1}))\right)\mathrm{du}\\
&= \sum_{m=-\infty}^{\infty} J^{\ell}_{m,H}(z_1)\ \frac{1}{2\pi i}\int^{(0+)} u^{m-n-1}\ e^{\ell}_{H}\left(\frac{z_2}{2}(u+_{\ell}(-u^{-1}))\right)\mathrm{du}.
\end{aligned}$$

Once again making an appeal to (2.11), we are led to (3.23). □

Corollary 3.12. *The following series relation holds.*

$$J^{\ell}_{n,H}(2z) = \sum_{m=0}^{n} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z) + 2\sum_{m=1}^{\infty}(-1)^m\ J^{\ell}_{m,H}(z)\ J^{\ell}_{n+m,H}(z). \tag{3.24}$$

Proof. If $z_1 = z_2 = z$ in (3.23), then

$$\begin{aligned}
J^{\ell}_{n,H}(2z) &= \sum_{m=-\infty}^{\infty} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z)\\
&= \sum_{m=-\infty}^{-1} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z) + \sum_{m=0}^{\infty} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z)\\
&= \sum_{m=-\infty}^{-1} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z) + \sum_{m=0}^{n} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z)\\
&\quad + \sum_{m=n+1}^{\infty} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z).
\end{aligned}$$

This in view of (1.16) gives

$$J^{\ell}_{n,H}(2z) = \sum_{m=0}^{n} J^{\ell}_{m,H}(z)\ J^{\ell}_{n-m,H}(z) + \sum_{m=1}^{\infty}(-1)^m\ J^{\ell}_{m,H}(z)\ J^{\ell}_{n+m,H}(z)$$

$$+\sum_{m=1}^{\infty}(-1)^m\, J^{\ell}_{m,H}(z)\, J^{\ell}_{n+m,H}(z)$$
$$=\sum_{m=0}^{n} J^{\ell}_{m,H}(z)\, J^{\ell}_{n-m,H}(z)+2\sum_{m=1}^{\infty}(-1)^m\, J^{\ell}_{m,H}(z)\, J^{\ell}_{n+m,H}(z). \qquad \square$$

Remark 3.13. The newly defined function ℓ-HBF (1.9) can be considered as an extension to the hypergeometric function ${}_0F_q$ where q in the second index goes to infinity together with the summation index k in the power series. Consequently, this leads us the construction of the hyper-Bessel type differential operator which helps in establishing an infinite order differential equation satisfied by the new class of Bessel functions defined by (1.9). Noticing that the differential equations of infinite order appear in the perturbative approach to the p-adic string theory [1] as well as in the tachyon field in open string field theory [1], it may be of the interest to examine the occurrence of the ℓ-HBF in these theories.

The graphs of the ℓ-H Bessel functions of different order with different scales are shown in Figures 1-4.

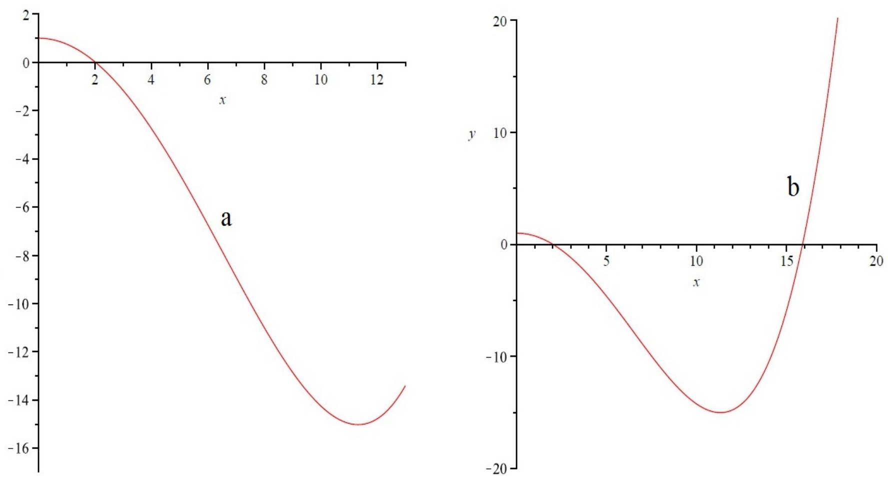


FIGURE 1. a,b: Graphs of $J^1_{0,H}(x)$ with different scales

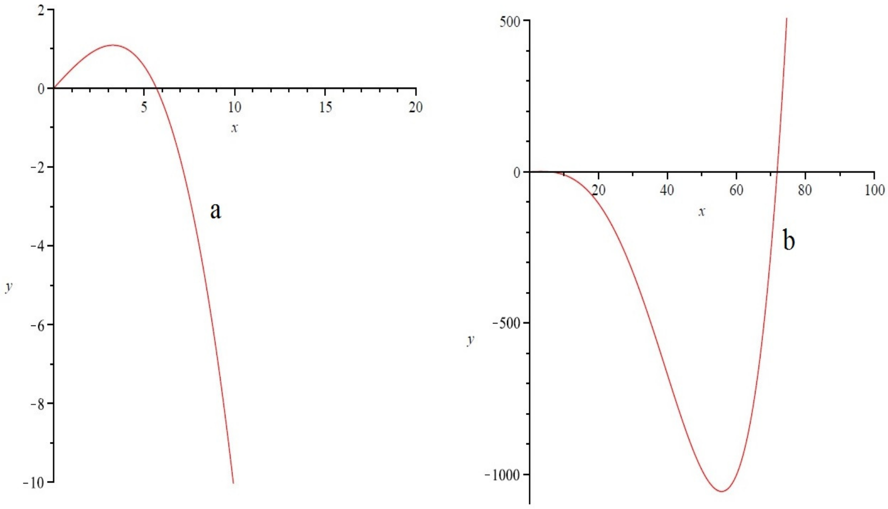


FIGURE 2. a,b: Graphs of $J^1_{1,H}(x)$ with different scales

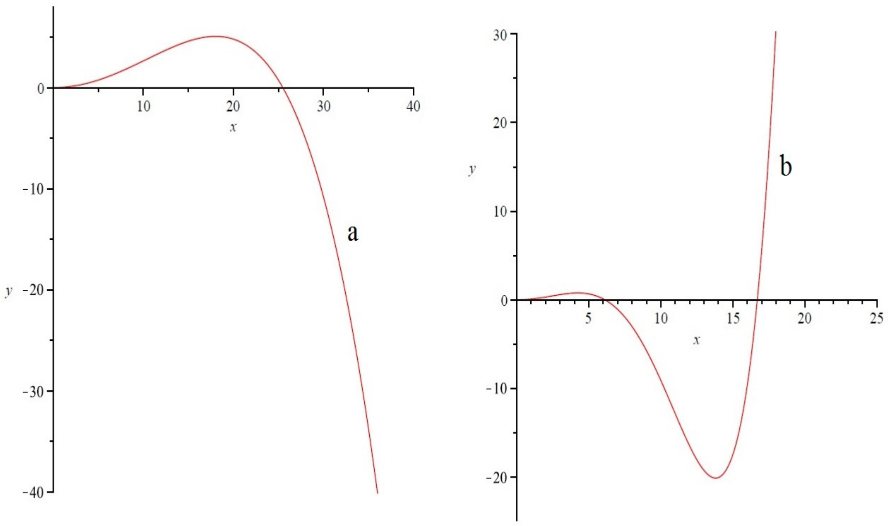


FIGURE 3. a,b: Graphs of $J^1_{2,H}(x)$ with different scales

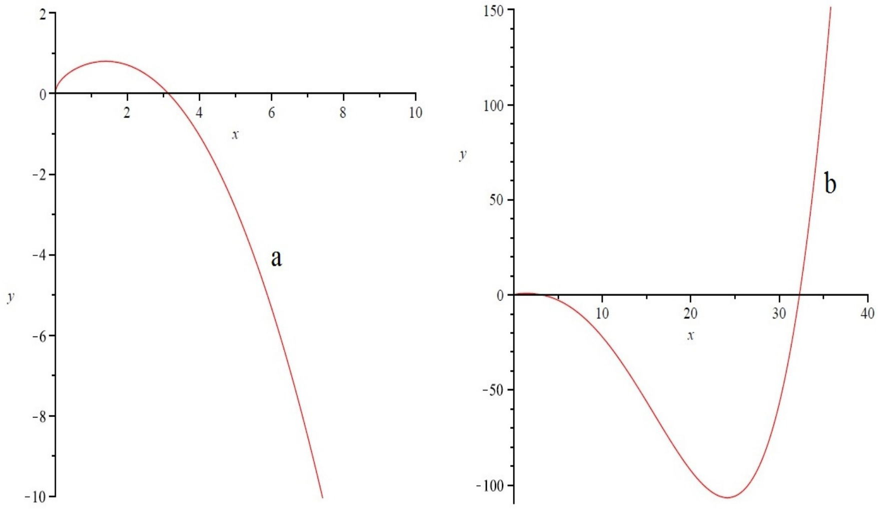


FIGURE 4. a,b: Graphs of $J^{1}_{\frac{1}{2},H}(x)$ with different scales

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