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BIHARMONIC-KIRCHHOFF TYPE EQUATION INVOLVING CRITICAL SOBOLEV EXPONENT WITH SINGULAR TERM

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Abstract. Using variational methods, we show the existence of a unique weak solution of the following singular biharmonic problems of Kirchhoff type involving critical Sobolev exponent:

$$(\mathcal{P}_\lambda)\begin{cases}\Delta^2u-(a\int_\Omega|\nabla u|^2dx+b)\Delta u+cu=f(x)\,|u|^{-\gamma}-\lambda\,|u|^{p-2}u & \text{in }\Omega,\\ \Delta u=u=0 & \text{on }\partial\Omega,\end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$ $(n\geq 5)$, Δ^2 is the biharmonic operator, and ∇u denotes the spatial gradient of u and $0<\gamma<1$, $\lambda>0$, $0<p\leq 2^\sharp$ and a,b,c are three positive constants with $a+b>0$ and f belongs to a given Lebesgue space.

1. Introduction

In this work, we are concerned with a class of singular biharmonic problems of Kirchhoff type involving critical Sobolev exponent:

$$(\mathcal{P}_\lambda)\quad\begin{cases}\Delta^2u-(a\int_\Omega|\nabla u|^2dx+b)\Delta u+cu=f(x)\,|u|^{-\gamma}-\lambda\,|u|^{p-2}u & \text{in }\Omega,\\ \Delta u=u=0 & \text{on }\partial\Omega,\end{cases}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n\,(n\geq 5)$, Δ^2 is the biharmonic operator, and ∇u denotes the spatial gradient of u and $0<\gamma<1$, $\lambda>0$, $0<p\leq 2^\sharp$ and a,b,c are three positive constants with $a+b>0$ and $f\in L^q(\Omega)$ with $q:=\frac{2^\sharp}{2^\sharp+\gamma-1}$ satisfying $f(x)>0$ for almost every $x\in\Omega$, and $2^\sharp:=\frac{2n}{n-4}$ denotes the critical Sobolev exponent for the embedding $H^2(\Omega)\hookrightarrow L^{2^\sharp}(\Omega)$.

Historically, important developments have been achieved regarding the existence of solutions of critical biharmonic elliptic problems. We recall briefly few of them.

In [7], using the upper and lower solutions and monotone iterative methods Wang has proved the existence of nontrivial solution of the following fourth

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order elliptic problem:

$$(\mathcal{P}_1)\begin{cases}\Delta^2 u + a(x)\Delta u + c(x)u = f(x,u,\nabla u,\Delta u) & \text{in } \Omega,\\ \Delta u = u = 0 & \text{on } \partial\Omega,\end{cases}$$

when Ω is a bounded domain of $\mathbb{R}^n$.

The problem $(\mathcal{P}_\lambda)$ is called nonlocal because of the presence of the integral over the entire domain Ω, which implies that the equation in $(\mathcal{P}_\lambda)$ is no longer a pointwise identity.

The original one-dimensional Kirchhoff equation was introduced by Kirchhoff himself [8] in 1883. His model takes into account the changes in length of the strings produced by transverse vibrations. This problem $(\mathcal{P}_\lambda)$ is related to the stationary analog of the evolution equation of Kirchhoff type:

$$(\mathcal{P}_2)\begin{cases}u_{tt} + \Delta^2 u - \lambda(a\int_\Omega |\nabla u|^2 dx + b)\Delta u = h(x,u) & \text{in } \Omega\times(0,\ T),\\ u = 0 & \text{in } \partial\Omega\times(0,\ T),\\ u(x,\ 0) = v(x),\ u_t(x,\ 0) = w(x),\end{cases}$$

where T is a positive constant, v and w are given functions. Dimensions one and two are relevant from the point of view of physics and engineering because in those situations model is considered a good approximation for describing nonlinear vibrations of beams or plate.

Nonlocal problems arise not only from mathematical and physical fields but also from several other branches. When they appear in biological systems, u describes a process depending on the average of itself as population density. Their theoretical study has attracted a lot of interests from mathematicians for a long time and many works have been done. We quote in particular the famous article of Lions [9]. However in most of papers, the used approach relies on topological methods.

In addition, in the last two decades the nonlocal fourth-order equation:

$$(\mathcal{P}_3)\begin{cases}\Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\cdot\Delta u = f(x,u) & \text{in } \Omega,\\ \Delta u = u = 0 & \text{on } \partial\Omega,\end{cases}$$

has also been studied by many authors, in many domains like micro-electro-mechanical systems, surface diffusion on solids, thin film theory. We refer the readers to [1,3,4,6,8,12–14]. Particularly, in [15] Wang has studied the following fourth-order equation of Kirchhoff type equation:

$$(\mathcal{P}_4)\begin{cases}\Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\cdot\Delta u = f(x,u) & \text{in } \Omega,\\ \Delta u = u = 0 & \text{on } \partial\Omega,\end{cases}$$

where λ is a positive parameter. The authors showed that there exists such that the fourth-order elliptic equation has a nontrivial solution for $0<\lambda<\lambda^*$ by using the mountain pass iterative techniques and the truncation method. Massar and al. [10] employing a smooth version of Ricceri's variational principle [11], the authors ensured the existence of infinitely many solutions for fourth-order Kirchhoff-type elliptic problems.

In [5], using variational methods and critical point theory, the authors established multiplicity results of nontrivial and nonnegative solutions for a fourth-order Kirchhoff-type elliptic problem, by combining an algebraic condition on the nonlinear term with the classical Ambrosetti-Rabinowitz condition.

After that, many authors studied the following nonlocal elliptic boundary value problem:

$$(\mathcal{P}_5)\begin{cases}-M\left(\int_\Omega |\nabla u|^2 dx\right).\Delta u = f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Problems like $(\mathcal{P}_5)$ can be used for modeling several physical and biological systems where u describes a process which depends on the average of itself, such as the population density, see [3].

Motivated by these works in [2], to study problem $(\mathcal{P}_\lambda)$, we combine the term dominant of bi-Laplacian with the Kirchhoff coefficient.

Notation 1. In this paper, we make use of the following notation:

$L^p(\Omega)$ for $1 \le p < \infty$, denote Lebesgue spaces; the norm in $L^p(\Omega)$ is given by

$$||u||_{L^p(\Omega)} := \left(\int_\Omega |u|^p dx\right)^{1/p},$$

$H_0^1(\Omega)$ denotes the completion of the space $C_c^\infty(\Omega)$ in the norm

$$||u||^2_{H_0^1(\Omega)} := \int_\Omega |\nabla u|^2 dx$$

and $H^2(\Omega)$ denotes the completion of the space $C_c^\infty(\Omega)$ in the norm

$$||u||^2_{H^2(\Omega)} := \int_\Omega |\Delta u|^2 dx$$

S is the best Sobolev constant to the Sobolev embedding $H^2(\Omega) \hookrightarrow L^{2^\sharp}(\Omega)$, that is

$$S := \inf_{\left\{u\in H^2(\Omega) \text{ and } \int_\Omega |u|^{2^\sharp} dx = 1\right\}} \int_\Omega |\Delta u|^2 dx.$$

It is well known that $\Omega = \mathbb{R}^n$ the best constant is attained by the functions

$$u_\epsilon(x) := \alpha_n \left(\frac{\epsilon}{|x-x_0|^2+\epsilon^2}\right)^{\frac{n-4}{2}},$$

where

$$\alpha_n := \left(n(n-4)\left(n^2-4\right)\right)^{\frac{n-4}{8}}.$$

Let X be the space $H^2(\Omega)\cap H_0^1(\Omega)$ endowed with the norm:

$$||u||_X^2 := \int_\Omega |\Delta u|^2 + a|\nabla u|^2 + cu^2 dx.$$

For $u \in X$, we define the functional J_λ by:

$$J_\lambda(u) = \frac{1}{2}||u||_X^2 + \frac{a}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 + \frac{\lambda}{p}\int_\Omega |u|^p \, dx + \frac{1}{\gamma - 1}\int_\Omega f\left(x\right)|u|^{1-\gamma} \, dx.$$

Note that a function u is called a weak solution of problem $(\mathcal{P}_\lambda)$ if $u \in X$ such that, for all $\varphi \in X$:

$$\begin{aligned}&\int_\Omega \left(\Delta u \Delta \varphi + cu\varphi\right) dx + (a\int_\Omega |\nabla u|^2 dx + b) \times \int_\Omega \nabla u \nabla \varphi dx \\ = & -\lambda \int_\Omega |u|^{p-2} \, u\varphi dx + \int_\Omega f\left(x\right)|u|^{-\gamma}\, \varphi dx.\end{aligned}$$

Throughout this article, we make the following assumptions:

(H^1) $0 < \gamma < 1$ and $0 < p \leq 2^\sharp := \frac{2n}{n-4}$.

(H^2) $f \in L^q\left(\Omega\right)$ with $q := \frac{2^\sharp}{2^\sharp + \gamma - 1}$ satisfying $f(x) > 0$ for almost every $x \in \Omega$.

(H^3) For all $x \in X$:

$$||u||_X^2 := \int_\Omega |\Delta u|^2 + a|\nabla u|^2 + cu^2 dx.$$

The main result can be described as follows.

Theorem 1.1. *Assume (H^1), (H^2) and (H^3) hold. Then the problem possesses a positive and a unique weak solution. Moreover, this solution is a global minimizer.*

This work is organized as follows: In Section 2 we give some preliminary results which we will use later. Section 3 is devoted to the proof of the main Theorem.

2. Some preliminary results

We give the following useful lemmas.

Lemma 2.1. *The energy functional J_λ is coercive and bounded from below on X.*

Proof. Since $0 < \gamma < 1$, $\lambda > 0$, by Hölder inequality, we have

$$\begin{aligned}&\left|\int_\Omega f\left(x\right)|u|^{(1-\gamma)}\, dx\right| \\ \leq & \left(\int_\Omega |f\left(x\right)|^{\frac{2^\sharp}{2^\sharp+\gamma-1}}\, dx\right)^{\frac{2^\sharp+\gamma-1}{2^\sharp}} \times \left(\int_\Omega |u|^{(1-\gamma)\frac{2^\sharp}{(1-\gamma)}}\, dx\right)^{\frac{(1-\gamma)}{2^\sharp}} \\ = & \, ||f||_{L^q(\Omega)} \times ||u||_{L^{2^\sharp}(\Omega)}^{(1-\gamma)} \quad \text{with} \quad q := \frac{2^\sharp}{2^\sharp + \gamma - 1}.\end{aligned}$$

Furthermore, by the Sobolev embedding $H^2(\Omega) \hookrightarrow L^{2^\sharp}(\Omega)$ and using $X \hookrightarrow H^2(\Omega)$, we obtain that there exists a constant $\Lambda > 0$ such that

$$||u||_{L^{2^\sharp}(\Omega)}^{(1-\gamma)} \leq (1-\gamma)\,\Lambda ||u||_X^{(1-\gamma)}.$$

Hence

$$J_\lambda(u) = \frac{1}{2}||u||_X^2 + \frac{a}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 + \frac{\lambda}{p}\int_\Omega h(x)\,|u|^p\,dx + \frac{1}{\gamma-1}\int_\Omega \frac{f(x)}{|u|^{\gamma-1}}dx,$$

$$J_\lambda(u) \geq \frac{1}{2}||u||_X^2 + \frac{a}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 - \Lambda ||u||_X^{(1-\gamma)}.$$

This implies that J_λ is coercive.

To prove that J_λ is bounded from below on X, we divided in two cases:
Case I: If $||u||_X \geq 1$ and $0 < \gamma < 1$, then

$$J_\lambda(u) \geq \left(\frac{1}{2} - \Lambda\right) ||u||_X^{(1-\gamma)}.$$

Case II: If $0 < ||u||_X \leq 1$ and $0 < \gamma < 1$, then

$$J_\lambda(u) \geq -\Lambda.$$

Thus, J_λ is bounded from below on X. □

Lemma 2.2. *The functional J_λ has a minimum m_λ in X with $m_\lambda < 0$.*

Proof. Since J_λ is coercive and bounded from below on X in Lemma 2.1,

$$m_\lambda := \inf_{u \in X} J_\lambda(u)$$

is well defined.

Moreover, since $0 < \gamma < 1$ and $f(x) > 0$ almost every $x \in \Omega$, we have $J_\lambda(t\rho) < 0$ for all $\rho \neq 0$ and small $t > 0$.

Thus, we obtain

$$m_\lambda := \inf_{u \in X} J_\lambda(u) < 0.$$

The proof is complete. □

We have the following important result.

Lemma 2.3. *Assume that conditions (H^1) and (H^2) hold. Then the functional J_λ attains the global minimizer in X, that is, there exists $u^* \in X$ such that*

$$m_\lambda := J_\lambda(u^*) < 0.$$

Proof. From Lemma 2.1, there exists a minimizing sequence $(u_m)_m$ in X such that

$$J_\lambda(u_m) = m_\lambda + o(1).$$

Using Lemma 2.2, the sequence $(u_m)_m$ is bounded in X and from the reflexivity of X and the compact embedding theorem, up to a subsequence still noted $(u_m)_m$ there exists $u \in X$ such that

(1) $u_m \to u^*$ weakly in X.
(2) $u_m \to u^*$ strongly in $L^p(\Omega)$ for $1 \leq p < 2^\sharp$.
(3) $u_m(x) \to u^*(x)$ a.e in Ω.

After these preliminaries we can prove that $w_m = u_m - u^*$ converges to 0 strongly in X.

By Vitali's theorem, we find that

$$\int_\Omega f(x)\,|u_m|^{1-\gamma}\,dx = \int_\Omega f(x)\,|u^*|^{1-\gamma}\,dx + o(1).$$

Moreover, by using Brézis-Lieb Lemma, we obtain

$$\|u_m\|_X^2 - \|w_m\|_X^2 = \|u^*\|_X^2 + o(1)$$

and

$$\int_\Omega h(x)\,|u_m|^{2^\sharp}\,dx = \int_\Omega h(x)\,|w_m|^{2^\sharp}\,dx + \int_\Omega h(x)\,|u^*|^{2^\sharp}\,dx + o(1),$$

where $o(1)$ is an infinitesimal as $m \to +\infty$.

Hence, in the case that $0 < p < 2^\sharp$, we deduce that

$$\begin{aligned}
m_\lambda &= \lim_{m\to+\infty} J_\lambda(u_m) \\
&= \lim_{m\to+\infty} \left[\frac{1}{2}||u_m||_X^2 + \frac{a}{4}\left(\int_\Omega |\nabla u_m|^2 dx\right)^2 + \frac{\lambda}{p}\int_\Omega h(x)\,|u_m|^p\,dx \right.\\
&\qquad \left. + \frac{1}{\gamma-1}\int_\Omega f(x)\,|u_m|^{1-\gamma}\,dx\right] \\
&= \lim_{m\to+\infty} \frac{1}{2}\left(\|w_m\|_X^2 + \|u^*\|_X^2\right) + \frac{a}{4}\left(\int_\Omega |\nabla u^*|^2 dx\right)^2 \\
&\quad + \frac{\lambda}{p}\int_\Omega h(x)\,|u^*|^p\,dx + \frac{1}{\gamma-1}\int_\Omega f(x)\,|u^*|^{1-\gamma}\,dx,
\end{aligned}$$

$$m_\lambda = J_\lambda(u^*) + \frac{1}{2}\lim_{m\to+\infty} \|w_m\|_X^2 .$$

Then

$$m_\lambda = J_\lambda(u^*) + \frac{1}{2}\lim_{m\to+\infty} \|w_m\|_X^2 \geq J_\lambda(u^*) \geq \inf_{u_m\in X} J_\lambda(u_m) = m_\lambda$$

which implies that

$$m_\lambda = J_\lambda(u^*).$$

In the case $p = 2^\sharp$, it follows that

$$m_\lambda = J_\lambda(u^*) + \lim_{m\to+\infty}\left(\frac{1}{2}\|w_m\|_X^2 + \frac{\lambda}{p}\int_\Omega h(x)\,|w_m|^p\,dx\right).$$

Then

$$m_\lambda \geq J_\lambda(u^*) \geq m_\lambda$$

which yields that

$$m_\lambda = J_\lambda(u^*).$$

Thus

$$\inf_{u_m \in X} J_\lambda(u_m) = J_\lambda(u^*). \qquad \square$$

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We only need to prove that u^* is a weak solution of the problem $(\mathcal{P}_\lambda)$ in Ω. Firstly, we show that u^* is a weak solution of problem $(\mathcal{P}_\lambda)$. From Lemma 2.2, we see that

$$\min J_\lambda(u^* + tv) = J_\lambda(u^* + tv)|_{t=0} = J_\lambda(u^*), \forall v \in X.$$

This implies that

$$\begin{aligned} &\int_\Omega (\Delta u^* \Delta v + cu^* v)\, dx + (a \int_\Omega |\nabla u^*|^2 dx + b) \times \int_\Omega \nabla u^* \nabla v dx \\ = &-\lambda \int_\Omega h(x) |u^*|^{p-2} u^* v dx + \int_\Omega \frac{f(x)}{|u^*|^\gamma} v dx, \end{aligned}$$

$$\begin{aligned} 0 &\le \frac{J_\lambda(u^* + tv) - J_\lambda(u^*)}{t} \\ &= \frac{1}{2}\left(\frac{||u^* + tv||_X^2 - ||u^*||_X^2}{t}\right) \\ &+ \frac{a}{4t}\left[\left(\int_\Omega |\nabla u^* + tv|^2 dx\right)^2 - \left(\int_\Omega |\nabla u^*|^2 dx\right)^2\right] \\ &- \frac{1}{\gamma - 1}\int_\Omega f(x)\left(\frac{|u^* + tv|^{1-\gamma} - |u^*|^{1-\gamma}}{t}\right) dx \\ &+ \frac{\lambda}{p}\int_\Omega h(x)\left(\frac{|u^* + tv|^p - |u^*|^p}{t}\right) dx. \end{aligned} \tag{2.1}$$

Using the Lebesgue Dominated Convergence Theorem, we have

$$\frac{1}{p}\lim_{t\to 0^+}\int_\Omega h(x)\left(\frac{|u^* + tv|^p - |u^* + tv|^p}{t}\right) dx = \int_\Omega h(x) |u^*|^{p-2} u^* v dx. \tag{2.2}$$

For any $x \in \Omega$, we denote

$$g(t) := f(x)\left(\frac{|u^*(x) + tv(x)|^{1-\gamma} - |u^*(x)|^{1-\gamma}}{(\gamma - 1)t}\right).$$

Then

$$g'(t) := f(x)\left(\frac{(u^*(x))^{1-\gamma} - (\gamma t v(x) + u^*(x)) |u^*(x) + tv(x)|^{-\gamma}}{(1-\gamma)t^2}\right) \le 0,$$

which implies that $g(t)$ is non increasing for $t > 0$. Moreover, we have

$$\lim_{t\to 0^+} g(t) = f(x)(u^*(x))^{-\gamma} v(x)$$

for every $x \in \Omega$, which may be $+\infty$ when $u^*(x) = 0$ and $v(x) > 0$. Consequently, by the Monotone Convergence Theorem, we obtain

$$\begin{aligned} &\lim_{t\to 0^+}\left[\frac{1}{\gamma-1}\int_\Omega f(x)\left(\frac{|u^*+tv|^{1-\gamma}-|u^*|^{1-\gamma}}{t}\right)dx\right] \\ &= \int_\Omega f(x)(u^*(x))^{-\gamma}v(x)\,dx, \end{aligned}$$

which possibly equals to $+\infty$. Combining this with (2.1), let $t \to 0^+$, it follows from (2.2) that

$$\begin{aligned} &\int_\Omega f(x)(u^*(x))^{-\gamma}v(x)\,dx \\ &\le \int_\Omega \Delta u^*\Delta v(x)\,dx + (a\int_\Omega |\nabla u^*|^2dx + b)\int_\Omega \nabla u^*\nabla v(x)\,dx \\ &\quad + c\int_\Omega u^*v(x)\,dx + \lambda\int_\Omega h(x)|u^*|^{p-2}u^*v(x)\,dx \end{aligned} \tag{2.3}$$

for all $v \in X$ with $v > 0$.

Let $e_1 \in X$ be the first eigenfunction of the operator Δ^2 with $e_1 > 0$ and $\|e_1\| = 1$. Particularly, taking $v = e_1$ in (2.3), one gets that

$$\begin{aligned} &\int_\Omega f(x)(u^*(x))^{-\gamma}e_1dx \\ &\le \int_\Omega \Delta u^*\Delta e_1dx + (a\int_\Omega |\nabla u^*|^2dx + b)\int_\Omega \nabla u^*\nabla e_1dx \\ &\quad + c\int_\Omega u^*e_1dx + \lambda\int_\Omega h(x)|u^*|^{p-2}u^*e_1dx < \infty, \end{aligned}$$

which implies that $u^* > 0$ for almost every $x \in \Omega$.

Finally, we prove the uniqueness of solutions of problem $(\mathcal{P}_\lambda)$. Assume that w^* is another solution of problem $(\mathcal{P}_\lambda)$, then it follows from (2.3) that

$$\begin{aligned} &\int_\Omega \Delta u^*\Delta w^*dx + (a\int_\Omega |\nabla u^*|^2dx + b)\int_\Omega \nabla u^*\nabla w^*dx + c\int_\Omega u^*w^*dx \\ &+ \lambda\int_\Omega h(x)|u^*|^{p-2}u^*w^*dx - \int_\Omega f(x)(u^*(x))^{-\gamma}w^*dx = 0 \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &\int_\Omega \Delta u^*\Delta w^*dx + (a\int_\Omega |\nabla w^*|^2dx + b)\int_\Omega \nabla u^*\nabla w^*dx + c\int_\Omega u^*w^*dx \\ &+ \lambda\int_\Omega h(x)|w^*|^{p-2}u^*w^*dx - \int_\Omega f(x)(w^*(x))^{-\gamma}u^*dx = 0. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we obtain

$$a\int_\Omega |\nabla(u^*-w^*)|^2dx\int_\Omega \nabla u^*\nabla w^*dx + \lambda\int_\Omega h(x)\left(|u^*|^{p-2}-|w^*|^{p-2}\right)u^*w^*dx$$

$$-\int_{\Omega} f(x)\left((u^*(x))^{-\gamma} w^* - (w^*(x))^{-\gamma} u^*\right) dx = 0.$$

Put

$$\begin{aligned} H(u^*, w^*) := a \int_{\Omega} |\nabla (u^* - w^*)|^2 dx \int_{\Omega} \nabla u^* \nabla w^* dx \\ + \lambda \int_{\Omega} h(x)\left(|u^*|^{p-2} - |w^*|^{p-2}\right) u^* w^* dx \\ - \int_{\Omega} f(x)\left[(u^*(x))^{-\gamma} w^* - (w^*(x))^{-\gamma} u^*\right] dx = 0. \end{aligned}$$

Using the Hölder inequality, one has

$$H(u^*, w^*) \geq 0.$$

Since $0 < \gamma < 1$ and $p > 0$, it is well clear the following inequalities:

$$\forall m, n > 0 : \begin{cases} (m^p - n^p)(m - n) \geq 0, \\ (m^{-\gamma} - n^{-\gamma})(m - n) \leq 0. \end{cases}$$

Thus

$$\int_{\Omega} h(x)\left(|u^*|^{p-2} - |w^*|^{p-2}\right) u^* w^* dx \leq 0$$

and

$$\int_{\Omega} f(x)\left[(u^*(x))^{-\gamma} w^* - (w^*(x))^{-\gamma} u^*\right] dx \geq 0.$$

Consequently we obtain,

$$\|u^* - w^*\|_X^2 < 0.$$

This completes the proof of the Theorem. □

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