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A GENERALIZATION OF THE LAGUERRE POLYNOMIALS

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Abstract. The main aim of this paper is to introduce and study the generalized Laguerre polynomials and prove that these polynomials are characterized by the generalized hypergeometric function. Also we investigate some properties and formulas for these polynomials such as explicit representations, generating functions, recurrence relations, differential equation, Rodrigues formula, and orthogonality.

1. Introduction and preliminaries

Laguerre polynomials are among the most important and useful polynomials in mathematics and mathematical physics. Most of monographs and books related to special functions include Laguerre polynomials (see, e.g., [3, 15, 16]). Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by (see, e.g., [15, Chapter 12])

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_1F_1(-n\,;\, 1+\alpha\,;\, x) \tag{1}$$

$$\left(n \in \mathbb{N}_0,\, 1+\alpha \in \mathbb{C}\setminus\mathbb{Z}_0^-,\, x \in \mathbb{C}\right),$$

where ${}_1F_1$ is a particular case of the well-known generalized hypergeometric series ${}_pF_q$ $(p,\, q \in \mathbb{N}_0)$ given by (see, e.g., [15, p. 73]):

$$\begin{aligned} {}_pF_q\left[\begin{matrix}\lambda_1,\,\ldots,\,\lambda_p\,; \\ \mu_1,\,\ldots,\,\mu_q\,;\end{matrix}\; z\right] &= \sum_{n=0}^{\infty} \frac{(\lambda_1)_n \cdots (\lambda_p)_n}{(\mu_1)_n \cdots (\mu_q)_n}\frac{z^n}{n!} \\ &= {}_pF_q(\lambda_1,\,\ldots,\,\lambda_p;\, \mu_1,\,\ldots,\,\mu_q;\, z). \end{aligned} \tag{2}$$

Here $(\alpha)_\beta$ denotes the Pochhammer symbol defined (for $\alpha,\, \beta \in \mathbb{C}$) by

$$(\alpha)_\beta := \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} = \begin{cases} 1 & (\beta = 0;\ \alpha \neq 0) \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & (\beta = n \in \mathbb{N}), \end{cases} \tag{3}$$

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Γ being the familiar Gamma function and it being read traditionally that $(\alpha)_0 := 1$. Here and elsewhere, let $\mathbb{N}$, $\mathbb{Z}_0^-$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of positive integers, non-positive integers, real numbers, and complex numbers, respectively, and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The particular case $\alpha = 0$ of (1)

$$L_n(x) = L_n^{(0)}(x) = {}_1F_1(-n\,;\,1\,;\,x) \tag{4}$$

$$(n \in \mathbb{N}_0,\ x \in \mathbb{C})$$

is called as simple Laguerre (or Laguerre) polynomial which has also attracted much attention. For certain formulas and properties including these polynomials, one may be referred (for example) to [1], [3, Section 6.2], [5, 6, 12–14], [15, pp. 201–202], [7–11, 17, 18].

Among numerous generating functions which can produce (1) or (4), we recall the following (see, e.g., [15, p. 202])

$$\frac{1}{(1-t)^{1+\alpha}} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n. \tag{5}$$

Ali et al. [2] brought in a generalization of Bateman polynomial and presented some interesting and presumably useful properties and formulas involving it. In the same vein, in this paper, we introduce a generalization of Laguerre polynomials and investigate certain properties and formulas associated with it such as recurrence relation, differential formula, generating function, Rodrigues formula, and orthogonality.

2. Generalized Laguerre polynomials

We begin by introducing generalized Laguerre polynomials, which are denoted by $L_{p,n}^{(\alpha)}(x)$ whose generating function is given as in Definition 1.

Definition 1. Let $p \in \mathbb{N};\ x,\, \alpha \in \mathbb{C}$.

$$\frac{1}{(1-t)^{1+\alpha}} \exp\left(\frac{-x^p t^p}{(1-t)^p}\right) = \sum_{n=0}^{\infty} L_{n,p}^{(\alpha)}(x)\, t^n \tag{6}$$

$$(p \in \mathbb{N};\ x,\, \alpha \in \mathbb{C})\,.$$

Obviously $L_{n,1}^{(\alpha)}(x) = L_n^{(\alpha)}(x)$. Hereafter we explore certain formulas and properties involving the generalized Laguerre polynomials in (6). Throughout, let $F(p; x, t)$ be the left-handed generating function in (6).

Explicit representation

An explicit expression of the generalized Laguerre polynomials $L_{n,p}^{(\alpha)}(x)$ in the following theorem.

Theorem 2.1. *Let $x, \alpha \in \mathbb{C}$, $p \in \mathbb{N}$, and $n \in \mathbb{N}_0$. Then*

$$L_{n,p}^{(\alpha)}(x) = (1+\alpha)_n \sum_{k=0}^{[n/p]} \frac{(-1)^k}{k!\,(1+\alpha)_{pk}\,(n-pk)!}\, x^{pk} \tag{7}$$

$$= \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{[n/p]} \frac{(-1)^{(p+1)k}\,(-n)_{pk}}{k!\,(1+\alpha)_{pk}}\, x^{pk}. \tag{8}$$

Here and throughout, $[m]$ denotes the greatest integer less than or equal to $m \in \mathbb{R}$. Or, equivalently,

$$L_{n,p}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_pF_p \left[\begin{array}{c} \frac{-n}{p}, \frac{-n+1}{p}, \ldots, \frac{-n-1+p}{p}\,; \\ \frac{\alpha+1}{p}, \frac{\alpha+2}{p}, \ldots, \frac{\alpha+p}{p}\,; \end{array} (-1)^{p+1}\, x^p \right]. \tag{9}$$

Proof. Expanding the exponential in the left-hand side of (6), we find

$$F(p;x,t) = \frac{1}{(1-t)^{1+\alpha+pk}} \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{pk}\, t^{pk}}{k!}.$$

Employing the binomial theorem

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!}\, z^n = {}_1F_0\,(a\,;\,—;\,z) \quad (a \in \mathbb{C};\; |z| < 1), \tag{10}$$

we obtain the following double series

$$F(p;x,t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k\,(1+\alpha+pk)_n\, x^{pk}}{k!\,n!}\, t^{n+pk}. \tag{11}$$

Recall a known double series manipulation (see, e.g., [4, Eq. (1.1)])

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} A_{k,n-pk} \quad (p \in \mathbb{N}) \tag{12}$$

$\Longleftrightarrow$

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} A_{k,n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A_{k,n+pk} \quad (p \in \mathbb{N}), \tag{13}$$

where $A_{x,y}$ denotes a function of two variables x and y and the involved double series is assumed to be absolutely convergent.

An application of (12) in (11) gives

$$F(p;x,t) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} \frac{(-1)^k\,(1+\alpha+pk)_{n-pk}\, x^{pk}}{k!\,(n-pk)!}\, t^n. \tag{14}$$

Equating the coefficients of t^n in the right members of (6) and (14) yields

$$L_{n,p}^{(\alpha)}(x) = \sum_{k=0}^{[n/p]} \frac{(-1)^k\,(1+\alpha+pk)_{n-pk}}{k!\,(n-pk)!}\,x^{pk}. \tag{15}$$

Using (3) and a known identity

$$(n-k)! = \frac{(-1)^k\,n!}{(-n)_k} \quad (k,\,n\in\mathbb{N}_0;\ 0\le k\le n)\,, \tag{16}$$

we derive

$$(1+\alpha+pk)_{n-pk} = \frac{(1+\alpha)_n}{(1+\alpha)_{pk}} \quad \text{and} \quad (n-pk)! = \frac{(-1)^{pk}\,n!}{(-n)_{pk}}. \tag{17}$$

Hence, use of (17) in (15) leads to the desired identity (8).

Finally, applying the multiplication formula

$$(\lambda)_{mn} = m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n \quad (\lambda\in\mathbb{C};\ m\in\mathbb{N};\ n\in\mathbb{N}_0) \tag{18}$$

to (8) provides the equivalent expression (9). □

Remark 2.2. Eq. (8) reveals that, for each $n\in\mathbb{N}_0$, $L_{n,p}^{(\alpha)}(x)$ is a polynomial in the variable x of degree at most $p[n/p]$. In fact, the degree of $L_{n,p}^{(\alpha)}(x)$ is a step function in the following manner:

$$\deg L_{n,p}^{(\alpha)}(x) = \ell p \quad (\ell p\le n<(\ell+1)p;\ \ell\in\mathbb{N}_0)\,. \tag{19}$$

Generating function

Establish two generating functions for the generalized Laguerre polynomials $L_{n,p}^{(\alpha)}(x)$ in Theorem 2.3.

Theorem 2.3. *Let* t, x, α, $c\in\mathbb{C}$ *and* $p\in\mathbb{N}$. *Then*

$$\begin{aligned} &e^t\,{}_0F_p\left(\text{---};\ \frac{\alpha+1}{p},\ \frac{\alpha+2}{p},\ \ldots,\ \frac{\alpha+p}{p};\ -\left(\frac{xt}{p}\right)^p\right)\\ &= \sum_{n=0}^{\infty}\frac{L_{n,p}^{(\alpha)}(x)\,t^n}{(1+\alpha)_n} \end{aligned} \tag{20}$$

and

$$\begin{aligned} &\frac{1}{(1-t)^c}\,{}_pF_p\left(\begin{matrix}\frac{c}{p},\ \frac{c+1}{p},\ \ldots,\ \frac{c+p-1}{p}\ ;\\ \frac{\alpha+1}{p},\ \frac{\alpha+2}{p},\ \ldots,\ \frac{\alpha+p}{p}\ ;\end{matrix}\ -\left(\frac{xt}{1-t}\right)^p\right)\\ &= \sum_{n=0}^{\infty}\frac{(c)_n\,L_{n,p}^{(\alpha)}(x)\,t^n}{(1+\alpha)_n} \quad (|t|<1). \end{aligned} \tag{21}$$

Proof. Using (7), (13), and (18), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{L_{n,p}^{(\alpha)}(x)\,t^n}{(1+\alpha)_n} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-x^p t^p)^k}{k!\,(1+\alpha)_{pk}} \\ &= e^t \sum_{k=0}^{\infty} \frac{1}{k!\,\prod_{j=1}^{p}\left(\frac{\alpha+j}{p}\right)_k} \left(-\left(\frac{xt}{p}\right)^p\right)^k. \end{aligned} \tag{22}$$

In view of (2), the rightmost term of (22) can be expressed as the left-hand side of (20).

Employing (7), (13), and (10), we find

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(c)_n\,L_{n,p}^{(\alpha)}(x)\,t^n}{(1+\alpha)_n} &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c+pk)_n\,t^n}{n!} \cdot \frac{(c)_{pk}\,\{-(xt)^p\}^k}{k!\,(1+\alpha)_{pk}} \\ &= \frac{1}{(1-t)^c} \sum_{k=0}^{\infty} \frac{(c)_{pk}}{k!\,(1+\alpha)_{pk}} \left\{-\left(\frac{xt}{1-t}\right)^p\right\}^k, \end{aligned}$$

which, upon using (18) and (2), leads to the left-hand member of (21). □

It is noted that the case $c = 1+\alpha$ of (21) yields the generating function (6).

Recurrence relation

Present some recurrence relations involving the generalized Laguerre polynomials $L_{n,p}^{(\alpha)}(x)$ and their derivative in the following theorem.

Theorem 2.4. *Let x, $\alpha \in \mathbb{C}$ and p, $n \in \mathbb{N}$. Also let $D = \frac{d}{dx}$. Then*

$$x\,DL_{n,p}^{(\alpha)}(x) - n\,L_{n,p}^{(\alpha)}(x) + (\alpha+n)\,L_{n-1,p}^{(\alpha)}(x) = 0; \tag{23}$$

$$DL_{n,p}^{(\alpha)}(x) = \begin{cases} 0 & (0 \le n \le p-1) \\ -p\,x^{p-1}\,L_{n-p,p}^{(\alpha+p)}(x) & (n \ge p); \end{cases} \tag{24}$$

$$(\alpha+n)\,L_{n-1,p}^{(\alpha)}(x) - n\,L_{n,p}^{(\alpha)}(x) = p\,x^p\,L_{n-p,p}^{(\alpha+p)}(x) \quad (n \ge p). \tag{25}$$

Proof. From (22), we can set

$$G(p;x,t) := \sum_{n=0}^{\infty} \frac{L_{n,p}^{(\alpha)}(x)\,t^n}{(1+\alpha)_n} = e^t\,\Phi\left(-\left(\frac{xt}{p}\right)^p\right), \tag{26}$$

where the function

$$\Phi\left(-\left(\frac{xt}{p}\right)^p\right) = \sum_{k=0}^{\infty} \frac{1}{k!\,\prod_{j=1}^{p}\left(\frac{\alpha+j}{p}\right)_k} \left(-\left(\frac{xt}{p}\right)^p\right)^k.$$

Differentiating $G(p;x,t)$ with respect to x and t, respectively, gives

$$G_x(p;x,t) = e^t \Phi' \left(-\left(\frac{xt}{p}\right)^p \right) \cdot \frac{-x^{p-1}\, t^p}{p^{p-1}}$$

and

$$G_t(p;x,t) = e^t \Phi \left(-\left(\frac{xt}{p}\right)^p \right) + e^t \Phi' \left(-\left(\frac{xt}{p}\right)^p \right) \cdot \frac{-x^p\, t^{p-1}}{p^{p-1}}.$$

Combining $G_x(p;x,t)$ and $G_t(p;x,t)$ yields

$$x\, G_x(p;x,t) - t\, G_t(p;x,t) + t\, G(p;x,t) = 0. \tag{27}$$

Applying the series in (26) to (27), we obtain

$$\sum_{n=1}^{\infty} \frac{x\, DL_{n,p}^{(\alpha)}(x)\, t^n}{(1+\alpha)_n} - \sum_{n=1}^{\infty} \frac{n\, L_{n,p}^{(\alpha)}(x)\, t^n}{(1+\alpha)_n} + \sum_{n=1}^{\infty} \frac{L_{n-1,p}^{(\alpha)}(x)\, t^n}{(1+\alpha)_{n-1}} = 0. \tag{28}$$

We observe from (28) that each coefficient of t^n should be zero, which gives (23).

Differentiating both sides of (6) provides

$$\begin{aligned}
\sum_{n=1}^{\infty} DL_{n,p}^{(\alpha)}(x)\, t^n &= \frac{1}{(1-t)^{1+\alpha+p}} \exp\left(\frac{-x^p t^p}{(1-t)^p} \right) \cdot \left(-p\, x^{p-1}\, t^p \right) \\
&= -p\, x^{p-1} \sum_{n=0}^{\infty} L_{n,p}^{(\alpha+p)}(x)\, t^{n+p} \\
&= -p\, x^{p-1} \sum_{n=p}^{\infty} L_{n-p,p}^{(\alpha+p)}(x)\, t^{n},
\end{aligned}$$

which, upon equating the coefficients of t^n $(n \geq p)$ in the leftmost and rightmost members, produces (24).

Setting (24) in (23) provides (25). □

Differential equation

Provide a differential equation which is satisfied by the generalized Laguerre polynomials $L_{n,p}^{(\alpha)}(x)$ in Theorem 2.5 (for differential equation whose solution is ${}_pF_q$, see, e.g., [15, Section 47]).

Theorem 2.5. *Let x, $\alpha \in \mathbb{C}$ and p, $n \in \mathbb{N}$. Also let $\theta = x\frac{d}{dx}$. Then*

$$\begin{aligned}
&\left[\frac{1}{p}\theta \prod_{j=1}^{p} \left(\frac{1}{p} \left(\theta/p - 1 + \alpha + j \right) \right) \right. \\
&\qquad \left. + (-1)^p x^p \prod_{j=1}^{p} \frac{1}{p} (\theta + j - n - 1) \right] L_{n,p}^{(\alpha)}(x) = \eta(\alpha, p, n),
\end{aligned} \tag{29}$$

where $\frac{\alpha+j}{p} \in \mathbb{C} \setminus \mathbb{Z}_0^-$ *and no two* $\frac{\alpha+j}{p}$ *differ by an integer* $(j = 1, \ldots, p)$*, and*

$$\eta(\alpha,p,n) = (-1)^{p+(p+1)[n/p]}\, \frac{p^p\,(1+\alpha)_n \cdot (-n+p[n/p])_p}{n!\,[n/p]!} \times \frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}\right)_{[n/p]}}{\prod_{j=1}^{p}\left(\frac{j+\alpha}{p}\right)_{[n/p]}}\, x^{p([n/p]+1)}.$$

Proof. We derive from (8) that

$$L_{n,p}^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{[n/p]} \frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}\right)_k}{k!\,\prod_{j=1}^{p}\left(\frac{j+\alpha}{p}\right)_k} (-1)^{(p+1)k}\, x^{pk}. \tag{30}$$

Since $\frac{\theta}{p} x^{pk} = k\,x^{pk}$, we have

$$\frac{1}{p}(\theta/p + j + \alpha - 1)x^{pk} = \frac{k+j+\alpha-1}{p} x^{pk}. \tag{31}$$

Applying (30) to the following differential operator with the aid of (31), we get

$$\begin{aligned} \mathcal{L}_{DE} &:= \left[\frac{1}{p}\theta \prod_{j=1}^{p}\left(\frac{1}{p}\,(\theta/p - 1 + \alpha + j)\right)\right] L_{n,p}^{(\alpha)}(x) \\ &= \frac{(1+\alpha)_n}{n!} \sum_{k=1}^{[n/p]} \frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}\right)_k \cdot k \prod_{j=1}^{p} \frac{k+j+\alpha-1}{p}}{k!\,\prod_{j=1}^{p}\left(\frac{j+\alpha}{p}\right)_k} (-1)^{(p+1)k}\, x^{pk}. \end{aligned} \tag{32}$$

We then obtain that

$$\mathcal{L}_{DE} = \frac{(1+\alpha)_n}{n!} \sum_{k=1}^{[n/p]} \frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}\right)_k}{(k-1)!\,\prod_{j=1}^{p}\left(\frac{j+\alpha}{p}\right)_{k-1}} (-1)^{(p+1)k}\, x^{pk}.$$

Putting $k-1 = k'$ and cancelling the prime on k provides

$$\mathcal{L}_{DE} = (-1)^{p+1} x^p\, \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{[n/p]-1} \frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}\right)_k \cdot \prod_{j=1}^{p}\left(\frac{j-n-1}{p} + k\right)}{k!\,\prod_{j=1}^{p}\left(\frac{j+\alpha}{p}\right)_k} (-1)^{(p+1)k}\, x^{pk}.$$

We get

$$\mathcal{L}_{DE} = (-1)^{p+1}x^p \,\frac{(1+\alpha)_n}{n!} \sum_{k=0}^{[n/p]} \frac{\prod_{j=1}^{p}\left(\frac{j-n-1}{p}\right)_k \cdot \prod_{j=1}^{p}\left(\frac{j-n-1}{p}+k\right)}{k!\,\prod_{j=1}^{p}\left(\frac{j+\alpha}{p}\right)_k}(-1)^{(p+1)k}\, x^{pk}$$
$$+\,\eta(\alpha,p,n).$$

Noting

$$\prod_{j=1}^{p}\left(\frac{\theta}{p}+\frac{j-n-1}{p}\right) x^{pk} = \prod_{j=1}^{p}\left(k+\frac{j-n-1}{p}\right) x^{pk},$$

we find from (30) that

$$\mathcal{L}_{DE} = (-1)^{p+1}x^p \left[\prod_{j=1}^{p}\left(\frac{\theta+j-n-1}{p}\right)\right] L_{n,p}^{(\alpha)}(x) + \eta(\alpha,p,n). \tag{33}$$

Finally, matching the first equality of (32) with (33) gives (29). □

The Rodrigues formula

Here and throughout, let $D^k = \frac{d^k}{dx^k}$ $(k \in \mathbb{N}_0)$. We give the Rodrigues formula for the generalized Laguerre polynomials $L_{n,p}^{(\alpha)}(x)$ in the following theorem.

Theorem 2.6. *Let x, $\alpha \in \mathbb{C}$ and p, $n \in \mathbb{N}$. Then*

$$L_{n,p}^{(\alpha)}(x) = \frac{x^{-\alpha}\, \exp\left(-(-1)^{\frac{1}{p}}x\right)}{n!}\, D^n \left[\exp\left((-1)^{\frac{1}{p}}x\right)\, x^{n+\alpha}\right], \tag{34}$$

where n is a multiple of p.

Proof. Here (7) is written:

$$L_{n,p}^{(\alpha)}(x) = \sum_{k=0}^{[n/p]} \frac{(-1)^k\,(1+\alpha)_n}{k!\,(1+\alpha)_{pk}\,(n-pk)!}\, x^{pk}. \tag{35}$$

Noting

$$D^{n-pk}x^{n+\alpha} = \frac{(1+\alpha)_n\, x^{\alpha+pk}}{(1+\alpha)_{pk}}$$

and

$$D^{pk}\exp\left((-1)^{\frac{1}{p}}x\right) = (-1)^k\,\exp\left((-1)^{\frac{1}{p}}x\right),$$

we may get

$$\begin{aligned}L_{n,p}^{(\alpha)}(x) =& \frac{x^{-\alpha}\,\exp\left(-(-1)^{\frac{1}{p}}x\right)}{n!}\\ &\times \sum_{k=0}^{[n/p]} \frac{n!\left\{D^{pk}\exp\left((-1)^{\frac{1}{p}}x\right)\right\}\left\{D^{n-pk}x^{n+\alpha}\right\}}{k!\,(n-pk)!}\\ =& \frac{x^{-\alpha}\,\exp\left(-(-1)^{\frac{1}{p}}x\right)}{n!}\\ &\times \sum_{k=0}^{[n/p]} \binom{n}{pk}\left\{D^{pk}\exp\left((-1)^{\frac{1}{p}}x\right)\right\}\left\{D^{n-pk}x^{n+\alpha}\right\}\\ =& \frac{x^{-\alpha}\,\exp\left(-(-1)^{\frac{1}{p}}x\right)}{n!}\,D^n\left[\exp\left((-1)^{\frac{1}{p}}x\right)\,x^{n+\alpha}\right]. \qquad \square\end{aligned}$$

Orthogonality

Explore orthogonality for the generalized Laguerre polynomials $L_{n,p}^{(\alpha)}(x)$ in Theorem 2.7.

Theorem 2.7. *Let x, $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 1$ and p, m, $n \in \mathbb{N}$ be such that p is odd. Then*

$$\int_0^\infty x^\alpha\,\exp\left((-1)^{\frac{1}{p}}x\right)\,L_{n,p}^{(\alpha)}(x)\,L_{m,p}^{(\alpha)}(x)\,dx = 0 \quad (m \neq n)\,. \tag{36}$$

Also

$$\int_0^\infty x^\alpha\,\exp\left((-1)^{\frac{1}{p}}x\right)\,\left\{L_{n,p}^{(\alpha)}(x)\right\}^2\,dx = \frac{(-1)^{n+[n/p]}}{[n/p]!}\,\Gamma(1+\alpha+n), \tag{37}$$

where n is a multiple of p.

Proof. Let $\mathcal{L}_{m,n}(p;\alpha)$ be the left member of (36). Applying the Rodrigues formula (35) gives

$$\mathcal{L}_{m,n}(p;\alpha) = \frac{1}{n!}\int_0^\infty D^n\left[\exp\left((-1)^{\frac{1}{p}}x\right)\,x^{n+\alpha}\right]\,L_{m,p}^{(\alpha)}(x)\,dx.$$

Integrating by parts n times, we obtain

$$\mathcal{L}_{m,n}(p;\alpha) = \frac{(-1)^n}{n!}\int_0^\infty \exp\left((-1)^{\frac{1}{p}}x\right)\,x^{n+\alpha}\,\left[D^n L_{m,p}^{(\alpha)}(x)\right]\,dx. \tag{38}$$

In the process of integrating by parts, the integrated section

$$D^{n-k}\left[\exp\left((-1)^{\frac{1}{p}}x\right)\,x^{n+\alpha}\right]D^{k-1}\left[L_{m,p}^{(\alpha)}(x)\right] \quad (1 \leq k \leq n)$$

vanishes both at $x = 0$ and as $x \to \infty$ when p is odd and $\Re(\alpha) > -1$. Since $L_{m,p}^{(\alpha)}(x)$ is of degree at most m, $D^n L_{m,p}^{(\alpha)}(x) = 0$ for $n > m$. We find from (38) that $\mathcal{L}_{m,n}(p;\alpha)=0$ for $n>m$. Since the integral $\mathcal{L}_{m,n}(p;\alpha)$ is symmetric in n and m, $\mathcal{L}_{m,n}(p;\alpha)=0$ for $n<m$. This proves (36).

From (7), we have

$$L_{n,p}^{(\alpha)}(x)=\frac{(-1)^{[n/p]}(1+\alpha)_n}{[n/p]!\,(1+\alpha)_{p[n/p]}\,(n-p[n/p])!}\,x^{p[n/p]}+\varpi_n(x),$$

where $\varpi_n(x)$ is a polynomial in x of degree at most $p[n/p]-1$. In particular, when n is a multiple of p,

$$L_{n,p}^{(\alpha)}(x)=\frac{(-1)^{[n/p]}}{[n/p]!}\,x^n+\varpi_{n-1}(x),$$

where $\varpi_{n-1}(x)$ is a polynomial in x of degree at most $n-1$. Therefore we find

$$D^n L_{n,p}^{(\alpha)}(x)=\frac{(-1)^{[n/p]}}{[n/p]!}\,n!, \tag{39}$$

where n is a multiple of p. Setting (39) in the case $m=n$ of (38) yields

$$\begin{aligned}\mathcal{L}_{n,n}(p;\alpha)&=\frac{(-1)^{n+[n/p]}}{[n/p]!}\int_0^\infty e^{-x}\,x^{n+\alpha}\,dx\\&=\frac{(-1)^{n+[n/p]}}{[n/p]!}\,\Gamma(1+\alpha+n)\quad(\Re(\alpha)>-1),\end{aligned}$$

which p is an odd positive integer and $n\in\mathbb{N}$ is a multiple of p. □

Some other properties

Provide some other identities involving the generalized Laguerre polynomials $L_{n,p}^{(\alpha)}(x)$ in Theorem 2.8.

Theorem 2.8. *Let x, y, α, $\beta\in\mathbb{C}$ and p, $n\in\mathbb{N}$. Then*

$$\begin{aligned}\int_0^\infty x^\alpha e^{-x}\,L_{p,n}^{(\alpha)}(x)\,dx&=\frac{\Gamma(1+\alpha+n)}{n!}\\&\times{}_pF_0\left(\frac{-n}{p},\frac{-n+1}{p},\ldots,\frac{-n+p-1}{p};\,\text{—};\,(-1)^{p+1}p^p\right)\end{aligned} \tag{40}$$

$$(\Re(\alpha)>-1;\ n \textit{ is a multiple of } p);$$

$$L_{n,p}^{(\alpha)}(x)=\sum_{k=0}^{n}\frac{(\alpha-\beta)_k\,L_{n-k,p}^{(\beta)}(x)}{k!}; \tag{41}$$

$$L_{n,p}^{(\alpha+\beta+1)}(z)=\sum_{k=0}^{n}L_{k,p}^{(\alpha)}(x)\,L_{n-k,p}^{(\beta)}(y), \tag{42}$$

where $x^p + y^p \in \mathbb{C} \setminus \{0\}$ *and* $z := (x^p + y^p)^{\frac{1}{p}}$ *whose principal branch can be chosen*;

$$L_{n,p}^{(\alpha)}(xy) = \sum_{k=0}^{n} \frac{(1+\alpha)_n\,(1-y)^{n-k}\,y^k\,L_{k,p}^{(\alpha)}(x)}{(n-k)!\,(1+\alpha)_k}. \tag{43}$$

Proof. Using (8) and Euler's integral of the gamma function with the aid of (3), we have

$$\begin{aligned}\int_0^\infty x^\alpha e^{-x} L_{p,n}^{(\alpha)}(x)dx &= \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{[n/p]} \frac{(-1)^{(p+1)k}\,(-n)_{pk}}{k!\,(1+\alpha)_{pk}} \int_0^\infty e^{-x}\,x^{\alpha+pk}\,dx \\ &= \frac{\Gamma(1+\alpha+n)}{n!} \sum_{k=0}^{[n/p]} \frac{(-1)^{(p+1)k}\,(-n)_{pk}}{k!},\end{aligned}$$

which, upon using (18) and (2), yields (40).

From (6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} L_{n,p}^{(\alpha)}(x)\,t^n &= (1-t)^{-1-\alpha} \exp\left(\frac{-x^p t^p}{(1-t)^p}\right) \\ &= (1-t)^{-(\alpha-\beta)} \cdot (1-t)^{-1-\beta} \exp\left(\frac{-x^p t^p}{(1-t)^p}\right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha-\beta)_k}{k!} L_{n,p}^{(\beta)}(x) t^{n+k} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\alpha-\beta)_k}{k!} L_{n-k,p}^{(\beta)}(x) t^n,\end{aligned}$$

which, upon equating the coefficients of t^n, yields (41).

We find from (6) that

$$\begin{aligned}&\sum_{n=0}^{\infty} \sum_{k=0}^{n} L_{k,p}^{(\alpha)}(x)\,L_{n-k,p}^{(\beta)}(y)\,t^n \\ &= (1-t)^{-1-\alpha} \exp\left(\frac{-x^p t^p}{(1-t)^p}\right)\,(1-t)^{-1-\beta} \exp\left(\frac{-y^p t^p}{(1-t)^p}\right) \\ &= (1-t)^{-1-(\alpha+\beta+1)}\,\exp\left(\frac{-z^p t^p}{(1-t)^p}\right) \\ &= \sum_{n=0}^{\infty} L_{n,p}^{(\alpha+\beta+1)}(z)\,t^n,\end{aligned}$$

which, upon matching the coefficients of t^n, gives (42).

We consider

$$e^t\,{}_0F_p\left(\text{——};\,\frac{\alpha+1}{p},\,\frac{\alpha+2}{p},\,\ldots,\,\frac{\alpha+p}{p}\,;\,\left(-\frac{xyt}{p}\right)^p\right)$$
$$=e^{(1-y)t}\,e^{yt}\,{}_0F_p\left(\text{——};\,\frac{\alpha+1}{p},\,\frac{\alpha+2}{p},\,\ldots,\,\frac{\alpha+p}{p}\,;\,\left(-\frac{x(yt)}{p}\right)^p\right),$$

which, in view of (20), produces

$$\sum_{n=0}^{\infty}\frac{L_{n,p}^{(\alpha)}(xy)\,t^n}{(1+\alpha)_n}=\left(\sum_{n=0}^{\infty}\frac{(1-y)^n\,t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{L_{k,p}^{(\alpha)}(x)\,y^k t^k}{(1+\alpha)_k}\right).$$

Then, from the last equality, we obtain (43). □

3. Conclusion remarks

Since $L_{n,1}^{(\alpha)}(x)=L_n^{(\alpha)}(x)$ and $L_{n,1}^{(0)}(x)=L_n(x)$, the results in Section 2 reduce to yield certain properties and formulas for the Laguerre polynomials $L_n^{(\alpha)}(x)$ and the simple Laguerre polynomials $L_n(x)$.

The identity (9) is rewritten as follows:

$$\frac{n!}{(1+\alpha)_n}\,L_{n,p}^{(\alpha)}(x)={}_pF_p\left[\frac{-n}{p},\,\frac{-n+1}{p},\,\ldots,\,\frac{-n-1+p}{p}\,;\;\frac{\alpha+1}{p},\,\frac{\alpha+2}{p},\,\ldots,\,\frac{\alpha+p}{p}\,;\,(-1)^{p+1}\,x^p\right], \tag{44}$$

which, upon setting $p=1$, yields a known expression for the Laguerre polynomials

$$\frac{n!}{(1+\alpha)_n}\,L_n^{(\alpha)}(x)={}_1F_1[-n\,;\,1+\alpha\,;\,x]. \tag{45}$$

It is known (see, e.g., [15, Section 48]) that there are $3p$ linearly independent contiguous function relations for ${}_pF_p$. Using the three contiguous relations for ${}_1F_1$ with the aid of (45), the three mixed recurrence relations for $L_n^{(\alpha)}(x)$ are established (see [15, p. 203, Eqs. (8), (9) and (10)]), for example,

$$L_n^{(\alpha)}(x)=L_{n-1}^{(\alpha)}(x)+L_n^{(\alpha-1)}(x). \tag{46}$$

Similarly, $3p$ different recurrence relations for $L_{n,p}^{(\alpha)}(x)$ may be obtained from the $3p$ contiguous relations for ${}_pF_p$. Unfortunately, no recurrence relations for $L_{n,p}^{(\alpha)}(x)$ $(p\geq 2)$ can be derived from the $3p$ contiguous relations for ${}_pF_p$. Indeed, if 1 is added or subtracted at one of the numerator or denominator parameters in the right member of (44), the other parameters cannot be expressed in the same fashion as in (44) whenever $p\geq 2$. The generalized Laguerre polynomials introduced here and their properties and formulas presented are hoped to be potentially useful.

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