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ON GENERIC SUBMANIFOLDS OF LP-SASAKIAN MANIFOLDS WITH CONCURRENT VECTOR FIELDS

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Abstract. The object of the present paper is to deduce some important results on generic submanifolds and also generic product of LP-Sasakian manifolds with concurrent vector fields. Also, we provide a necessary and sufficient condition for which the invariant distribution D and anti-invariant distribution $D^{\perp}$ of M are Einstein. Also, we deduce an interesting necessary and sufficient condition for submanifolds of LP-Sasakian manifolds to be totally umbilical submanifolds. Especially we deal with the generic submanifolds admitting a Ricci soliton in LP-Sasakian manifolds endowed with concurrent vector fields.

1. Introduction

Nowadays submanifold theory has become an amusing area of research in differential geometry and plays an essential role in the development of the subject. The results of this field are mainly used in applied mathematics and theoretical physics [9, 19, 26]. For instance, the method of invariant submanifolds is used in the study of non-linear autonomous systems [26]. Semi-Riemannian geometry has remarkable applications in relativity theory [19]. Many authors have worked on invariant submanifolds [5, 20, 26], and deduced a large number of significant results. Some of them have studied semi-invariant submanifolds which are generalizations of invariant and anti-invariant submanifolds. The first study on semi-invariant submanifolds of Sasakian manifolds was done by Bejancu and Papaghiuc in [4]. Semi-invariant submanifolds have been studied by several authors [1,3,12]. These types of submanifolds help us to explain the beauty of the subject. Generic semi-invariant submanifolds are an exceptional category of semi-invariant submanifolds which give us more attractive and extraordinary results. For generic submanifolds we refer [24, 25]. In this paper we study generic semi-invariant submanifolds as well as generic semi-invariant products which are more specific.

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Lorentzian manifolds form one of the most important sub-classes of pseudo Riemannian manifolds. It plays a crucial role in mathematical physics, more specifically in the development of the theory of relativity and cosmology. The notion of Lorentzian almost para-contact manifolds was introduced by K. Matsumoto [15]. Subsequently many authors worked on this type of manifolds [1, 2, 16]. Lorentzian para-Sasakian manifolds are special kind of Lorentzian almost para contact manifolds. In brief, Lorentzian para-Sasakian manifolds are called LP-Sasakian manifolds. A. Sarkar and M. Sen have worked on invariant submanifolds of LP-Sasakian manifolds [20].

There has been several papers on Riemannian manifolds and pseudo Riemannian manifolds which admit concircular vector fields and concurrent vector fields. Recently B. Y. Chen and S. W. Wei studied Riemannian submanifolds with concircular canonical vector fields in [8]. Many papers have been published on related topics [7, 13, 14, 21, 23]. In 2015, B. Y. Chen deduced some results on concircular vector fields and their applications to Ricci solitons [6]. In the paper [25], the authors discussed on generic submanifolds of Sasakian manifolds with concurrent vector fields. Keeping these works in mind we establish some interesting results on generic semi-invariant submanifolds of LP-Sasakian manifolds with concurrent vector fields.

The concept of Ricci solitons in Riemannian geometry was introduced by Hamilton as a self similar solution of the Ricci flow in 1982 [11]. A Ricci soliton is known as quasi-Einstein metric in physics literature. This concept has been studied in many fields of the manifold theory by several geometers. For more details we refer [10, 17, 18, 22].

A Ricci soliton is a pseudo-Riemannian manifold $(\widetilde{M}, g)$ that admits a smooth vector field V on $\widetilde{M}$ such that

$$\text{Ł}_V g + 2\widetilde{S} = 2\lambda g \tag{1.1}$$

where, $\text{Ł}_V g$ is the Lie-derivative of the metric tensor g in the direction of the vector field V, which is called a potential vector field of the Ricci soliton, λ is a constant and $\widetilde{S}$ is the Ricci tensor of $\widetilde{M}$. A Ricci soliton is denoted by $(\widetilde{M}, g, V, \lambda)$. If $\text{Ł}_V g = 0$, then the potential vector field V is called Killing. Also if $\text{Ł}_V g = \rho g$, then the vector field V is called conformal Killing, where ρ is a smooth function. If $V = 0$ or Killing, then the Ricci soliton is called trivial and in this case, the metric g is Einstein. So, a Ricci soliton is viewed as a generalization of Einstein metric. The present paper is organized as follows:

After the introduction in Section 1, we give some basic definitions, notations and formulas of generic submanifolds and Lorentzian almost para-contact manifolds in Section 2. In Section 3, we construct two interesting examples of 5-dimensional generic submanifolds of 7-dimensional LP-Sasakian manifolds. In Section 4, we deal with the generic submanifolds of LP-Sasakian manifolds with concurrent vector fields. In Section 5, we deduce an interesting necessary and sufficient condition for submanifolds of LP-Sasakian manifold to be umbilical submanifolds. In the last section we study the generic semi-invariant products admitting a Ricci soliton.

2. Preliminaries

A vector field V on a Riemannian or pseudo-Riemannian manifold $\widetilde{M}$ is called a concircular vector field if it satisfies

$$\widetilde{\nabla}_X V = fX$$

for any X tangent to $\widetilde{M}$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{M}$ and f is a real valued function on $\widetilde{M}$. In particular, if $f = 1$, then the concircular vector field V is called a concurrent vector field and also if $f = 0$, then the concircular vector field V is called a parallel vector field.

Throughout this paper we consider the vector field V is concurrent and from definition it follows

$$\widetilde{\nabla}_X V = X \tag{2.1}$$

for any X tangent to $\widetilde{M}$.

Let, $\widetilde{M}$ be an m-dimensional real differentiable manifold of differentiability class C^∞ endowed with a C^∞-vector valued linear function ϕ, a C^∞-vector field ξ, 1-form η and Lorentzian metric g of type $(0, 2)$ such that for each point $p \in \widetilde{M}$, the tensor $g_p : T_p\widetilde{M} \times T_p\widetilde{M} \longrightarrow R$ is a non-degenerate inner product of signature $(-, +, +, +, \dots, +)$, where $T_p\widetilde{M}$ denotes the tangent vector space of $\widetilde{M}$ at p and R is the real number space, which satisfies

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2.3}$$

for all differentiable vector fields X, Y tangent to $\widetilde{M}$. Such a structure (ϕ, ξ, η, g) is termed as Lorentzian para-contact structure. In a Lorentzian para-contact structure the following relations also hold

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \text{rank}\phi = n - 1. \tag{2.4}$$

A Lorentzian para-contact manifold $\widetilde{M}$ is called a Lorentzian para-Sasakian manifold if the following condition holds

$$(\widetilde{\nabla}_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.5}$$

for all X, Y tangent to $\widetilde{M}$, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$ with respect to g. From (2.5) it follows that

$$\widetilde{\nabla}_X \xi = \phi X, \tag{2.6}$$

$$(\widetilde{\nabla}_X \eta)Y = g(X, Y) + \eta(X)\eta(Y). \tag{2.7}$$

Let M be an n-dimensional submanifold of an m-dimensional LP-Sasakian manifold $\widetilde{M}$. Here, M is also an n-dimensional LP-Sasakian manifold. Then we have [7]

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.8}$$

$$\widetilde{\nabla}_X N = \nabla^{\perp}_X N - A_N X \tag{2.9}$$

for all vector fields X, Y tangent to M and normal vector field N on M, where ∇ is the pseudo Riemannian connection on M defined by the induced metric g and $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$ of M, σ is the second fundamental form of M and A_N is a shape operator.

It is well known that the relation between the second fundamental form σ and the shape operator A_N are given by

$$g(\sigma(X, Y), N) = g(A_N X, Y) \tag{2.10}$$

for any vector fields X, Y tangent to M. Here we denote by the same symbol g the Lorentzian metric induced by g on $\widetilde{M}$.

Let M be a real n-dimensional submanifold of an m-dimensional LP-Sasakian manifold $\widetilde{M}$ such that ξ is tangent to M. Then, M is called a semi-invariant submanifold of $\widetilde{M}$, if there exist two orthogonal differentiable distributions D and $D^{\perp}$ on M satisfying following conditions.

a. the distribution D is invariant by ϕ, i.e., $\phi(D_x) = D_x$ for all $x \in M$.

b. the distribution $D^{\perp}$ is anti-invariant by ϕ, i.e., $\phi(D^{\perp}_x) \subset T^{\perp}_x M$ for all $x \in M$.

Suppose we consider dimension of $\widetilde{M} = m$, dimension of $M = n$, dimension of the distribution $D = p$ and dimension of the distribution $D^{\perp} = q$. Now, if $q = m - n$, then the semi-invariant submanifold M is called a generic semi-invariant submanifold of $\widetilde{M}$.

Let M be a semi-invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$. By using the definition of semi-invariant submanifold, the tangent bundle and normal bundle of a semi-invariant submanifold M have the orthogonal decomposition

$$TM = D \oplus D^{\perp} \oplus \langle \xi \rangle,\ T^{\perp}M = \phi(D^{\perp}) \oplus \mu,\ \phi(\mu) = \mu, \tag{2.11}$$

where μ is the complementary subbundle orthogonal to $\phi(D^{\perp})$ in $\Gamma(T^{\perp}M)$ and $\langle \xi \rangle$ is the 1-dimensional distribution which is spanned by ξ. Also, if the distributions $D \oplus \xi$ and $D^{\perp}$ are totally geodesics in M, then the submanifold M is called a semi-invariant product.

Furthermore, on a semi-invariant submanifold M of an LP-Sasakian manifold $\widetilde{M}$, the following lemma holds.

Lemma 2.1 ([24]). *The following properties are equivalent*:

(i) *M is a semi-invariant product*;
(ii) $A_{\phi(Z)} X = 0$;
(iii) *the second fundamental form of M satisfies* $\sigma(\phi X, Y) = \phi\sigma(X, Y)$

for any $X \in \Gamma(D), Z \in \Gamma(D^\perp)$ *and* $Y \in \Gamma(TM)$.

Again, a semi-invariant product is called a generic semi-invariant product if $m - n = q$ is satisfied. Then, we have $\mu = \{0\}$ in (2.11). Therefore, we get the following decomposition

$$TM = D \oplus D^\perp \oplus \langle \xi \rangle, \qquad T^\perp M = \phi(D^\perp). \tag{2.12}$$

For, a generic semi-invariant product, we can write

$$V = V^T + V^\perp + \phi(V^\perp) + f\xi, \tag{2.13}$$

where $V \in \Gamma(T\widetilde{M}), V^T \in \Gamma(D), V^\perp \in \Gamma(D^\perp)$ and here we consider the function f is constant.

Let M be a semi-invariant submanifold of an LP-Sasakian manifold $\widetilde{M}$. The semi-invariant submanifold M is called a D-geodesic if it satisfies the following

$$\sigma(X, Y) = 0 \tag{2.14}$$

for all $X, Y \in \Gamma(D)$.

Similarly, for any $X, Y \in \Gamma(D^\perp)$ if the relation (2.14) is satisfied on M, then the semi-invariant submanifold is called a $D^\perp$-geodesic. Furthermore, for any $X \in \Gamma(D), Y \in \Gamma(D^\perp)$ if the relation (2.14) is satisfied on M, the semi-invariant submanifold M is called a $(D, D^\perp)$-geodesic or a mixed geodesic.

Again, for any $X, Y \in \Gamma(TM)$ if we get $\sigma(X, Y) = 0$, then the semi-invariant submanifold is called a totally-geodesic.

Now the distribution D is called parallel with respect to $\widetilde{\nabla}$, if it satisfies $\widetilde{\nabla}_X Y \in \Gamma(D)$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{M}$, for any $X \in \Gamma(T\widetilde{M})$ and $Y \in \Gamma(D)$.

3. Examples

Let us construct two examples of generic submanifolds of LP-Sasakian manifolds which help us to verify the obtained results.

Example 3.1. Let us consider the 7-dimensional manifold $\widetilde{M}=\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7\}$, where $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ are the standard coordinates in $\mathbb{R}^7$. The vector fields $e_i = \frac{\partial}{\partial x_i}$, for $i = 1, 2, 3, 4, 5, 6, 7$, are linearly independent at each point of $\widetilde{M}$.

Let, g be the Lorentzian metric defined by

$$g(e_i, e_j) = 0 \text{ for } i \neq j \text{ and } i, j = 1, 2, 3, 4, 5, 6, 7,$$
$$g(e_i, e_j) = 1 \text{ for } i = j \text{ and } i, j = 1, 2, 3, 4, 5, 6,$$
$$g(e_7, e_7) = -1.$$

Let ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = e_1,\ \phi(e_2) = e_2,\ \phi(e_3) = e_3,\ \phi(e_4) = -e_4,$$
$$\phi(e_5) = -e_5,\ \phi(e_6) = -e_6,\ \phi(e_7) = 0.$$

Therefore (ϕ, ξ, η, g) is a Lorentzian almost para-contact manifold.

Now let us consider an immersed submanifold M in $\widetilde{M}$ satisfying the equations

$$x_1^2 + x_2^2 = x_5^2 + x_6^2, \ x_3 + x_4 = 0.$$

By direct computation, it is easy to check that the tangent bundle of M is spanned by the vectors.

$$\begin{aligned}
E_1 &= \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} + \cos\alpha \frac{\partial}{\partial x_5} + \sin\alpha \frac{\partial}{\partial x_6}, \\
E_2 &= -u\sin\theta \frac{\partial}{\partial x_1} + u\cos\theta \frac{\partial}{\partial x_2}, \\
E_3 &= \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \\
E_4 &= -u\sin\alpha \frac{\partial}{\partial x_5} + u\cos\alpha \frac{\partial}{\partial x_6}, \\
E_5 &= \frac{\partial}{\partial x_7},
\end{aligned}$$

where θ, α, u denote arbitrary parameters. From the definition of almost Lorentzian para-contact structure ϕ, we can derive

$$\begin{aligned}
\phi E_1 &= \cos\theta \frac{\partial}{\partial x_1} + \sin\theta \frac{\partial}{\partial x_2} - \cos\alpha \frac{\partial}{\partial x_5} - \sin\alpha \frac{\partial}{\partial x_6}, \\
\phi E_2 &= E_2, \ \phi E_3 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}, \ \phi E_4 = -E_4, \ \phi E_5 = 0.
\end{aligned}$$

Since ϕE_1 and ϕE_3 are orthogonal to TM and ϕE_2 and ϕE_4 are tangent to TM, we find that $D = \text{span}\{E_2, E_4\}$ is an invariant distribution of M and $D^\perp = \text{span}\{E_1, E_3\}$ is an anti-invariant distribution of M. Then $TM = D \oplus D^\perp \oplus \langle E_5 \rangle$. Thus M is a 5-dimensional semi-invariant submanifold of $\widetilde{M}$ with its usual metric structure (ϕ, ξ, η, g).

Here, $\dim(\widetilde{M}) = 7$, $\dim(M) = 5$, $\dim(D) = 2$, and $\dim(D^\perp) = 2$.

Also $\dim(D^\perp) = 2 = 7 - 5 = \dim(\widetilde{M}) - \dim(M)$.

Therefore, $TM = D \oplus D^\perp \oplus \langle \xi \rangle$, $T^\perp M = \phi(D^\perp)$. So this semi-invariant submanifold M is also a generic semi-invariant submanifold of $\widetilde{M}$.

Example 3.2. Let us consider the 7-dimensional manifold $\widetilde{M}=\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7\}$, where $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ are the standard coordinates in $\mathbb{R}^7$. The vector fields

$$\begin{aligned}
e_1 &= -2\frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2}, \ e_2 = \frac{\partial}{\partial x_2}, \ e_3 = \frac{\partial}{\partial x_3}, \\
e_4 &= \frac{\partial}{\partial x_4}, \ e_5 = -2\frac{\partial}{\partial x_5} + 2x_6 \frac{\partial}{\partial x_4}, \ e_6 = \frac{\partial}{\partial x_6}, \ e_7 = \frac{\partial}{\partial x_7}
\end{aligned}$$

are linearly independent at each point of $\widetilde{M}$.

Let, g be the Lorentzian metric defined by

$$g(e_i, e_j) = 0 \text{ for } i \neq j \text{ and } i, j = 1, 2, 3, 4, 5, 6, 7,$$
$$g(e_i, e_j) = 1 \text{ for } i = j \neq 4,$$
$$g(e_4, e_4) = -1.$$

Let, η be the 1-form defined by

$$\eta(X) = g(X, e_4)$$

for any $X \in T\widetilde{M}$.

Let ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = e_2,\ \phi(e_2) = e_1,\ \phi(e_3) = e_7,\ \phi(e_4) = 0,$$
$$\phi(e_5) = e_6,\ \phi(e_6) = e_5,\ \phi(e_7) = e_3.$$

Then for $e_4 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian almost para-contact manifold on $\widetilde{M}$.

Let $\widetilde{\nabla}$ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = -2e_2,\ [e_5, e_6] = -2e_4,\ \text{remaining } [e_i, e_j] = 0.$$

Taking $e_4 = \xi$ and using Koszul formula for the Lorentzian metric g, we can easily calculate the following:

$$\widetilde{\nabla}_{e_2} e_1 = 2e_2,\ \widetilde{\nabla}_{e_2} e_2 = -2e_1,\ \widetilde{\nabla}_{e_5} e_4 = -e_6,\ \widetilde{\nabla}_{e_6} e_4 = e_5,$$
$$\widetilde{\nabla}_{e_4} e_5 = e_6,\ \widetilde{\nabla}_{e_6} e_5 = e_4,\ \widetilde{\nabla}_{e_4} e_6 = e_5,\ \widetilde{\nabla}_{e_5} e_6 = -e_4,$$
$$\text{remaining } \widetilde{\nabla}_{e_i} e_j = 0.$$

From the above it can be easily seen that $\widetilde{M}^7(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold.

Let f be an isometric immersion from M to $\widetilde{M}$ defined by

$$f(x_1, x_3, x_4, x_5, x_7) = (x_1, 0, x_3, x_4, x_5, 0, x_7).$$

Let $M = \{(x_1, x_3, x_4, x_5, x_7) \in \mathbb{R}^5\}$, where $(x_1, x_3, x_4, x_5, x_7)$ are the standard coordinates in $\mathbb{R}$.

The vector fields $\{e_1, e_3, e_4, e_5, e_7\}$ are linearly independent at each point of M.

Let ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = e_2,\ \phi(e_3) = e_7,\ \phi(e_4) = 0,\ \phi(e_5) = e_6,\ \phi(e_7) = e_3.$$

Since $\phi(e_1)$ and $\phi(e_5)$ are orthogonal to TM and $\phi(e_3)$ and $\phi(e_7)$ are tangent to TM, we find that $D = \text{span}\{e_3, e_7\}$ is an invariant distribution of M and $D^\perp = \text{span}\{e_1, e_5\}$ is an anti-invariant distribution of M. Then $TM = D \oplus D^\perp \oplus \langle e_4 \rangle$. Thus M is a 5-dimensional semi-invariant submanifold of $\widetilde{M}$ with its usual metric structure (ϕ, ξ, η, g).

Here, $\dim(\widetilde{M}) = 7$, $\dim(M) = 5$, $\dim(D) = 2$, and $\dim(D^\perp) = 2$.

Also $\dim(D^{\perp}) = 2 = 7 - 5 = \dim(\widetilde{M}) - \dim(M)$.

Therefore, $TM = D \oplus D^{\perp} \oplus \langle\xi\rangle$, $T^{\perp}M = \phi(D^{\perp})$. So this semi-invariant submanifold M is also a generic semi-invariant submanifold of $\widetilde{M}$.

4. Generic submanifolds of LP-Sasakian manifolds with concurrent vector fields

In this section first we discuss some important lemma and propositions, then we deduce the main results on this topic.

Lemma 4.1. *Let M be a generic submanifold of an LP-Sasakian manifold $\widetilde{M}$ with concurrent vector field V. Then we get the following*

$$\nabla_X V^T + \nabla_X V^{\perp} - A_{\phi V^{\perp}} X = X - f\phi X,$$

$$\sigma(X, V^T) + \sigma(X, V^{\perp}) + \nabla^{\perp}_X \phi V^{\perp} = 0$$

for $X \in \Gamma(D)$ and others are usual notations discussed in Section 2.

Proof. Since V is a concurrent vector field then from (2.13) we get

$$\widetilde{\nabla}_X V^T + \widetilde{\nabla}_X V^{\perp} + \widetilde{\nabla}_X \phi V^{\perp} + \widetilde{\nabla}_X f\xi = X.$$

Now from (2.8) and (2.9) we get,

$$\nabla_X V^T + \sigma(X, V^T) + \nabla_X V^{\perp} + \sigma(X, V^{\perp}) + \nabla^{\perp}_X \phi V^{\perp} - A_{\phi V^{\perp}} X + f\widetilde{\nabla}_X \xi = X.$$

Putting the value of $\widetilde{\nabla}_X \xi$ from (2.6) and comparing the tangential and normal components we have the required results. ☐

Proposition 4.1. *Let M be a generic submanifold of an LP-Sasakian manifold $\widetilde{M}$ with concurrent vector field V. Then we get*

$$\nabla_X \phi V^T + \nabla_X V^{\perp} - A_{\phi V^{\perp}} X = g(X, V^T)\xi - fX + \phi X,$$

$$\sigma(X, \phi V^T) + \sigma(X, V^{\perp}) + \nabla^{\perp}_X \phi V^{\perp} = 0.$$

Proof. Since $\widetilde{M}$ is an LP-Sasakian manifold then putting $Y = V$ in (2.5), we get

$$\widetilde{\nabla}_X \phi V - \phi X = g(X, V)\xi + \eta(V)X + 2\eta(X)\eta(V)\xi.$$

Now from (2.13), we have

$$\widetilde{\nabla}_X \phi V^T + \widetilde{\nabla}_X \phi V^{\perp} + \widetilde{\nabla}_X \phi^2 V^{\perp} - \phi X = g(X, V^T)\xi - fX.$$

Now using (2.8), (2.9) and (2.2) in the above equation, we obtain

$$\begin{aligned} &\nabla_X \phi V^T + \sigma(X, \phi V^T) + \nabla^{\perp}_X \phi V^{\perp} - A_{\phi V^{\perp}} X + \nabla_X V^{\perp} + \sigma(X, V^{\perp}) - \phi X \\ &= g(X, V^T)\xi - fX. \end{aligned}$$

From the tangential and normal components of the above equation, we have the required results. Hence the proposition is proved. ☐

Proposition 4.2. *Let M be a generic submanifold of an LP-Sasakian manifold $\widetilde{M}$ with concurrent vector field V. And also M is a D-geodesic. Then we get*

$$\nabla_X \phi V^T - \phi(\nabla_X V^T) = g(X, V^T)\xi.$$

Proof. Since $\widetilde{M}$ is an LP-Sasakian manifold then putting $Y = V^T$ in (2.5), we get

$$\widetilde{\nabla}_X \phi V^T - \phi(\widetilde{\nabla}_X V^T) = g(X, V^T)\xi + \eta(V^T)X + 2\eta(X)\eta(V^T)\xi.$$

Now applying (2.8) in the above equation, we have

$$\nabla_X \phi V^T + \sigma(X, \phi V^T) - \phi(\nabla_X V^T + \sigma(X, V^T)) = g(X, V^T)\xi.$$

Since, M is a D-geodesic, we get

$$\nabla_X \phi V^T - \phi(\nabla_X V^T) = g(X, V^T)\xi. \tag{4.1}$$

Hence the proposition is proved. □

Proposition 4.3. *Let M be a generic submanifold of an LP-Sasakian manifold $\widetilde{M}$ with concurrent vector field V, M is a mixed-geodesic and $\nabla_X V^\perp \in \Gamma(D)$ for any $X \in \Gamma(D)$. Then we get*

$$\phi(\nabla_X V^\perp) = -A_{\phi V^\perp} X, \ \nabla^\perp_X \phi V^\perp = 0.$$

Proof. Since $\widetilde{M}$ is an LP-Sasakian manifold then putting $Y = V^\perp$ in (2.5), we get

$$\widetilde{\nabla}_X \phi V^\perp - \phi(\widetilde{\nabla}_X V^\perp) = g(X, V^\perp)\xi + \eta(V^\perp)X + 2\eta(X)\eta(V^\perp)\xi.$$

Now applying (2.8) and (2.9) in the above equation, we obtain

$$\nabla^\perp_X \phi V^\perp - A_{\phi V^\perp} X - \phi(\nabla_X V^\perp + \sigma(X, V^\perp)) = 0.$$

Since, M is a mixed-geodesic, we get

$$\nabla^\perp_X \phi V^\perp - A_{\phi V^\perp} X - \phi(\nabla_X V^\perp) = 0.$$

Now, comparing the tangential and normal components, we have

$$\phi(\nabla_X V^\perp) = -A_{\phi V^\perp} X, \ \nabla^\perp_X \phi V^\perp = 0. \tag{4.2}$$

Hence the proposition is proved. □

Theorem 4.1. *Let M be a generic submanifold of an LP-Sasakian manifold $\widetilde{M}$ with concurrent vector field V. If the submanifold M is a D-geodesic and $\nabla_X V^T$ is not orthogonal to ξ, then V^T on D is concurrent if and only if $f = 0$.*

Proof. From Proposition 4.1. we get

$$\nabla_X \phi V^T + \nabla_X V^\perp - A_{\phi V^\perp} X = g(X, V^T)\xi - fX + \phi X. \tag{4.3}$$

Since, M is a D-geodesic, then from (4.1) we obtain

$$\nabla_X \phi V^T = \phi(\nabla_X V^T) + g(X, V^T)\xi. \tag{4.4}$$

Combining (4.3) and (4.4)

$$\phi(\nabla_X V^T) + \nabla_X V^\perp - A_{\phi V^\perp} X = -fX + \phi X. \tag{4.5}$$

From Lemma 4.1, we have the following

$$\nabla_X V^T + \nabla_X V^\perp - A_{\phi V^\perp} X = X - f\phi X. \tag{4.6}$$

Subtracting (4.6) from (4.5), we have

$$\phi(\nabla_X V^T) - \nabla_X V^T = (1+f)\phi X - (1+f)X. \tag{4.7}$$

Now, the equation (4.7) can be written as follows

$$\phi(\nabla_X V^T - X - fX) = (\nabla_X V^T - X - fX). \tag{4.8}$$

Applying ϕ on both sides of (4.8), we have the following

$$\eta(\nabla_X V^T - X - fX) = 0.$$

Hence, by the given condition we get

$$\nabla_X V^T = X + fX.$$

Therefore V^T is concurrent on D if and only if f is a zero function. □

In the next theorem we study $V^\perp$ on M.

Theorem 4.2. *Let M be a generic semi-invariant product of an LP-Sasakian manifold $\widetilde{M}$ with concurrent vector field V. If the submanifold M is a mixed-geodesic and $\nabla_X V^\perp \in \Gamma(D)$ for any $X \in \Gamma(D)$, then $V^\perp$ is a parallel vector field.*

Proof. From Proposition 4.3. we get

$$\phi(\nabla_X V^\perp) = -A_{\phi V^\perp} X.$$

Using Lemma 2.1,

$$\phi(\nabla_X V^\perp) = 0.$$

Applying ϕ on both sides of this we have

$$-\nabla_X V^\perp + \eta(\nabla_X V^\perp)\xi = 0.$$

Now, from the given condition we get

$$\nabla_X V^\perp = 0.$$

Therefore, $V^\perp$ is a parallel vector field. □

5. Generic submanifolds with conformal vector fields

In this section we study an interesting relation between conformal vector fields and umbilical submanifolds.

Definition 5.1. The mean curvature vector H of an n-dimensional submanifold M is defined by

$$H = \frac{1}{n}\text{trace } \sigma,$$

where σ is the second fundamental form defined in Section 2.

Definition 5.2. A submanifold M is called totally umbilical if its second fundamental form σ satisfies

$$\sigma(X,Y) = g(X,Y)H.$$

Definition 5.3. A submanifold M is said to be umbilical with respect to a normal vector field N if its second fundamental form σ satisfies

$$g(\sigma(X,Y),N) = \mu g(X,Y) \tag{5.1}$$

for a function μ.

Theorem 5.1. *Let M be a generic submanifold of an LP-Sasakian manifold $\widetilde{M}$ endowed with a concurrent vector field V. If $f = 0$ and $\nabla_X Y = -\nabla_Y X$, then V^T is a conformal vector field if and only if M is umbilical with respect to $\phi V^\perp$.*

Proof. It is well known that the Lie derivative on M satisfies

$$(Ł_V g)(X,Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X) \tag{5.2}$$

for any vector fields X, Y, V tangent to M.

Putting $V = V^T$ in the above equation we get

$$(Ł_{V^T} g)(X,Y) = g(\nabla_X V^T, Y) + g(\nabla_Y V^T, X). \tag{5.3}$$

Using Lemma 4.1 in (5.3) we have

$$\begin{aligned}(Ł_{V^T} g)(X,Y) &= 2g(X,Y) - 2fg(\phi X, Y) + 2g(\sigma(X,Y), \phi V^\perp) \\ &\quad - g(\nabla_X V^\perp, Y) - g(\nabla_Y V^\perp, X).\end{aligned} \tag{5.4}$$

Again we know that

$$(\nabla_X g)(V^\perp, Y) = \nabla_X g(V^\perp, Y) - g(\nabla_X V^\perp, Y) - g(V^\perp, \nabla_X Y). \tag{5.5}$$

Since, M is a generic submanifold, we have from (5.5)

$$g(\nabla_X V^\perp, Y) = -g(V^\perp, \nabla_X Y). \tag{5.6}$$

Similarly we get

$$g(\nabla_Y V^\perp, X) = -g(V^\perp, \nabla_Y X). \tag{5.7}$$

Using (5.6) and (5.7) in (5.4) we have

$$\begin{aligned}(Ł_{V^T} g)(X,Y) &= 2g(X,Y) - 2fg(\phi X, Y) + 2g(\sigma(X,Y), \phi V^\perp) \\ &\quad + g(V^\perp, \nabla_X Y + \nabla_Y X).\end{aligned} \tag{5.8}$$

Applying the given condition we get

$$(Ł_{V^T} g)(X,Y) = 2g(X,Y) + 2g(\sigma(X,Y), \phi V^\perp). \tag{5.9}$$

Now let us assume that, the vector field V^T is conformal. Then from the definition of conformal vector field we have

$$(Ł_{V^T} g)(X,Y) = \rho g(X,Y) \tag{5.10}$$

for a function ρ. From (5.9) and (5.10), we get

$$g(\sigma(X,Y),\phi V^{\perp}) = (\frac{\rho}{2} - 1)g(X,Y).$$

From above it is clear that M is umbilical with respect to $\phi V^{\perp}$.

Conversely, we assume that M is umbilical with respect to $\phi V^{\perp}$, so we have

$$g(\sigma(X,Y),\phi V^{\perp}) = \mu g(X,Y) \tag{5.11}$$

for a function μ.

From (5.9) and (5.11), we get

$$(\L_{V^T} g)(X,Y) = 2(1+\mu)g(X,Y). \tag{5.12}$$

Therefore we have V^T is a conformal vector field.

Hence the theorem is proved. □

6. Ricci solitons in generic submanifolds

In this section we study the generic submanifolds admitting Ricci solitons of LP-Sasakian manifolds with concurrent vector fields. First we prove two important lemmas.

Lemma 6.1. *Let M be a generic submanifold admitting a Ricci soliton of an LP-Sasakian manifold $\widetilde{M}$ endowed with a concurrent vector field V. Then the Ricci tensor S_D of the invariant distribution D is given by*

$$\begin{aligned} S_D(X,Y) = {} & (\lambda - 1)g(X,Y) + fg(\phi X,Y) - g(\sigma(X,Y),\phi V^{\perp}) \\ & - \frac{1}{2}g(V^{\perp},\nabla_X Y + \nabla_Y X), \end{aligned}$$

where ∇ is the Levi-Civita connection on M for any $X,Y \in \Gamma(D)$.

Proof. From the definition of Lie derivative, we get

$$(\L_{V^T} g)(X,Y) = g(\nabla_X V^T, Y) + g(\nabla_Y V^T, X). \tag{6.1}$$

Using Lemma 4.1 in (6.1) we have

$$\begin{aligned} (\L_{V^T} g)(X,Y) = {} & 2g(X,Y) - 2fg(\phi X,Y) + 2g(\sigma(X,Y),\phi V^{\perp}) \\ & - g(\nabla_X V^{\perp},Y) - g(\nabla_Y V^{\perp},X). \end{aligned} \tag{6.2}$$

Again we know that

$$(\nabla_X g)(V^{\perp},Y) = \nabla_X g(V^{\perp},Y) - g(\nabla_X V^{\perp},Y) - g(V^{\perp},\nabla_X Y). \tag{6.3}$$

Since M is a generic submanifold, we have from (6.3)

$$g(\nabla_X V^{\perp},Y) = -g(V^{\perp},\nabla_X Y). \tag{6.4}$$

Similarly we get

$$g(\nabla_Y V^{\perp},X) = -g(V^{\perp},\nabla_Y X). \tag{6.5}$$

Using (6.4) and (6.5) in (6.2) we have

$$(\L_{V^T} g)(X,Y) = 2g(X,Y) - 2fg(\phi X,Y) + 2g(\sigma(X,Y),\phi V^{\perp})$$

$$+ g(V^{\perp}, \nabla_X Y + \nabla_Y X). \tag{6.6}$$

Again, since the generic submanifold M admits a Ricci soliton, using the equation (1.1) we have the following

$$(\text{Ł}_{V^T} g)(X,Y) + 2S_D(X,Y) = 2\lambda g(X,Y). \tag{6.7}$$

Combining the equations (6.6) and (6.7) we get

$$\begin{aligned} S_D(X,Y) = (\lambda - 1)g(X,Y) + fg(\phi X, Y) - g(\sigma(X,Y), \phi V^{\perp}) \\ - \frac{1}{2} g(V^{\perp}, \nabla_X Y + \nabla_Y X), \end{aligned}$$

which is the required result. □

Lemma 6.2. *Let M be a generic submanifold admitting a Ricci soliton of an LP-Sasakian manifold $\widetilde{M}$ endowed with a concurrent vector field V. Then the Ricci tensor $S_{D^{\perp}}$ of the anti-invariant distribution $D^{\perp}$ is given by*

$$S_{D^{\perp}}(X,Y) = (\lambda - 1)g(X,Y) - g(\sigma(X,Y), \phi V^{\perp}) - \frac{1}{2} g(V^{\perp}, \nabla_X Y + \nabla_Y X),$$

where ∇ is the Levi-civita connection on M for any $X, Y \in \Gamma(D^{\perp})$.

Proof. The proof is similar to the proof of Lemma 6.1. □

Theorem 6.1. *Let M be a generic submanifold admitting a Ricci soliton of an LP-Sasakian manifold $\widetilde{M}$ endowed with a concurrent vector field V. If the invariant distribution D is D-parallel and $f = 0$, then the invariant distribution D is Einstein.*

Proof. It is an easy consequence from Lemma 6.1. □

Theorem 6.2. *Let M be a generic submanifold admitting a Ricci soliton of an LP-Sasakian manifold $\widetilde{M}$ endowed with a concurrent vector field V. If the anti-invariant distribution $D^{\perp}$ is $D^{\perp}$-parallel, then the anti-invariant distribution $D^{\perp}$ is Einstein.*

Proof. It follows from Lemma 6.2. □

Theorem 6.3. *Let M be a generic semi-invariant product admitting a Ricci soliton of an LP-Sasakian manifold $\widetilde{M}$ endowed with a concurrent vector field V and also let $f = 0$. Then the following are satisfied*:

(i) *The vector field V^T is conformal Killing on D.*
(ii) *The invariant distribution D is Einstein.*

Proof. From (6.6) we have

$$\begin{aligned} (\text{Ł}_{V^T} g)(X,Y) = 2g(X,Y) - 2fg(\phi X, Y) + 2g(\sigma(X,Y), \phi V^{\perp}) \\ + g(V^{\perp}, \nabla_X Y + \nabla_Y X). \end{aligned} \tag{6.8}$$

From the given condition it follows that

$$(\text{Ł}_{V^T} g)(X,Y) = 2g(X,Y). \tag{6.9}$$

Therefore, the vector field V^T is conformal Killing on D.

Since M be a generic submanifold admitting a Ricci soliton we get from (1.1)

$$(\text{Ł}_{V^T} g)(X,Y) + 2S_D(X,Y) = 2\lambda g(X,Y). \tag{6.10}$$

Combining the equations (6.9) and (6.10) we get

$$S_D(X,Y) = (\lambda - 1)g(X,Y).$$

So, we can conclude that D is Einstein. Hence the theorem is proved. □

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