Commun. Korean Math. Soc. **36** (2021), No. 2, pp. 209–227
https://doi.org/10.4134/CKMS.c200209
pISSN: 1225-1763 / eISSN: 2234-3024

SYMMETRICITY AND REVERSIBILITY FROM THE PERSPECTIVE OF NILPOTENTS

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Abstract. In this paper, we deal with the question that what kind of properties does a ring gain when it satisfies symmetricity or reversibility by the way of nilpotent elements? By the motivation of this question, we approach to symmetric and reversible property of rings via nilpotents. For symmetricity, we call a ring R *middle right-*(resp. *left-*)*nil symmetric* (mr-nil (resp. ml-nil) symmetric, for short) if $abc = 0$ implies $acb = 0$ (resp. $bac = 0$) for a, $c \in R$ and $b \in \mathrm{nil}(R)$ where $\mathrm{nil}(R)$ is the set of all nilpotent elements of R. It is proved that mr-nil symmetric rings are abelian and so directly finite. We show that the class of mr-nil symmetric rings strictly lies between the classes of symmetric rings and weak right nil-symmetric rings. For reversibility, we introduce *left* (resp. *right*) *N-reversible ideal* I of a ring R if for any $a \in \mathrm{nil}(R)$, $b \in R$, being $ab \in I$ implies $ba \in I$ (resp. $b \in \mathrm{nil}(R)$, $a \in R$, being $ab \in I$ implies $ba \in I$). A ring R is called *left* (resp. *right*) *N-reversible* if the zero ideal is left (resp. right) N-reversible. Left N-reversibility is a generalization of mr-nil symmetricity. We exactly determine the place of the class of left N-reversible rings which is placed between the classes of reversible rings and CNZ rings. We also obtain that every left N-reversible ring is nil-Armendariz. It is observed that the polynomial ring over a left N-reversible Armendariz ring is also left N-reversible.

1. Introduction

Throughout this paper, all rings are associative with identity. A ring is called *reduced* if it has no nonzero nilpotent elements. A weaker condition was defined by Lambek in [18]. A ring R is said to be *symmetric* if for any a, $b, c \in R$, $abc = 0$ implies $acb = 0$. The class of weak symmetric rings was discussed in [26] and also studied in [12]. A ring R is called *weak symmetric* if $abc \in \mathrm{nil}(R)$ implies $acb \in \mathrm{nil}(R)$ for all a, b, $c \in R$. Generalized weakly symmetric rings (or GWS, for short) were studied in [30]. A ring R is called *GWS* if $abc = 0$ implies that bac is nilpotent for all a, b, $c \in R$. In [15], nil-symmetric rings

Received June 19, 2020; Accepted September 7, 2020.
2010 *Mathematics Subject Classification.* 16N40, 16S99, 16U80, 16U99.
Key words and phrases. Symmetric ring, middle right-nil symmetric ring, nil-symmetric ring, reversible ring, left N-reversible ring.

were weakened to weak nil-symmetric rings. A ring R is called *weak right nil-symmetric* if $abc = 0$ implies $acb = 0$ for all nilpotent a, b, $c \in R$ and it is called *weak left nil-symmetric* if $abc = 0$ implies $cab = 0$ for all nilpotent a, b, $c \in R$, and R is called *weak nil-symmetric* if it is both weak right nil-symmetric and weak left nil-symmetric. In [7], Chakraborty and Das called a ring R *right* (resp. *left*) *nil-symmetric* if $abc = 0$ (resp. $cab = 0$) implies $acb = 0$ for all nilpotent a, $b \in R$ and $c \in R$ and the ring R is *nil-symmetric* if it is both right and left nil-symmetric.

As an another generalization of the symmetric property of a ring, Cohn [9] called a ring R *reversible* if for a, $b \in R$, $ab = 0$ implies $ba = 0$. Anderson and Camillo [3] observed the rings whose zero products commute, and used the term ZC_2 for what is called reversible. Prior to Cohn's work, reversible rings were studied under the names of *completely reflexive* by Mason in [21] and *zero commutative* by Habeb in [10], and Tuganbaev [28] investigated reversible rings in the name of *commutative at zero*. Following [17], a ring R is called *central reversible* if $ab = 0$ for any $a, b \in R$ implies ba is a central element of R. Every reversible ring is central reversible. The reversible property of a ring is also generalized as: A ring R is said to satisfy the *commutativity of nilpotent elements at zero* [1, Definition 2.1] if $ab = 0$ for a, $b \in \mathrm{nil}(R)$ implies $ba = 0$. For simplicity, a ring is called *CNZ* if it satisfies the commutativity of nilpotent elements at zero. CNZ rings were generalized in [16]. A ring is called *central CNZ* if for any nilpotent a, $b \in R$, $ab = 0$ implies ba is central in R. Another generalization of reversible rings is nil-reversible. In [24], a ring R is called *nil-reversible* if for every $a \in R$, $b \in \mathrm{nil}(R)$, $ab = 0$ if and only if $ba = 0$.

Nilpotent elements are important tools for studying the structures of rings. In the light of aforementioned notions, we focus on the symmetricity and reversibility from the perspective of nilpotents. Motivated by the works on symmetric rings and reversible rings, the goal of this paper is to extend the notions of symmetric rings and reversible rings via nilpotents, namely, mr-nil symmetric rings and left N-reversible rings. We present that the concept of left N-reversible rings also generalizes that of mr-nil symmetric rings. We exactly determine the places of these classes of rings in ring theory, in the meantime we give various examples. We study the properties of mentioned classes of rings. It is obtained that being a von Neumann regular ring and being a strongly regular ring coincide for left N-reversible rings. Also, for semiprime rings and right p.p.-rings the concepts of reduced, symmetric, reversible, mr-nil symmetric and left N-reversible rings are the same. On the other hand, some extensions such as Dorroh extensions, Nagata extensions, polynomial rings of left N-reversible rings are also studied.

In what follows, $\mathbb{Z}$ denotes the ring of integers and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. For a ring R, $U(R)$, $\mathrm{Id}(R)$, $C(R)$, $P(R)$ and $J(R)$ denote the group of units, the set of all idempotents in R, the center of R, the prime radical and the Jacobson radical of R, respectively. Also, $M_n(R)$ stands for the ring of all $n \times n$ matrices, $U_n(R)$ is the ring of upper triangular matrices over R for a positive integer $n \geq 2$, $D_n(R)$ is the ring of all matrices in $U_n(R)$ having main diagonal entries equal, and $V_n(R)$ is the subring of $U_n(R)$:

$$V_n(R) = \left\{ \sum_{i=j}^{n} \sum_{j=1}^{n} a_j e_{(i-j+1)i} \mid a_j \in R \right\}.$$

For instance, elements of $V_4(R)$ has the form:

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{bmatrix},$$

where $a_1, a_2, a_3, a_4 \in R$. Let (x^n) denote the ideal generated by x^n in $R[x]$. It is obvious that $R[x]/(x^n) \cong V_n(R)$. Also,

$$V_n^k(R) = \left\{ \sum_{i=j}^{n} \sum_{j=1}^{k} a_j e_{(i-j+1)i} + \sum_{i=j}^{n-k} \sum_{j=1}^{n-k} a_{ij} e_{j(k+i)} \mid a_j, a_{ij} \in R \right\},$$

where $a_i \in R$, $a_{js} \in R$, $1 \leq i \leq k$, $1 \leq j \leq n-k$ and $k+1 \leq s \leq n$. For instance, elements of $V_4^2(R)$ are of the form:

$$\begin{bmatrix} a_1 & a_2 & a_{13} & a_{14} \\ 0 & a_1 & a_2 & a_{24} \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{bmatrix},$$

where $a_1, a_2, a_{13}, a_{14}, a_{24} \in R$ and

$$D_n^k(R) = \left\{ \sum_{i=1}^{k} \sum_{j=k+1}^{n} a_{ij} e_{ij} + \sum_{j=k+2}^{n} b_{(k+1)j} e_{(k+1)j} + cI_n \mid a_{ij}, b_{ij}, c \in R \right\},$$

where $k = [n/2]$, i.e., k satisfies $n = 2k$ when n is an even integer, and $n = 2k+1$ when n is an odd integer. Elements of $D_n^k(R)$ for $n = 4$ and $n = 5$ are of the form:

$$\begin{bmatrix} a_1 & 0 & a_{13} & a_{14} \\ 0 & a_1 & a_{23} & a_{24} \\ 0 & 0 & a_1 & a_{34} \\ 0 & 0 & 0 & a_1 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & a_1 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_1 & a_{34} & a_{35} \\ 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & a_1 \end{bmatrix},$$

respectively.

2. Middle right-(left-)nil symmetric rings

In this section, we introduce and study a class of rings, middle-nil symmetric rings which is weaker than the class of symmetric rings and stronger than the classes of nil-symmetric rings, weak symmetric rings and weak right nil-symmetric rings.

Definition 2.1. A ring R is called *middle right-nil symmetric* (simplicity, *mr-nil symmetric*) if $abc = 0$ implies $acb = 0$ for a, $c \in R$, $b \in \mathrm{nil}(R)$. Similarly, R is called *middle left-nil symmetric* (*ml-nil symmetric*, for short) if $abc = 0$ implies $bac = 0$ for a, $c \in R$, $b \in \mathrm{nil}(R)$.

We have the following hierarchy:

$$\begin{aligned}\{\text{symmetric rings}\} &\subseteq \{\text{mr-nil symmetric rings}\}\\ &\subseteq \{\text{weak right nil-symmetric rings}\}.\end{aligned}$$

The following examples show that the aforementioned implications are strict. The next example also shows that the middle-nil symmetricity is not left-right symmetric.

Example 2.2. Let F be a field and define the free algebra $A = F\langle a, b\rangle$ where a and b are noncommuting indeterminates. Let I be the two-sided ideal generated by the elements ab and b^2 and consider the ring $R = A/I$. Then R is mr-nil symmetric but not symmetric and not ml-nil symmetric.

Proof. Since $ab = 0$ and $ba \neq 0$, R is not symmetric. Also, $aba = 0$ but $baa \neq 0$. Hence R is not ml-nil symmetric. Next we prove that R is mr-nil symmetric. Note that nilpotent elements of R have the form bra, b and brb where $r \in R$. We write a and b for their images in the factor ring R. The elements of R is the finite sum of the some of the monomials of the form $\alpha a^i b^j$, $\beta b^l a^m$, γa^k and δb where $\alpha, \beta, \gamma, \delta \in F$ and i, j, l, m and k are positive integers. Let x, $y \in R$ and $n \in \mathrm{nil}(R)$ with $xny = 0$. Then n is one of the form b, brb or bra for some $r \in R$. Note that $(a^i b^j)b = 0$, $(b^l a^m)b = 0$, $a^k b = 0$ and $bb = 0$ where i, j, l, m and k are positive integers. This implies $yb = 0$ for any $y \in R$. It follows that $xyn = 0$. Therefore R is mr-nil symmetric. This completes the proof. □

Example 2.3. Let R be a reduced ring. Then $U_2(R)$ is weak right nil-symmetric but not mr-nil symmetric.

Proof. Let E_{ij} denote the matrix units in $U_2(R)$. Then $U_2(R)$ is weak right nil-symmetric since the product of two nilpotents is zero in $U_2(R)$. Let $A = E_{12} \in \mathrm{nil}(U_2(R))$, $B = E_{11} \in U_2(R)$ and I be the unit matrix. Then $IAB = 0$ but $IBA \neq 0$. Hence $U_2(R)$ is not mr-nil symmetric. □

An idempotent e of a ring R is called *right* (resp. *left*) *semicentral* if $ex = exe$ (resp. $xe = exe$) for each $x \in R$. The ring R is called *right* (resp. *left*) *semicentral* in case every idempotent is right (resp. left) semicentral. A ring R is called *abelian* if R is both left and right semicentral. A ring R is called *2-primal* if $P(R) = \mathrm{nil}(R)$, and R is said to be an *NI ring* if $\mathrm{nil}(R)$ forms an ideal. 2-primal rings are NI. In [6], Bell called a ring R to satisfy the *Insertion-of-Factors Property* (in short, *IFP*) if $ab = 0$ implies $aRb = 0$ for a, $b \in R$. In [23], a ring R is called *nil-semicommutative* if for every a, $b \in \mathrm{nil}(R)$, $ab = 0$ implies $aRb = 0$. In [1], a nil-semicommutative ring is called *nil-IFP*.

Proposition 2.4. *Let R be an mr-nil symmetric ring. Then the following hold.*

(1) *R is abelian.*
(2) *R is nil-semicommutative.*
(3) *R is 2-primal.*
(4) *Subrings of R is mr-nil symmetric.*

Proof. (1) Let $e \in \text{Id}(R)$ and $r \in R$. Then $er - ere, re - ere \in \text{nil}(R)$ and $1(er - ere)e = 0$. Hence by hypothesis, $er = ere$. Similarly, $1(re - ere)(1 - e) = 0$. By hypothesis, $(1 - e)(re - ere) = 0$. We have $re = ere$. Thus $er = re$ for each $r \in R$. Therefore R is abelian.

(2) Let a, $b \in \text{nil}(R)$ with $ab = 0$. For any $r \in R$, $abr = 0$. By hypothesis $arb = 0$. Hence $aRb = 0$.

(3) Let $a \in \text{nil}(R)$ and $a^n = 0$ for some positive integer n. For any $r_1 \in R$, $a^{n-1}ar_1 = 0$. By hypothesis, $a^{n-1}r_1a = 0$. Let $r_2 \in R$. Then $a^{n-1}r_1ar_2 = 0$. By hypothesis, $a^{n-2}r_1ar_2a = 0$. Let $r_3 \in R$. Then $a^{n-2}r_1ar_2ar_3 = 0$. By hypothesis, again $a^{n-3}r_1ar_2ar_3a = 0$. Continuing in this way $ar_1ar_2ar_3a \cdots ar_na = 0$ for all $r_1, r_2, r_3, \ldots, r_n \in R$. Hence $ar_1ar_2ar_3a \cdots r_{n-1}aRa = 0$. Let P be any prime ideal of R. Then $ar_1ar_2ar_3a \cdots r_{n-1}aRa \subseteq P$. If $a \in P$, there is nothing to do. Otherwise, $ar_1ar_2ar_3a \cdots r_{n-2}aRa \subseteq P$. Since $a \notin P$, $ar_1ar_2ar_3a \cdots r_{n-3}aRa \subseteq P$. Continuing in this way we reach $aRa \subseteq P$. Hence $a \in P$. This contradiction proves that all nilpotents belong to $P(R)$.

(4) It is clear. □

The first three conditions of Proposition 2.4 are all left-right agnostic, so we have the following result.

Proposition 2.5. *Let R be an ml-nil symmetric ring. Then the following hold.*

(1) *R is abelian.*
(2) *R is nil-semicommutative.*
(3) *R is 2-primal.*
(4) *Subrings of R is ml-nil symmetric.*

Proof. Similar to the proof of Proposition 2.4. □

Examples 2.6. (1) For any ring R and any positive integer $n \geq 2$, $U_n(R)$ and $M_n(R)$ are not mr-nil symmetric.
(2) For a commutative ring R and a positive integer n, $V_n(R)$ is mr-nil symmetric.
(3) There are abelian rings that are not mr-nil symmetric.

Proof. (1) It is enough to show for $n = 2$. Let R be a ring and consider $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in U_2(R)$ and $B = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in \text{nil}(U_2(R))$. Then $ABC = 0$. However, $ACB = B \neq 0$.

(2) Clear from the fact that R is commutative if and only if $V_n(R)$ is commutative.

(3) We consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \mid a, b, c, d \in \mathbb{Z},\ a \equiv d \pmod 2), b \equiv c \equiv 0 \pmod 2) \right\}.$$

The idempotents of R are zero and identity matrices. So R is an abelian ring. Let $A = \left[\begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix}\right] \in R$, $B = \left[\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{nil}(R)$, $C = \left[\begin{smallmatrix} 2 & 2 \\ 0 & 0 \end{smallmatrix}\right] \in R$. Then $ABC = 0$. But $ACB \neq 0$. So R is abelian but not mr-nil symmetric. □

The class of mr-nil symmetric rings is closed under taking subrings and isomorphisms of rings, however, it is not closed forming factor rings as the next example shows.

Example 2.7. Let F be a field and $R = F\langle a, b, c\rangle$ be the free algebra with noncommuting indeterminates a, b, c. Then R does not contain nonzero nilpotents and so is an mr-nil symmetric ring. Let I be the ideal of R generated by ab, a^2 and b^2. Let $\overline{R} = R/I$ and $\overline{a} = a + I$, $\overline{b} = b + I$, $\overline{r} = r + I \in \overline{R}$. By the definition of the ideal I, $\overline{a}$ is nilpotent in $\overline{R}$ and $\overline{c}\overline{a}\overline{b} = 0$, but $\overline{c}\overline{b}\overline{a} \neq 0$. Hence $\overline{R}$ is not mr-nil symmetric.

For any ring R and a positive integer $n \geq 2$, $M_n(R)$ is not mr-nil symmetric. However, there are subrings of $M_n(R)$ that are mr-nil symmetric as shown below.

Proposition 2.8. *Let R be a ring with no zero divisors. Then $V_n(R)$ is mr-nil symmetric for every positive integer n.*

Proof. The cases $n = 1$ and $n = 2$ are clear. Firstly, we give a proof for $n = 3$. Let $n = 3$ and $A = \left[\begin{smallmatrix} a & x & y \\ 0 & a & x \\ 0 & 0 & a \end{smallmatrix}\right]$, $C = \left[\begin{smallmatrix} c & d & t \\ 0 & c & d \\ 0 & 0 & c \end{smallmatrix}\right] \in V_3(R)$, $B = \left[\begin{smallmatrix} 0 & b & z \\ 0 & 0 & b \\ 0 & 0 & 0 \end{smallmatrix}\right] \in \mathrm{nil}(V_3(R))$ with $ABC = 0$. Then

$$ABC = \begin{bmatrix} 0 & abc & abd + azc + xbc \\ 0 & 0 & abc \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Hence $abc = 0$ and $abd + azc + xbc = 0$. Note that

$$ACB = \begin{bmatrix} 0 & acb & acz + adb + xcb \\ 0 & 0 & acb \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $abc = 0$, by hypothesis, $a = 0$ or $bc = 0$. If $a = 0$, then $ABC = 0$ implies $xbc = 0$. Then $x = 0$ or $bc = 0$. Hence $x = 0$ or $cb = 0$. Thus $ACB = 0$.

Assume that $a \neq 0$ and $bc = 0$. Then $cb = 0$. Hence $ABC = 0$ implies $abd + azc = 0$. We consider this equality in two cases:
Case I. If $b = 0$ and $c \neq 0$, then $z = 0$. Hence $ACB = 0$.
Case II. If $b \neq 0$ and $c = 0$, then $d = 0$. Hence $ACB = 0$.
To complete the proof, we generalize the discussion for the integer $n \geq 4$. Let $A = \sum\limits_{i=j}^{n} \sum\limits_{j=1}^{n} a_j e_{(i-j+1)i} \in V_n(R)$, $B = \sum\limits_{i=j}^{n} \sum\limits_{j=2}^{n} b_j e_{(i-j+1)i} \in \mathrm{nil}(V_n(R))$ and

$C = \sum_{i=j}^{n} \sum_{j=1}^{n} c_j e_{(i-j+1)i} \in V_n(R)$ with $ABC = 0$. Then we have the following equalities:

$$
\begin{aligned}
&(1) && a_1b_2c_1 = 0,\\
&(2) && a_1b_2c_2 + (a_1b_3 + a_2b_2)c_1 = 0,\\
&(3) && a_1b_2c_3 + (a_1b_3 + a_2b_2)c_2 + (a_1b_4 + a_2b_3 + a_3b_2)c_1 = 0,\\
&(4) && a_1b_2c_4 + (a_1b_3 + a_3b_2)c_3 + (a_1b_4 + a_2b_3 + a_3b_2)c_2\\
& && + (a_1b_5 + a_2b_4 + a_3b_3 + a_4b_2)c_1 = 0,\\
& && \vdots\\
&(\text{n-1}) && a_1b_2c_{n-2} + (a_1b_3 + a_2b_2)c_{n-3} + (a_1b_4 + a_2b_3 + a_3b_2)c_{n-4}\\
& && + \cdots + (AB)_{(1,n-1)}c_1 = 0,\\
&(\text{n}) && a_1b_2c_{n-1} + (a_1b_3 + a_2b_2)c_{n-2} + (a_1b_4 + a_2b_3 + a_3b_2)c_{n-3}\\
& && + \cdots + (AB)_{(1,n-1)}c_2 + (AB)_{(1,n)}c_1 = 0,
\end{aligned}
$$

where $(AB)_{(1,n-1)}$ stands for $a_1b_{n-1} + a_2b_{n-2} + a_3b_{n-3} + \cdots + a_{n-3}b_3 + a_{n-2}b_2$ the $(1, n-1)$ entry of AB, $(AB)_{(1,n)}$ is $a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \cdots + a_{n-3}b_4 + a_{n-2}b_3 + a_{n-1}b_2$ the $(1, n)$ entry of AB.

Since $a_1b_2c_1 = 0$ and R has no zero divisors, we have the following cases.
Case I. $a_1 = 0$ and $b_2c_1 \neq 0$. Then (2) implies $a_2 = 0$, (3) implies $a_3 = 0, \dots,$ (n-1) implies $a_{n-2} = 0$ and from (n) we have $a_{n-1} = 0$. Hence $ACB = 0$.
Case II. $a_1 \neq 0$ and $b_2c_1 = 0$.
Subcase (i) $c_1 = 0$. Then (2) implies $b_2c_2 = 0$ and $c_2b_2 = 0$. (3) implies $b_2c_3 + b_3c_2 = 0$. Multiplying the latter from the left by c_2 yields $(b_3c_2)^2 = 0$. Hence $b_3c_2 = 0$, $c_2b_3 = 0$, similarly, we have $b_2c_3 = 0$, $c_3b_2 = 0$. By (4) we get $b_2c_4 + b_3c_3 + b_4c_2 = 0$. Multiplying the latter from the left by c_2, we get $c_2b_4 = 0$, $b_4c_2 = 0$ and $b_2c_4 + b_3c_3 = 0$. Multiplying the latter from the left by c_3 and using $c_3b_2 = 0$, we get $b_3c_3 = 0$, therefore $c_3b_3 = 0$, $b_2c_4 = 0$ and $c_4b_2 = 0$. Continuing in this way, (n-1) implies $b_ic_j = 0$ and $c_jb_i = 0$ for $1 \leq i, j \leq n-2$. Then (n) reads $b_2c_{n-1} + b_3c_{n-2} + b_4c_{n-3} + \cdots + b_{n-2}c_3 + b_{n-1}c_2 = 0$. Multiplying the latter from the left by c_2 we have $b_{n-1}c_2 = 0$ and $c_2b_{n-1} = 0$. The remaining is $b_2c_{n-1} + b_3c_{n-2} + b_4c_{n-3} + \cdots + b_{n-2}c_3 = 0$. Again multiplying the latter from the left by c_3, we have $b_{n-2}c_3 = 0$ and $c_3b_{n-2} = 0$. Continuing in this way, $b_ic_{n-(i-1)} = 0$ and $c_{n-(i-1)}b_i = 0$ for $2 \leq i \leq n-1$. It follows that $ACB = 0$.
Subcase (ii) $c_1 \neq 0$ and $b_2 = 0$. Then (2) implies $a_1b_3c_1 = 0$. Since $a_1 \neq 0$ and $c_1 \neq 0$, $b_3 = 0$. (3) implies $a_1b_4c_1 = 0$. So $b_4 = 0$. (4) implies $a_1b_5c_1 = 0$. So $b_5 = 0$. Continuing in this way we reach to $b_6 = \cdots = b_{n-1} = 0$. Thus $ACB = 0$. This completes the proof. □

Proposition 2.9. *Let R be a ring and S denote any one of the subrings $V_n(R)$, $V_n^k(R)$, $D_n(R)$ and $D_n^k(R)$ of $M_n(R)$. If S is mr-nil symmetric, then R is mr-nil symmetric.*

Proof. Let $a, c \in R$ and $b \in \text{nil}(R)$ with $abc = 0$. Consider $A = aI_n$, $B = bI_n$ and $C = cI_n$. Then $B \in \text{nil}(S)$ and $ABC = 0$. By hypothesis $ACB = 0$. Hence $acb = 0$. It follows that R is mr-nil symmetric. □

3. Left (Right) N-reversible rings

As noted in Introduction, the reversible ring property is generalized as central reversible, CNZ and central CNZ ring properties. In this section, we approach to reversibility from the perspective of nilpotents, namely, left (right) N-reversible rings. This notion is also a generalization of the mr-nil symmetric ring.

Definition 3.1. A ring R is called *left N-reversible* if for any nilpotent $a \in R$ and $b \in R$, $ab = 0$ implies $ba = 0$. Right N-reversible ring is defined similarly. A ring R is called *N-reversible* if it is both left N-reversible and right N-reversible.

The concept of a left (right) N-reversible ring is placed between that of reversible rings and CNZ rings, i.e.,

$$\begin{aligned}\{\text{symmetric rings}\} &\subseteq \{\text{reversible rings}\}\\ &\subseteq \{\text{left (right) N-reversible rings}\}\\ &\subseteq \{\text{CNZ rings}\}.\end{aligned}$$

Note that the subring of a left N-reversible ring is left N-reversible. In [29], a ring R is called *central reduced* if all nilpotent elements of R are central. Clearly, every central reduced ring is N-reversible. The property of N-reversibility of a ring is not left-right symmetric as shown below.

Examples 3.2. (1) There are CNZ rings that are not left N-reversible.
(2) There are left N-reversible rings but not right N-reversible and so not reversible.
(3) A ring is nil-reversible if and only if it is N-reversible.
(4) Every mr-nil symmetric ring is left N-reversible.
(5) There are left N-reversible rings which are not mr-nil symmetric.

Proof. (1) Let F be a field and $R = U_2(F)$. Then $\text{nil}(R) = \left[\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right]$. Let A, $B \in \text{nil}(R)$, then $AB = 0$ implies $BA = 0$. So R is CNZ. For $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in R$, we have $AB = 0$ but $BA \neq 0$. Hence R is not left N-reversible.

(2) Let F be a field and $A = F\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a and b. Let I denote the ideal of A generated by ba and a^2. Then the ring $R = A/I$ is left N-reversible but not right N-reversible and so not reversible. The ring R has a and ab^t as only nonzero nilpotent elements where t is any positive integer. Any element of R has the form $f(a, b) = k_0 + k_1 a + k_2 b^n + k_3 ab^m$. Assume that $af(a, b) = 0$. Then

$k_0a + k_2ab^n = 0$. Hence $k_0 = k_2 = 0$ since a and ab^n are not in I. So $f(a, b)a = k_1a^2 + k_3ab^ma = 0$. Similarly, $ab^t f(a, b) = 0$ implies $f(a, b)ab^t = 0$. Thus R is left N-reversible. Since $ba = 0$ and $ab \neq 0$, R is not right N-reversible. This also shows R is not reversible.

(3) Let R be a nil-reversible ring and $a \in \text{nil}(R)$ and $b \in R$ with $ab = 0$. Since $ab = 0$ if and only if $ba = 0$, $ab = 0$ implies $ba = 0$ and then R is left N-reversible. If $ba = 0$ and R is nil-reversible, then $ab = 0$. Hence R is right N-reversible. Conversely, assume that R is N-reversible. Let $a \in R$ and $b \in \text{nil}(R)$. If $ab = 0$, then R being right N-reversible implies $ba = 0$. If $ba = 0$, then R being left N-reversible implies $ab = 0$. Therefore R is nil-reversible.

(4) Assume that R is an mr-nil symmetric ring. Let $a \in \text{nil}(R)$ and $b \in R$ with $1ab = 0$ and 1 denote the identity of R. Then $1ba = 0$. So R is left N-reversible.

(5) Let $R = A/I$ denote the ring considered in [20, Example 5], where $A = F\langle x, y, z\rangle$ is the free algebra with F a field and the ideal I is defined by $I = (AxA)^2 + (AyA)^2 + (AzA)^2 + AxyzA + AyzxA + AzxyA$. Also it is noted that R is a local, 13-dimensional F-algebra, with vector space basis

$$\{1, x, y, z, xy, yx, xz, zx, yz, zy, xzy, zyx, yxz\}.$$

It is mentioned that, obviously, R is not symmetric. In fact, it is not mr-nil symmetric. Namely, $xyz = 0$ with $y^2 = 0$ but $xzy \neq 0$. It is also proved that R is reversible, therefore it is left N-reversible. □

In [11], an ideal I of a ring R is called *left N-reflexive* if for any $a \in \text{nil}(R)$, $b \in R$, being $aRb \subseteq I$ implies $bRa \subseteq I$, and the ring R is called *left N-reflexive* if the zero ideal is left N-reflexive.

Theorem 3.3. *Let R be a left N-reversible ring. Then the following hold.*

(1) *R is a nil-semicommutative ring.*
(2) *R is a left N-reflexive ring.*
(3) *R is a CNZ ring.*

Proof. Assume that R is a left N-reversible ring.

(1) Let a, $b \in \text{nil}(R)$ with $ab = 0$. Then $ba = 0$. So $bar = 0$ for all $r \in R$. By assumption $arb = 0$. Hence R is nil-semicommutative.

(2) To show that R is left N-reflexive, let $a \in \text{nil}(R)$, $b \in R$ with $aRb = 0$. Then $ab = 0$. For any $r \in R$, $abr = 0$. By assumption $bra = 0$. Hence $bRa = 0$. Thus R is left N-reflexive.

(3) Let a, $b \in \text{nil}(R)$ with $ab = 0$. By hypothesis $ba = 0$. □

The converse statement of Theorem 3.3(1) need not hold in general by the following example.

Example 3.4. For every reduced ring R, $U_3(R)$ is a nil-semicommutative ring [23, Example 2.2] which is neither left nor right N-reversible. Indeed for $A = E_{23} \in \text{nil}(U_3(R))$ and $B = E_{11} + E_{12} + E_{23} \in U_3(R)$, we have $AB = 0$ but

$BA \neq 0$. So $U_3(R)$ is not left N-reversible. For $A = E_{11} + E_{13} + E_{33} \in U_3(R)$, $B = E_{23} \in \text{nil}(U_3(R))$, we get $AB = 0$ but $BA \neq 0$. Hence $U_3(R)$ is not right N-reversible.

Proposition 3.5. *Every left N-reversible ring is abelian.*

Proof. Assume that R is a left N-reversible ring. Let $e \in Id(R)$ and $x \in R$. Then $ex - exe, xe - exe$ are nilpotent and $(ex - exe)e = 0$. By assumption, $e(ex - exe) = 0$. Hence $ex = exe$ or e is right semicentral. On the other hand, $(xe - exe)(1 - e) = 0$ and the assumption implies $(1 - e)(xe - exe) = 0$ from which we get $xe = exe$. Hence $ex = xe$. Therefore R is abelian. □

We now give an example of an abelian ring which is not N-reversible.

Example 3.6. We consider the ring R in Examples 2.6(3). The idempotents of R are only zero and identity matrices. So R is an abelian ring. For $A = \left[\begin{smallmatrix} 0 & 0 \\ 2 & 0 \end{smallmatrix}\right] \in \text{nil}(R)$ and $B = \left[\begin{smallmatrix} 0 & 0 \\ 2 & 2 \end{smallmatrix}\right] \in R$, we have $AB = 0$ but $BA \neq 0$. Also for $C = \left[\begin{smallmatrix} 2 & 0 \\ 2 & 0 \end{smallmatrix}\right] \in R$, we get $CA = 0$ but $AC \neq 0$. Hence R is neither left N-reversible nor right N-reversible.

Corollary 3.7. *Every left N-reversible ring is directly finite.*

In the next result, we show that von Neumann regularity and strongly regularity are the same for left N-reversible rings.

Theorem 3.8. *Let R be a left N-reversible ring. Then R is von Neumann regular if and only if it is strongly regular.*

Proof. Assume that R is left N-reversible and von Neumann regular and $x \in R$. There exists $y \in R$ such that $x = xyx$. Then $e = xy$ is central by Proposition 3.5. We have $x = ex = xe = x^2y$. So R is strongly regular. The converse is obvious. □

Compare the following proposition with Proposition 2.7 in [22].

Proposition 3.9. *Let R be a ring. For any $e \in \text{Id}(R)$ and a, $b \in R$, $e = ab$ implies $e = eba$ if and only if R is abelian.*

Proof. For the necessity, let $e^2 = e, a \in R$. Let $g = ea(1 - e) + e$. Then $g \in \text{Id}(R)$, $eg = g$ and $ge = e$. By assumption $eg = g$ implies $gge = e$. So $g = e$ and then $ea(1 - e) = 0$. Now let $g = (1 - e)ae + (1 - e)$. Then g is idempotent, $(1 - e)g = g$ and $g(1 - e) = 1 - e$. By invoking assumption, $gg(1 - e) = g$. It implies $g = 1 - e$. So $(1 - e)ae = 0$. Hence $ea = ae$. Thus R is abelian. For the sufficiency, assume that R is abelian. By [22, Proposition 2.7], $e = ab$ implies $e = bae$. Thus $e = eba$. □

Theorem 3.10. *R is a left N-reversible ring if and only if for any $e \in$Id(R) and $a \in$ nil(R), $b \in R$, $e = ab$ implies $e = ba$.*

Proof. For the necessity, let $e = ab \in \mathrm{Id}(R)$ where $a \in \mathrm{nil}(R), b \in R$. By Proposition 3.5, R is abelian. So we have $e = eba = bae$ by Proposition 3.9. Since $ab(1-e) = 0$, by assumption $b(1-e)a = 0$. Hence $ba = bea$. Since R is abelian, $ba = bea = bae = e$. For the sufficiency, let $a \in \mathrm{nil}(R)$, $b \in R$ with $ab = 0$. Since $0 \in \mathrm{Id}(R)$, by assumption, $ba = 0$. □

Theorem 3.11. *For a semiprime ring R, the following statements are equivalent.*

(1) *R is reduced.*
(2) *R is reversible.*
(3) *R is left N-reversible.*
(4) *R is mr-nil symmetric.*

Proof. (1) ⇒ (2) ⇒ (3) and (1) ⇒ (4) are clear.

(3) ⇒ (1) Let $a \in R$ such that $a^2 = 0$. Then $a(ar) = 0$ for all $r \in R$. Since R is left N-reversible, $ara = 0$. Then $a = 0$ since R is semiprime. Thus R is reduced.

(4) ⇒ (3) By Examples 3.2(4). □

By Kaplansky [13], a ring R is called *a right p.p.-ring* if each principal right ideal of R is projective, or equivalently, if the right annihilator of each element of R is generated by an idempotent. A ring R is called a *p.p.-ring* if it is both a right and a left p.p.-ring.

Theorem 3.12. *Let R be a right p.p.-ring. Then the following statements are equivalent.*

(1) *R is reduced.*
(2) *R is reversible.*
(3) *R is left N-reversible.*
(4) *R is mr-nil symmetric.*

Proof. (1) ⇒ (2) ⇒ (3) and (1)⇒ (4) are clear.

(3) ⇒ (1) Let $a \in R$ such that $a^2 = 0$. Then $a \in r_R(a) = eR$ for some $e^2 = e \in R$. We have $ae = ea = a = 0$ since R is left N-reversible.

(4)⇒ (3) By Examples 3.2(4). □

Corollary 3.13. *Let R be a von Neumann regular ring. Then the following are equivalent.*

(1) *R is reduced.*
(2) *R is reversible.*
(3) *R is left N-reversible.*
(4) *R is mr-nil symmetric.*

Proof. Since every regular ring is semiprime, the results are obtained from Theorem 3.11. □

According to the next result, symmetricity and mr-nil symmetricity coincide for semiprime rings and right p.p.-rings.

Corollary 3.14. *Let R be a semiprime or right p.p.-ring. Then R is symmetric if and only if it is mr-nil symmetric.*

Proof. Clear by Theorem 3.11 and Theorem 3.12. ☐

Proposition 3.15. *For a reduced ring R, let $S = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \mid a, b, c \in R \right\}$ be a subring of $M_3(R)$. Then S is left N-reversible.*

Proof. We note that $\text{nil}(S) = \left\{ \begin{bmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid b, c \in R \right\}$ since R is reduced. Let $A = \begin{bmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \text{nil}(S)$, $B = \begin{bmatrix} x & y & z \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} \in S$ with $AB = 0$. Then we have $bx = cx = 0$. Hence $xb = xc = 0$ since R is reduced. Therefore $BA = 0$. ☐

Theorem 3.16. *Let R be a left N-reversible ring. Then Ra is a nil left ideal and aR is a nil right ideal for each $a \in$ nil(R).*

Proof. Let $b \in \text{nil}(R)$ with $b^n = 0$ for some positive integer n. Then $b^n a = b(b^{n-1}a) = 0$ for all $a \in R$. By hypothesis, $b^{n-1}ab = 0$. Multiplying the latter by a from the right we get $b(b^{n-2}aba) = 0$. Again by hypothesis $b^{n-2}abab = 0$. Continuing on this way, we may reach $b^2(ab)^{n-2} = 0$. Multiplying the latter by a from the right we get $b(b(ab)^{n-2}a) = 0$. Hence $b(ab)^{n-2}ab = 0$ for each $a \in R$. So $(ab)^n = 0$ and $(ba)^n = 0$. Hence bR is a nil right ideal and Rb is a nil left ideal. ☐

Corollary 3.17. *Every left N-reversible ring is weak symmetric.*

Proof. By Theorem 3.16 and [12, Theorem 2.2]. ☐

Corollary 3.18. *Every left N-reversible ring is 2-primal.*

Proof. Note that for any $a \in \text{nil}(R)$, by Theorem 3.16, $aR \subseteq \text{nil}(R)$ and $Ra \subseteq \text{nil}(R)$. We now show that $\text{nil}(R) \subseteq P(R)$. Let $b \in \text{nil}(R)$ with $b^n = 0$ for some positive integer n. For any $r_1, r_2, \ldots, r_n \in R$, by hypothesis, $b^{n-1}br_1 = 0$ implies $br_1b^{n-2}br_2 = 0$. Similarly, $br_2br_1b^{n-2}r_3 = 0$. We continue in this way, $br_nbr_{n-1} \cdots br_2br_1bRb = 0$. The rest is clear from the proof of Proposition 2.4 and therefore $\text{nil}(R) \subseteq P(R)$. ☐

In [8], W. Chen called a ring R *nil-semicommutative* if for any a, $b \in R$, $ab \in \text{nil}(R)$ implies $aRb \subseteq \text{nil}(R)$.

Corollary 3.19. *Every left N-reversible ring is nil-semicommutative.*

Proof. Assume that R is a left N-reversible ring. Let $ab \in \text{nil}(R)$ for a, $b \in R$. Then $ba \in \text{nil}(R)$. By Theorem 3.16, bar is nilpotent for each $r \in R$. Then there exists a positive integer k such that $(bar)^k = 0$. So we have $(arb)^{k+1} = 0$. Hence R is nil-semicommutative. ☐

In [14], a ring R is called to *satisfy the reflexive-nilpotents-property*, or simply called an *RNP* ring if $aRb = 0$ for $a, b \in \text{nil}(R)$ implies $bRa = 0$.

Proposition 3.20. *Every left N-reversible ring is RNP.*

Proof. Let $aRb = 0$ for any nilpotents $a, b \in R$. In particular, $ab = 0$. Then $a(br) = 0$ for all $r \in R$. Since R is left N-reversible $bra = 0$, and so $bRa = 0$. Thus R is RNP. □

The following example shows that there exists an RNP ring which is not left N-reversible. By this example, we also say that $U_n(R)$ is not left N-reversible, where $n \geq 2$ because each subring of a left N-reversible ring is also left N-reversible.

Example 3.21. Let R be a reduced ring. Then $U_2(R)$ is RNP but not left N-reversible.

Proof. Let R be a reduced ring. By [14, Proposition 2.12(2)], $U_2(R)$ is RNP. Let $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in \text{nil}(U_2(R))$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in U_2(R)$, we have $AB = 0$ but $BA = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \neq 0$. □

For a nonempty subset S of a ring R, $r_{nil(R)}(S) = \{x \in nil(R) \mid Sx = 0\}$ is called the *right annihilator* of S in $\text{nil}(R)$. The left annihilator is defined similarly and written by $l_{nil(R)}(S)$. If $S = \{a\}$, then we write $r_{nil(R)}(a)$ (resp., $l_{nil(R)}(a)$) instead of $r_{nil(R)}(\{a\})$ (resp., $l_{nil(R)}(\{a\})$).

Proposition 3.22. *For a ring R, the following are equivalent.*

(1) *R is left N-reversible.*
(2) *$l_{nil(R)}(S) \subseteq r_{nil(R)}(S)$ for any nonempty subset S of R.*
(3) *For each $b \in R$, $l_{nil(R)}(b) \subseteq r_{nil(R)}(b)$.*
(4) *$AB = 0$ implies $BA = 0$ for any nonempty subsets A of nil(R) and B of R.*

Proof. It is clear from the definition of a left N-reversible ring. □

Proposition 3.23. *Let R be a ring and I be a proper ideal of R. If R/I is left N-reversible and I is reduced as a ring without identity, then R is left N-reversible.*

Proof. Let $a \in \text{nil}(R)$, $b \in R$ with $ab = 0$. Then $\bar{a} \in \text{nil}(\bar{R})$, $\bar{b} \in \bar{R}$ and $\bar{a}\bar{b} = \bar{0}$ where $\bar{R} = R/I$. Since R/I is left N-reversible, $\bar{b}\bar{a} = \bar{0}$ and so $ba \in I$. We have $(ba)^2 = b(ab)a = 0$ and so $ba = 0$ since I is reduced. Hence R is left N-reversible. □

We say an ideal I of a ring R *left N-reversible* if $ab \in I$ implies $ba \in I$ for $a \in \sqrt{I}$, $b \in R$, where $\sqrt{I} = \{s \in R \mid s^n \in I$ for some positive integer $n\}$.

Proposition 3.24. *Let I be an ideal of a ring R. Then R/I is left N-reversible if and only if I is left N-reversible.*

Proof. Let $\overline{R}$ denote the ring R/I. Assume that I is a left N-reversible ideal. Let $\bar{a} \in \text{nil}(\bar{R})$ and $\bar{b} \in \bar{R}$ with $\bar{a}\bar{b} = \bar{0}$. Then there exists a positive integer n such that $a^n \in I$ and $ab \in I$. By assumption, $ba \in I$. So we have $\bar{b}\bar{a} = \bar{0}$.

Hence $\bar{R}$ is left N-reversible. Conversely, assume that $\bar{R}$ is left N-reversible. Let $a \in \sqrt{I}$, $b \in R$ such that $ab \in I$. The ring $\bar{R}$ being left N-reversible implies $ba \in I$. Thus I is left N-reversible. □

Proposition 3.25. *Let I be an index set and $\{R_i\}_{i\in I}$ be a class of left N-reversible rings and let $R = \prod_{i\in I} R_i$ be the direct product of $\{R_i\}_{i\in I}$. Then R is left N-reversible if and only if R_i is left N-reversible for each $i \in I$.*

Proof. We note that $\mathrm{nil}(\prod_{i\in I} R_i) = \prod_{i\in I}\mathrm{nil}(R_i)$. Assume that R_i is left N-reversible for each $i \in I$. Let $a = (a_i)_{i\in I} \in \mathrm{nil}(\prod_{i\in I} R_i)$ and $b = (b_i)_{i\in I} \in \prod_{i\in I} R_i$ with $ab = 0$. So $a_i b_i = 0$ for each $i \in I$. By assumption, $b_i a_i = 0$ for each $i \in I$, that is $ba = 0$. Conversely, assume that $\prod_{i\in I} R_i$ is left N-reversible. Let $a_i \in \mathrm{nil}(R_i)$ and $b_i \in R_i$ with $a_i b_i = 0$ for each $i \in I$. Let a denote the element of $\mathrm{nil}(\prod_{i\in I} R_i)$ having i^{th}-entry is a_i and all other entries are zero and b the element of $\prod_{i\in I} R_i$ having i^{th}-entry is b_i and all other are entries zero. Then $ab = 0$. By assumption, $ba = 0$ and so $b_i a_i = 0$ for each $i \in I$. □

4. Some extensions of left N-reversible rings

In this section, we study some kinds of extensions of left N-reversible rings. Let R be a ring. The *Dorroh extension* $D(\mathbb{Z}, R) = \{(n, r) \mid n \in \mathbb{Z}, r \in R\}$ of a ring R is the ring defined by the direct sum $\mathbb{Z} \oplus R$ with the ring operations $(n_1, r_1)+(n_2, r_2) = (n_1+n_2, r_1+r_2)$ and $(n_1, r_1)(n_2, r_2) = (n_1 n_2, r_1 r_2+n_1 r_2+n_2 r_1)$, where $r_i \in R$ and $n_i \in \mathbb{Z}$ for $i = 1, 2$. It is obvious that $\mathrm{nil}(D(\mathbb{Z}, R)) = \{(0, r) \mid r \in \mathrm{nil}(R)\}$.

Proposition 4.1. *A ring R is left N-reversible if and only if $D(\mathbb{Z}, R)$ is left N-reversible.*

Proof. Assume that R is left N-reversible. Let $(n, a) \in D(\mathbb{Z}, R)$ and $(0, r) \in \mathrm{nil}(D(\mathbb{Z}, R))$ with $(0, r)(n, a) = 0$. Then $(0, r)(n, a) = 0$ implies $r(a + n1_R) = 0$ and so $(a + n1_R)r = 0$ since R is left N-reversible. Thus $(n, a)(0, r) = (0,(n1_R + a)r) = 0$. Conversely, let $s \in R$ and $r \in \mathrm{nil}(R)$ with $rs = 0$. We have $(0, r) \in \mathrm{nil}(D(\mathbb{Z}, R))$ and $(0, r)(0, s) = 0$. By hypothesis, $(0, s)(0, r) = 0$. It follows that $sr = 0$. So R is left N-reversible. □

Let R be a commutative ring, M be an R-module, and σ be an endomorphism of R. Give $R\oplus M$ a ring structure with multiplication $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1)m_2+r_2 m_1)$, where $r_i \in R$ and $m_i \in M$. This extension is called the *Nagata extension* of R by M and σ, and denoted by $N(R, M; \sigma)$ (see [25]). We now give examples to show that the left N-reversibility of Nagata extension of the ring depends on the endomorphism σ. That is, there is a Nagata extension which is left N-reversible for some α and there is a Nagata extension which is not left N-reversible for some β.

Example 4.2. (1) Consider the direct sum $R = \mathbb{Z}_4 \oplus \mathbb{Z}_4$. Then R is a commutative ring, and so R is left N-reversible. Let $\sigma_1 : R \to R$ be the identity endomorphism. Then $N(R, R; \sigma_1)$ is left N-reversible.

(2) Consider the ring in (1). Let $\sigma_2 : R \to R$ be the endomorphism defined by $\sigma_2((a, b)) = (b, a)$. Then $N(R, R; \sigma_2)$ is not left N-reversible.

Proof. (1) It is clear.

(2) For $x = ((\overline{2}, \overline{0}), (\overline{0}, \overline{0})) \in \text{nil}(N(R, R; \sigma_2))$ and $y = ((\overline{0}, \overline{1}), (\overline{1}, \overline{0})) \in N(R, R; \sigma_2)$, we have $xy = 0$ but $yx \neq 0$. □

Let R be a ring and S be the subset of R consisting of central regular elements. Set $S^{-1}R = \{s^{-1}r \mid s \in S, r \in R\}$. Then $S^{-1}R$ is a ring with the usual addition and multiplication and it has an identity. Note that $\text{nil}(S^{-1}R) = \{1_R r \mid r \in \text{nil}(R)\}$.

Proposition 4.3. *A ring R is left N-reversible if and only if $S^{-1}R$ is left N-reversible.*

Proof. Assume that R is left N-reversible, $r \in \text{nil}(R)$ and $s^{-1}t \in S^{-1}R$ with $(1_R r)(s^{-1}t) = 0$. Since $(1_R r)(s^{-1}t) = s^{-1}rt$ and s is regular, $rt = 0$. Then left N-reversibility of R implies $tr = 0$. It follows that $(s^{-1}t)(1_R r) = s^{-1}tr = 0$. Conversely, let $r \in \text{nil}(R)$ and $t \in R$ with $rt = 0$. Then for $1_R r \in \text{nil}(S^{-1}R)$, $1_R t \in S^{-1}R$ we have $(1_R r)(1_R t) = 1_R rt = 0$. Since $S^{-1}R$ is left N-reversible, we get $(1_R t)(1_R r) = 0$ and so $tr = 0$. □

Corollary 4.4. *For a ring R, $R[x]$ is left N-reversible if and only if $R[x; x^{-1}]$ is left N-reversible.*

Let R be a ring and S a subring of R and

$$T[R, S] = \{(r_1, r_2, \ldots, r_n, s, s, \ldots) \mid r_i \in R, s \in S, 1 \leq n, 1 \leq i \leq n, i, n \in \mathbb{Z}\}.$$

Then $T[R, S]$ is a ring under the componentwise addition and multiplication. In the following we give necessary and sufficient conditions for $T[R, S]$ to be left N-reversible.

Proposition 4.5. *Let R be a ring and S a subring of R. Then the following are equivalent.*

(1) *$T[R, S]$ is left N-reversible.*
(2) *R is left N-reversible.*

Proof. (1) ⇒ (2) Let $a \in \text{nil}(R)$, $b \in R$ with $ab = 0$. Let $A = (a, 0, 0, 0, \ldots)$, $B = (b, 0, 0, 0, \ldots)$. Then $A \in \text{nil}(T[R, S])$ and $AB = 0$. By (1), $BA = 0$ in $T[R, S]$. Hence $ba = 0$ and so R is left N-reversible.

(2) ⇒ (1) Assume that $A = (a_1, a_2, \ldots, a_n, s, s, \ldots) \in \text{nil}(T[R, S])$ and $B = (b_1, b_2, \ldots, b_m, t, t, \ldots) \in T[R, S]$ with $AB = 0$. Then all components of A are nilpotent in R. Since R is left N-reversible, we obtain $BA = 0$. Hence $T[R, S]$ is left N-reversible. □

5. Polynomial rings over left (right) N-reversible rings

Recall that a ring R is called an *Armendariz ring* if whenever two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, we have $a_i b_j = 0$ for all i, j. This name is connected with the work of Armendariz [5] and studied by many authors [2, 4, 27].

Theorem 5.1 (See [23, Theorem 3.3]). *If R is a nil-semicommutative ring, then $nil(R[x]) = nil(R)[x]$.*

By Theorem 3.3(1), we have the following result.

Corollary 5.2. *If R is a left (right) N-reversible ring, then*

$$nil(R[x]) = nil(R)[x].$$

In [19], Liu and Zhao introduce weak Armendariz rings as a generalization of Armendariz rings. A ring R is said to be *weak Armendariz* if whenever two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j \in nil(R)$ for each i, j. In [4], Antoine introduced the notion of a nil-Armendariz ring. A ring R is called *nil-Armendariz* if whenever two polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) \in nil(R)[x]$, then $a_i b_j \in nil(R)$ for all i, j. Clearly, both Armendariz and nil-Armendariz rings are weak Armendariz.

Question 1: Is there any ring which weak Armendariz but not left (right) N-reversible?

In [19], Liu and Zhao proved that a ring R is weak Armendariz if and only if for any n, the upper triangular matrix ring $U_n(R)$ is weak Armendariz. However $U_n(R)$ is not left N-reversible for a reduced ring R.

Corollary 5.3 (See [4, Corollary 5.2]). *If R is an Armendariz ring, then $nil(R)[x] = nil(R[x])$.*

Example 5.4. There are left (right) N-reversible rings but not Armendariz.

Proof. The ring $D_2(\mathbb{Z}_4)$ is commutative so is left (right) N-reversible. Let $f(x) = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right] x \in D_2(\mathbb{Z}_4)[x]$. Then $f(x)f(x) = 0$ but $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right] \neq 0$. □

Theorem 5.5. *If a ring R is left (right) N-reversible, then R is nil-Armendariz.*

Proof. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ with $f(x)g(x) \in \text{nil}(R)[x]$. Then we have the following system of equations:

(0) $a_0 b_0 \in \text{nil}(R)$
(1) $a_0 b_1 + a_1 b_0 \in \text{nil}(R)$
(2) $a_0 b_2 + a_1 b_1 + a_2 b_0 \in \text{nil}(R)$
$\vdots$
(n) $a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 \in \text{nil}(R)$.

Since R is left N-reversible, $\mathrm{nil}(R)$ is an ideal of R by Corollary 3.18. Observing that Equation (0) yields that b_0a_0, a_0b_0 are in $\mathrm{nil}(R)$. If we multiply Equation (1) from the left by b_0, then $b_0a_1b_0 \in \mathrm{nil}(R)$, so b_0a_1, $a_1b_0 \in \mathrm{nil}(R)$. Similarly, a_0b_1 and b_1a_0 are nilpotent. If we multiply Equation (2) from the right by a_0, then $a_0b_2a_0 \in \mathrm{nil}(R)$, so a_0b_2 and b_2a_0 are in $\mathrm{nil}(R)$. Then $a_1b_1 + a_2b_0$ is in $\mathrm{nil}(R)$. If we multiply from the right by a_1 in this statement, we have $a_1b_1a_1$ is in $\mathrm{nil}(R)$ and then a_1b_1 and b_1a_1 are in $\mathrm{nil}(R)$. So we get a_2b_0 is in $\mathrm{nil}(R)$. To complete the proof for an arbitrary integer n, we proceed by induction on the sum of indices i, j. For $i + j = 0$, both a_0b_0, b_0a_0 are in $\mathrm{nil}(R)$. Assume that it holds for $i + j < n$. Multiplying Equation (n) by b_0 from the left gives an expression from $\mathrm{nil}(R)$. Then all $b_0a_0b_n, b_0a_1b_{n-1}, \ldots, b_0a_{n-1}b_1$ are in $\mathrm{nil}(R)$ by the induction step and the subtraction yields $b_0a_nb_0 \in \mathrm{nil}(R)$, so b_0a_n and a_nb_0 are nilpotent as well. For b_na_0, resp. a_0b_n, one proceeds analogically by multiplying Equation (n) by a_0 from the right. The induction terminates and thus R is nil-Armendariz. ☐

The converse statement of Theorem 5.5 need not hold in general by the following example.

Example 5.6. Consider the ring in [4, Example 4.12]. Let F be a field and $R = F\langle a \mid a^2 = 0\rangle$. Then the ring $T = \left[\begin{smallmatrix} R & aR \\ aR & R \end{smallmatrix}\right]$ is nil-Armendariz by the argument in [4, Example 4.12]. We note that the set of all nilpotent elements of T is $\left[\begin{smallmatrix} aR & aR \\ aR & aR \end{smallmatrix}\right]$. For $A = \left[\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{nil}(T)$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in T$, we have $AB = 0$ but $BA \neq 0$. Thus T is not left N-reversible.

Theorem 5.7. *Let R be a ring. If R is a left N-reversible and Armendariz ring, then $R[x]$ is left N-reversible.*

Proof. By Corollary 5.2, we note that $\mathrm{nil}(R[x]) = \mathrm{nil}(R)[x]$ since R is left N-reversible. Let $f(x) = \sum_{i=0}^{m} a_ix^i \in \mathrm{nil}(R[x])$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ with $f(x)g(x) = 0$. Then we have $a_ib_j = 0$ since R is Armendariz and $a_i \in \mathrm{nil}(R)$ for all i, j. Thus by the left N-reversibility of R, we get $b_ja_i = 0$ for all i, j which implies that $g(x)f(x) = 0$. Hence $R[x]$ is left N-reversible. ☐

References

[1] A. M. Abdul-Jabbar, C. A. K. Ahmed, T. K. Kwak, and Y. Lee, *On commutativity of nilpotent elements at zero*, Commun. Korean Math. Soc. **32** (2017), no. 4, 811–826. https://doi.org/10.4134/CKMS.c170003

[2] D. D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), no. 7, 2265–2272. https://doi.org/10.1080/00927879808826274

[3] ———, *Semigroups and rings whose zero products commute*, Comm. Algebra **27** (1999), no. 6, 2847–2852. https://doi.org/10.1080/00927879908826596

[4] R. Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), no. 8, 3128–3140. https://doi.org/10.1016/j.jalgebra.2008.01.019

[5] E. P. Armendariz, *A note on extensions of Baer and P.P.-rings*, J. Austral. Math. Soc. **18** (1974), 470–473.

[6] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368. https://doi.org/10.1017/S0004972700042052

[7] U. S. Chakraborty and K. Das, *On nil-symmetric rings*, J. Math. **2014** (2014), Art. ID 483784, 7 pp. https://doi.org/10.1155/2014/483784
[8] W. Chen, *On nil-semicommutative rings*, Thai J. Math. **9** (2011), no. 1, 39–47.
[9] P. M. Cohn, *Reversible rings*, Bull. London Math. Soc. **31** (1999), no. 6, 641–648. https://doi.org/10.1112/S0024609399006116
[10] J. M. Habeb, *A note on zero commutative and duo rings*, Math. J. Okayama Univ. **32** (1990), 73–76.
[11] A. Harmanci, H. Kose, Y. Kurtulmaz, and B. Ungor, *Reflexivity of rings via nilpotent elements*, accepted in Rev. Un. Mat. Argentina, also arXiv:1807.02333 [math.RA].
[12] A. Harmanci, H. Kose, and B. Ungor, *On weak symmetric property of rings*, Southeast Asian Bull. Math. **42** (2018), no. 1, 31–40.
[13] I. Kaplansky, *Rings of Operators*, W. A. Benjamin, Inc., New York, 1968.
[14] M. Kheradmand, H. Khabazian, T. K. Kwak, and Y. Lee, *Reflexive property restricted to nilpotents*, J. Algebra Appl. **16** (2017), no. 3, 1750044, 20 pp. https://doi.org/10.1142/S021949881750044X
[15] H. K. Kim, T. K. Kwak, S. I. Lee, Y. Lee, S. J. Ryu, H. J. Sung, and S. J. Yun, *A generalization of symmetric ring property*, Bull. Korean Math. Soc. **53** (2016), no. 5, 1309–1325. https://doi.org/10.4134/BKMS.b150589
[16] H. Kose and A. Harmancı, *Central CNZ rings*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **38** (2018), no. 1, Mathematics, 95–104.
[17] H. Kose, B. Ungor, S. Halicioglu, and A. Harmanci, *A generalization of reversible rings*, Iran. J. Sci. Technol. Trans. A Sci. **38** (2014), no. 1, 43–48.
[18] J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), 359–368. https://doi.org/10.4153/CMB-1971-065-1
[19] Z. Liu and R. Zhao, *On weak Armendariz rings*, Comm. Algebra **34** (2006), no. 7, 2607–2616. https://doi.org/10.1080/00927870600651398
[20] G. Marks, *Reversible and symmetric rings*, J. Pure Appl. Algebra **174** (2002), no. 3, 311–318. https://doi.org/10.1016/S0022-4049(02)00070-1
[21] G. Mason, *Reflexive ideals*, Comm. Algebra **9** (1981), no. 17, 1709–1724. https://doi.org/10.1080/00927878108822678
[22] F. Meng and J. Wei, *e-symmetric rings*, Commun. Contemp. Math. **20** (2018), no. 3, 1750039, 8 pp. https://doi.org/10.1142/S0219199717500390
[23] R. Mohammadi, A. Moussavi, and M. Zahiri, *On nil-semicommutative rings*, Int. Electron. J. Algebra **11** (2012), 20–37.
[24] ———, *On annihilations of ideals in skew monoid rings*, J. Korean Math. Soc. **53** (2016), no. 2, 381–401. https://doi.org/10.4134/JKMS.2016.53.2.381
[25] M. Nagata, *Local Rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York, 1962.
[26] L. Ouyang and H. Chen, *On weak symmetric rings*, Comm. Algebra **38** (2010), no. 2, 697–713. https://doi.org/10.1080/00927870902828702
[27] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17. http://projecteuclid.org/euclid.pja/1195510144
[28] A. A. Tuganbaev, *Semidistributive modules and rings*, Mathematics and its Applications, 449, Kluwer Academic Publishers, Dordrecht, 1998. https://doi.org/10.1007/978-94-011-5086-6
[29] B. Ungor, S. Halicioglu, H. Kose, and A. Harmanci, *Rings in which every nilpotent is central*, Algebras Groups Geom. **30** (2013), no. 1, 1–18.
[30] J. Wei, *Generalized weakly symmetric rings*, J. Pure Appl. Algebra **218** (2014), no. 9, 1594–1603. https://doi.org/10.1016/j.jpaa.2013.12.011

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