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SOME RESULTS ON η-RICCI SOLITONS IN QUASI-SASAKIAN 3-MANIFOLDS

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Abstract. In the present paper, we characterize quasi-Sasakian 3-manifolds admitting η-Ricci solitons. Finally, the existence of η-Ricci soliton in a quasi-Sasakian 3-manifold has been proved by a concrete example.

1. Introduction

Quasi-Sasakian manifold is a natural generalization of Sasakian manifold whose notion was introduced by Blair [2] to unify Sasakian and cosympletic structures. The properties of quasi-Sasakian manifolds have been studied by various authors such as Gonzalez and Chinea [13], Kanemaki ([16,17]), De and Sarkar [9], De et al. [10], Turan et al. [22] and many others. On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Olszak [19] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [20].

A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive, respectively.

As a generalization of Ricci soliton, the notion of η-Ricci soliton was introduced by Cho and Kimura [6]. This notion has also been studied for Hopf hypersurfaces in complex space forms by Calin and Crasmareanu [5]. An η-Ricci soliton is a quadruple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{2}$$

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where S is the Ricci tensor associated to g and η is a 1-form. Recently, an η-Ricci soliton has been studied by various authors in several ways to a different extent such as Blaga [1], De and Mondal [8], Ghosh [12], Haseeb and Prasad ([14, 15]), Majhi et al. [18], Prakasha and Hadimani [21] and many others. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ).

Let M be a $(2n + 1)$-dimensional Riemannain manifold. If there exists a one to one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the projective curvature tensor vanishes. Here the projective curvature tensor P with respect to the Levi-Civita connection ∇ is defined by ([11, 24])

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y] \tag{3}$$

for all $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor and S is the Ricci tensor with respect to the Levi-Civita connection.

The paper is organized as follows: In Section 2, we give a brief introduction of quasi-Sasakian manifolds. In Section 3, we study projectively flat quasi-Sasakian 3-manifolds admitting η-Ricci solitons. Section 4 deals with the study ϕ-projectively semisymmetric quasi-Sasakian 3-manifolds admitting η-Ricci solitons. Sections 5, 6 and 7 are devoted to study η-Ricci solitons in quasi-Sasakian 3-manifolds satisfying the curvature conditions $P(\xi, X) \cdot S = 0$, $Q \cdot P = 0$ and $Q \cdot R = 0$, respectively. In Section 8, we study η-Ricci solitons in recurrent quasi-Sasakian 3-manifolds. Finally, we construct an example of quasi-Sasakian 3-manifolds which admits an η-Ricci soliton.

2. Preliminaries

Let M be an almost contact metric manifold of dimension $(2n + 1)$ with an almost contact metric structure (ϕ, ξ, η, g), where ϕ, ξ, η are tensor fields of type $(1, 1), (1, 0), (0, 1)$ respectively and g is a Riemannian metric on M such that ([3, 4])

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \tag{4}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{5}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \tag{6}$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ denotes the collection of all smooth vector fields of M. M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η) is normal and the fundamental 2-form Φ is closed ($d\Phi = 0$). A three dimensional almost contact metric manifold M is quasi-Sasakian if and only if [19]

$$\nabla_X \xi = -\beta\phi X \tag{7}$$

for a certain function β on M, such that $\xi\beta = 0$, where ∇ is the Levi-Civita connection of M. Clearly, a quasi-Sasakian 3-manifold is cosymplectic if and only if $\beta = 0$. If $\beta =$ constant, then the manifold reduces to a β-Sasakian manifold and $\beta = 1$ gives the Sasakian structure. Throughout in the paper, we are using the fact that $\beta =$ constant.

In a quasi-Sasakian 3-manifold, we have [19]

$$(\nabla_X\phi)Y = \beta(g(X,Y)\xi - \eta(Y)X), \tag{8}$$

$$(\nabla_X\eta)Y = -\beta g(\phi X, Y), \tag{9}$$

$$\begin{aligned} R(X,Y)Z = {} & (\frac{r}{2} - 2\beta^2)(g(Y,Z)X - g(X,Z)Y) \\ & + (3\beta^2 - \frac{r}{2})(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ & + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y), \end{aligned} \tag{10}$$

$$S(X,Y) = (\frac{r}{2} - \beta^2)g(X,Y) + (3\beta^2 - \frac{r}{2})\eta(X)\eta(Y) \tag{11}$$

for all $X, Y, Z \in \chi(M)$.

Definition. A quasi-Sasakian manifold is said to be an η-Einstein manifold if its non-vanishing Ricci tensor S is of the form

$$S(X,Y)Y = ag(X,Y) + b\eta(X)\eta(Y), \tag{12}$$

where a and b are smooth functions on the manifold. If $b = 0$, then the manifold is said to be an Einstein manifold.

Proposition 2.1 ([12])**.** *A 3-dimensional non-cosymplectic quasi-Sasakian manifold admitting an η-Ricci soliton is an η-Einstein manifold given by*

$$S(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y) \tag{13}$$

for any $X, Y \in \chi(M)$.

Proposition 2.2 ([12])**.** *If a 3-dimensional non-cosymplectic quasi-Sasakian manifold admits an η-Ricci soliton, then we have*

$$\lambda + \mu = -2\beta^2, \tag{14}$$

where λ and μ are constants.

3. Projectively flat quasi-Sasakian 3-manifolds admitting η-Ricci solitons

Let M be a projectively flat quasi-Sasakian 3-manifold admitting an η-Ricci soliton. Therefore, $P(X,Y)Z = 0$ and thus from (3) it follows that

$$R(X,Y)Z = \frac{1}{2}[S(Y,Z)X - S(X,Z)Y]. \tag{15}$$

Taking the inner product of (15) with ξ and using (6), we have

$$g(R(X,Y)Z,\xi) = \frac{1}{2}[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)]$$

which by virtue of (10) takes the form

$$4\beta^2(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) = S(Y,Z)\eta(X) - S(X,Z)\eta(Y).$$

Putting $X = \xi$ in the last equation, then using (4), (6) and (13), we find

$$S(Y,Z) = 4\beta^2 g(Y,Z) - (4\beta^2 + \lambda + \mu)\eta(Y)\eta(Z). \tag{16}$$

By using (14) in (16), we get

$$S(Y,Z) = 4\beta^2 g(Y,Z) - 2\beta^2\eta(Y)\eta(Z). \tag{17}$$

Now from (13) and (17) it follows that $\lambda = -4\beta^2 < 0$ and $\mu = 2\beta^2$. Thus we can state the following:

Theorem 3.1. *A projectively flat quasi-Sasakian* 3*-manifold admitting an* η*-Ricci soliton is an* η*-Einstein manifold of the form* (17) *and the Ricci soliton is always shrinking.*

4. ϕ-projectively semisymmetric quasi-Sasakian 3-manifolds admitting η-Ricci solitions

Definition. A quasi-Sasakian manifold is said to be ϕ-projectively semisymmetric if [7]

$$P(X,Y)\cdot\phi = 0$$

for all X, Y on M.

Let M be a ϕ-projectively semisymmetric quasi-Sasakian 3-manifold admitting an η-Ricci soliton. Therefore $P(X,Y)\cdot\phi = 0$ turns into

$$(P(X,Y)\cdot\phi)Z = P(X,Y)\phi Z - \phi P(X,Y)Z = 0 \tag{18}$$

for any vector fields $X, Y, Z \in \chi(M)$. From (3), we find

$$P(X,Y)\phi Z = R(X,Y)\phi Z - \frac{1}{2}[S(Y,\phi Z)X - S(X,\phi Z)Y], \tag{19}$$

and

$$\phi P(X,Y)Z = \phi R(X,Y)Z - \frac{1}{2}[S(Y,Z)\phi X - S(X,Z)\phi Y]. \tag{20}$$

Combining the equations (18), (19) and (20), we have

$$R(X,Y)\phi Z - \phi R(X,Y)Z - \frac{1}{2}[S(Y,\phi Z)X - S(X,\phi Z)Y - S(Y,Z)\phi X + S(X,Z)\phi Y] = 0$$

which by taking $Y = \xi$ and using (6), (10), (13) and (14) reduces to

$$S(X,\phi Z)\xi - 4\beta^2 g(X,\phi Z)\xi - 2\beta^2\eta(Z)\phi X = 0. \tag{21}$$

Taking the inner product of (21) with ξ and using (4), we find

$$S(X,\phi Z) = 4\beta^2 g(X,\phi Z). \tag{22}$$

By replacing Z by ϕZ in (22) and making use of (4), we have

$$S(X,Z) - \eta(Z)S(X,\xi) - 4\beta^2 g(X,Z) + 4\beta^2\eta(X)\eta(Z) = 0$$

which by using (13) and (14) gives

$$S(X,Z) = 4\beta^2 g(X,Z) - 2\beta^2\eta(X)\eta(Z). \tag{23}$$

Now from (13) and (23) it follows that $\lambda = -4\beta^2 < 0$ and $\mu = 2\beta^2$. Thus we can state the following:

Theorem 4.1. *A ϕ-projectively semisymmetric quasi-Sasakian 3-manifold admitting an η-Ricci soliton is an η-Einstein manifold of the form (23) and the Ricci soliton is always shrinking.*

5. η-Ricci solitons in quasi-Sasakian 3-manifolds satisfying $P(\xi, X)\cdot S = 0$

Let M be a quasi-Sasakian 3-manifold admitting an η-Ricci soliton satisfying $P(\xi, X)\cdot S = 0$. Then we have

$$S(P(\xi,X)Y,Z) + S(Y,P(\xi,X)Z) = 0. \tag{24}$$

From (3), we find

$$P(\xi,X)Y = 2\beta^2(g(X,Y)\xi - \eta(Y)X) - \frac{1}{2}(S(X,Y)\xi + (\lambda+\mu)\eta(Y)X). \tag{25}$$

By virtue of (25), (24) takes the form

$$2\beta^2[g(X,Y)S(\xi,Z)+g(X,Z)S(\xi,Y)-\eta(Y)S(X,Z)-\eta(Z)S(X,Y)]$$
$$-\frac{1}{2}[S(X,Y)S(\xi,Z)+S(X,Z)S(\xi,Y)+(\lambda+\mu)(S(X,Y)\eta(Z)+S(X,Z)\eta(Y))]=0,$$

which by virtue of (13) reduces to

$$\begin{aligned} &2\beta^2[S(X,Z)\eta(Y) + S(X,Y)\eta(Z) \\ &+ (\lambda+\mu)(g(X,Y)\eta(Z) + g(X,Z)\eta(Y))] = 0. \end{aligned} \tag{26}$$

Taking $Z = \xi$ in (26) then using (6) and (13), we find

$$S(X,Y) = -(\lambda+\mu)g(X,Y). \tag{27}$$

Now by using (14) in (27), we get

$$S(X,Y) = 2\beta^2 g(X,Y). \tag{28}$$

From the equations (13) and (28) it follows that $\lambda = -2\beta^2 < 0$ and $\mu = 0$. Thus we can state the following:

Theorem 5.1. *If a quasi-Sasakian 3-manifold admitting an η-Ricci soliton (g,ξ,λ,η) satisfies $P(\xi,X)\cdot S = 0$, then the manifold is an Einstein manifold of the form (28), and the η-Ricci soliton becomes Ricci soliton (g,ξ,λ) which is always shrinking.*

Again it is known that a 3-dimensional Einstein manifold is a manifold of constant curvature [24]. Thus we have:

Corollary 5.2. *A quasi-Sasakian 3-manifold admitting an η-Ricci soliton satisfying $P(\xi,X)\cdot S = 0$ is a manifold of constant curvature.*

6. η-Ricci solitons in quasi-Sasakian 3-manifolds satisfying $Q \cdot P = 0$

Let M be a quasi-Sasakian 3-manifold admitting an η-Ricci soliton satisfying $Q \cdot P = 0$. Then we have

$$Q(P(X,Y)Z) - P(QX,Y)Z - P(X,QY)Z - P(X,Y)QZ = 0 \tag{29}$$

for all $X, Y, Z \in \chi(M)$. In view of (3), (29) becomes

$$\begin{aligned} &Q(R(X,Y)Z) - R(QX,Y)Z - R(X,QY)Z \\ &- R(X,Y)QZ + S(Y,QZ)X - S(X,QZ)Y = 0 \end{aligned}$$

which by taking the inner product with ξ takes the form

$$\begin{aligned} &\eta(Q(R(X,Y)Z)) - \eta(R(QX,Y)Z) - \eta(R(X,QY)Z) \\ &- \eta(R(X,Y)QZ) + S(Y,QZ)\eta(X) - S(X,QZ)\eta(Y) = 0. \end{aligned} \tag{30}$$

Putting $Y = \xi$ in (30), we have

$$\begin{aligned} &\eta(Q(R(X,\xi)Z)) - \eta(R(QX,\xi)Z) - \eta(R(X,Q\xi)Z) \\ &- \eta(R(X,\xi)QZ) + S(\xi,QZ)\eta(X) - S(X,QZ)\eta(\xi) = 0. \end{aligned} \tag{31}$$

From (10), we find

$$\begin{cases} \eta(Q(R(X,\xi)Z)) = \eta(R(X,Q\xi)Z) = 4\beta^4(\eta(X)\eta(Z) - g(X,Z)), \\ \eta(R(QX,\xi)Z) = \eta(R(X,\xi)QZ) = 2\beta^2(2\beta^2\eta(X)\eta(Z) - S(X,Z)), \\ S(\xi,QZ) = 4\beta^4\eta(Z). \end{cases} \tag{32}$$

By the use of (6) and (32), (31) takes the form

$$S(X,QZ) = 4\beta^2 S(X,Z) - 4\beta^4\eta(X)\eta(Z). \tag{33}$$

In view of (13), (33) turns to

$$(\lambda + 4\beta^2)S(X,Z) + 2\beta^2(\mu - 2\beta^2)\eta(X)\eta(Z) = 0 \tag{34}$$

which in view of (14) reduces to

$$S(X,Z) = 2\beta^2\eta(X)\eta(Z), \tag{35}$$

where $\mu \neq 2\beta^2$. Therefore, from (13) and (35), we get $\mu = -2\beta^2$ and hence $\lambda = 0$. Thus we can state the following theorem:

Theorem 6.1. *If a quasi-Sasakian* 3*-manifold admitting an* η*-Ricci soliton* (g, ξ, λ, η) *satisfies* $Q \cdot P = 0$*, then the manifold is a special type of* η*-Einstein manifold of the form* (35)*, and the Ricci solitons is always steady.*

7. η-Ricci solitons in quasi-Sasakian 3-manifolds satisfying $Q \cdot R = 0$

Let M be a quasi-Sasakian 3-manifold admitting an η-Ricci soliton satisfying $Q \cdot R = 0$. Then we have

$$Q(R(X,Y)Z) - R(QX,Y)Z - R(X,QY)Z - R(X,Y)QZ = 0 \tag{36}$$

for all $X, Y, Z \in \chi(M)$. By virtue of (10), (36) takes the form

$$\begin{aligned}&(r-2\beta^2)(S(X,Z)Y - S(Y,Z)X) + (3\beta^2 - \frac{r}{2})(g(Y,Z)\eta(X)Q\xi\\&- g(Y,Z)\eta(QX)\xi - g(X,Z)\eta(Y)Q\xi + g(X,Z)\eta(QY)\xi\\&+ 2S(X,Z)\eta(Y)\xi - 2S(Y,Z)\eta(X)\xi + \eta(QX)\eta(Z)Y\\&- \eta(QY)\eta(Z)X + \eta(X)\eta(QZ)Y - \eta(Y)\eta(QZ)X) = 0.\end{aligned} \tag{37}$$

From (13), it follows that

$$\begin{cases} QX = -\lambda X - \mu\eta(X)\xi \implies \eta(QX) = -(\lambda+\mu)\eta(X),\\ Q\xi = -(\lambda+\mu)\xi. \end{cases} \tag{38}$$

By using (38), (37) reduces to

$$\begin{aligned}&(r-2\beta^2)(S(X,Z)Y - S(Y,Z)X)\\&+ (6\beta^2 - r)(S(X,Z)\eta(Y)\xi - S(Y,Z)\eta(X)\xi)\\&- (\lambda+\mu)(\eta(X)Y - \eta(Y)X)\eta(Z) = 0.\end{aligned} \tag{39}$$

Taking the inner product of (39) with ξ, we have

$$\begin{aligned}&(r-2\beta^2)(S(X,Z)\eta(Y) - S(Y,Z)\eta(X))\\&+ (6\beta^2 - r)(S(X,Z)\eta(Y) - S(Y,Z)\eta(X)) = 0\end{aligned}$$

which by putting $Y = \xi$ and using (13) gives

$$4\beta^2(S(X,Z) + (\lambda+\mu)\eta(X)\eta(Z)) = 0. \tag{40}$$

Thus we have

$$S(X,Z) = -(\lambda+\mu)\eta(X)\eta(Z). \tag{41}$$

From (13) and (41), it follows that $\lambda = 0$ and hence from (14), we get $\mu = -2\beta^2$. Therefore, (41) turns to

$$S(X,Z) = 2\beta^2\eta(X)\eta(Z). \tag{42}$$

Thus we can state the following:

Theorem 7.1. *If a quasi-Sasakian* 3*-manifold admitting an η-Ricci soliton (g, ξ, λ, η) satisfies $Q \cdot R = 0$, then the manifold is a special type of η-Einstein manifold of the form* (42)*, and the Ricci solitons is always steady.*

8. Recurrent quasi-Sasakian 3-manifolds admitting η-Ricci solitons

Definition. A quasi-Sasakian 3-manifold is said to be recurrent if there exists a non-zero 1-form A such that [23]

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W \tag{43}$$

for all vector fields X, Y, Z and W on M. If the 1-form A vanishes, then the manifold reduces to a symmetric manifold.

Assume that M is a recurrent quasi-Sasakian manifold. Therefore, the curvature tensor of the manifold satisfies (43). By a suitable contraction of (43), we get

$$(\nabla_X S)(Z, W) = A(X)S(Z, W). \tag{44}$$

This implies that

$$\nabla_X S(Z, W) - S(\nabla_X Z, W) - S(Z, \nabla_X W) = A(X)S(Z, W) \tag{45}$$

which by taking $W = \xi$ and then using (7), (9) and (13) yields

$$\beta S(Z, \phi X) - \beta(\lambda + \mu)g(X, \phi Z) = -A(X)(\lambda + \mu)\eta(Z). \tag{46}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (46), we have

$$\beta S(Z, \phi X) - \beta(\lambda + \mu)g(X, \phi Z) = -(\lambda + \mu)\eta(X)\eta(Z). \tag{47}$$

By replacing Z by ϕZ in (47), then using (4) and (6), we get

$$\beta S(\phi Z, \phi X) + \beta(\lambda + \mu)(g(X, Z) - \eta(X)\eta(Z)) = 0. \tag{48}$$

In view of (5) and (13), it follows from (48) that

$$\beta\mu(g(X, Z) - \eta(X)\eta(Z)) = 0. \tag{49}$$

This implies that $\mu = 0$ and hence from (14), we get $\lambda = -2\beta^2$. Thus we can state the following:

Theorem 8.1. *In a recurrent quasi-Sasakian 3-manifold an η-Ricci soliton becomes a Ricci soliton which is always shrinking.*

Example. We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = \frac{\partial}{\partial x}, \; e_2 = \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}, \; e_3 = \frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \; g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.$$

Let η be the 1-form on M defined by $\eta(X) = g(X, e_3) = g(X, \xi)$ for all $X \in \chi(M)$. Let ϕ be the $(1, 1)$ tensor field on M defined by

$$\phi e_1 = -e_2, \ \phi e_2 = e_1, \ \phi e_3 = 0.$$

The linearity property of ϕ and g yields

$$\eta(\xi) = g(\xi, \xi) = 1, \ \phi^2 X = -X + \eta(X)\xi, \ \eta(\phi X) = 0,$$

$$g(X, \xi) = \eta(X), \ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$.

Now, by direct computations we obtain

$$[e_1, e_2] = 2e_3, \quad [e_1, e_3] = 0, \quad [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric tensor g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) = & Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ & - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = 0, \ \nabla_{e_2} e_1 = -e_3, \ \nabla_{e_3} e_1 = -e_2, \ \nabla_{e_1} e_2 = e_3, \ \nabla_{e_2} e_2 = 0,$$

$$\nabla_{e_3} e_2 = e_1, \ \nabla_{e_1} e_3 = -e_2, \ \nabla_{e_2} e_3 = e_1, \ \nabla_{e_3} e_3 = 0.$$

Also, one can easily verify that

$$\nabla_X \xi = -\beta \phi X \quad \text{and} \quad (\nabla_X \phi) Y = \beta(g(X, Y)\xi - \eta(Y)X), \ \text{where } \beta = -1.$$

Thus the manifold M is a quasi-Sasakian manifold. It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

By using the above results, one can easily obtain the components of the curvature tensors as follows:

$$R(e_1, e_2)e_1 = 3e_2, \ R(e_1, e_2)e_2 = -3e_1, \ R(e_1, e_2)e_3 = 0,$$

$$R(e_2, e_3)e_1 = 0, \ R(e_2, e_3)e_2 = -e_3, \ R(e_2, e_3)e_3 = e_2,$$

$$R(e_1, e_3)e_1 = -e_3, \ R(e_1, e_3)e_2 = 0, \ R(e_1, e_3)e_3 = e_1.$$

From these curvature tensors, we calculate the components of Ricci tensor as follows:

$$S(e_1, e_1) = S(e_2, e_2) = -2, \ S(e_3, e_3) = 2. \tag{50}$$

Therefore, $r = \sum_{i=1}^{3} S(e_i, e_i) = -2$. From the equations (13) and (50), we obtain $\lambda = 2$ and $\mu = -4$. Thus the data (g, ξ, λ, μ) for $\lambda = 2$ and $\mu = -4$ defines an η-Ricci solitons on (M, ϕ, ξ, η, g).

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