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EXTENDED GENERALIZED BATEMAN'S MATRIX POLYNOMIALS

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Abstract. In this article, a study of generalized Bateman's matrix polynomials is presented. We obtained partial differential equations by using differential operators in the generalized Bateman's matrix polynomials for two variables. Then we introduced some different recurrence relationships of the generalized Bateman's matrix polynomials. Finally present the relationship between the generalized Bateman's matrix polynomials of one and two variables.

1. Introduction

The study of orthogonal matrix polynomials for the last two decades includes an emerging topic of study, with important results in both theory and applications continuing to appear in the literature (e.g. [11]).

Bateman's polynomials B_n are a family orthogonal polynomials. A lot of researchers have generalized the classical results on the Bateman's polynomials (e.g. [3]).

In [2,4,7,14–16,18,19], a wide dedicated literature, numbers of related properties, extensions, generalizations and applications of Bateman's polynomials are available.

The study of the special matrix functions previous studies and both the theory of Lie group and the theory of numbers are well known [5,21]. Matrix polynomials have also appeared in recent studies in [8–10].

The theory of orthogonal polynomials extends, as in papers [6,7,17], to a polynomials matrix.

The generalized Bateman's polynomial $\mathrm{B}_n^{(\alpha,\beta)}(b,z)$ is defined by Khan and Shukla [13] as follows:

$$\mathrm{B}_n^{(\alpha,\beta)}(b,z) = {}_2F_2\left[-n, 1+\alpha+\beta+n; 1+\alpha, 1+b; z\right]. \tag{1.1}$$

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Kampe de Feriet series in the generalized form is defined by (e.g. [20]):

$$F^{p,q,k}_{l,m,n}\left[\ (a)_p : (b)_q ; (c)_k ; (\alpha)_l : (\beta)_m ; (\gamma)_n ; x, y\ \right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s x^r y^s}{r!s!\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}. \tag{1.2}$$

In [1] several formulas of generalized Bateman's polynomials of two variables were described by the authors as:

$$\mathrm{B}_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(b_1, x; b_2, y) = \sum_{h=0}^{n}\sum_{k=0}^{n-h} \frac{(-n)_{h+k}(1+\alpha_1+\beta_1+n)_h(1+\alpha_2+\beta_2+n)_k}{h!k!(1+\alpha_1)_h(1+b_1)_h(1+\alpha_2)_k(1+b_2)_k} x^h y^k. \tag{1.3}$$

The definition (1.3) can be expressed in terms of double the hypergeometric function as follows (e.g. [1]):

$$\mathrm{B}_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(b_1, x; b_2, y) = F^{1,1,1}_{0,2,2}\left[\begin{array}{l} -n, 1+\alpha_1+\beta_1+n, 1+\alpha_2+\beta_2+n; \\ -, 1+\alpha_1, 1+b_1, 1+\alpha_2, 1+b_2 \qquad ; x, y \end{array}\right], \tag{1.4}$$

where we have used a special case of the double hypergeometric function defined by (1.2).

Also, the definition (1.3) can be interpreted as:

$$\mathrm{B}_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(b_1, x; b_2, y) = \sum_{h=0}^{n}\sum_{k=0}^{n-h} \frac{(-n)_h(1+\alpha_1+\beta_1+n)_h x^h}{h!(1+\alpha_1)_h(1+b_1)_h} \mathrm{B}_{n-h}^{(\alpha_2,\beta_2+h)}(b_2, y), \tag{1.5}$$

where $\mathrm{B}_{n-h}^{(\alpha_2,\beta_2+h)}(b_2, y)$ is the well-known generalized Bateman's polynomials of single variable.

Finally the relationships between generalized Bateman's polynomials of two variables and generalized Bateman's polynomials of single variable as follows (e.g. [1]):

$$\mathrm{B}_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(b_1, z; b_2, -1) = B_n^{(\alpha_1,\beta_1)}(b_1, z), \tag{1.6}$$

$$\mathrm{B}_n^{(\alpha_1,\beta_1;\alpha_2,\beta_2)}(b_1, -1; b_2, w) = B_n^{(\alpha_2,\beta_2)}(b_2, w). \tag{1.7}$$

The generalized Bateman's matrix polynomials given as follows (e.g. [1, 12])

$$\mathrm{B}_n^{(A,B)}(C, z) = {}_2F_2\left[-nI, I+A+B+nI; I+A, I+C; z\right], \tag{1.8}$$

where the hypergeometric function is defined in the form (e.g. [9, 11]):

$${}_2F_1(A; B, C; z) = \sum_{n=0}^{\infty} \frac{(A)_n(B)_n(C)_n^{-1}}{n!} z^n \tag{1.9}$$

for matrices $A, B, C \in \mathbb{C}^{n\times n}$ such that $C+nI$ is an invertible for all integer $n \geq 0$ and for $|z|<1$.

If $A \in \mathbb{C}^{n\times n}$ is a matrix such that $A+nI$ is an invertible matrix for $n \geq 0$ we have the matrix version of the pochhammer symbol is

$$(A)_n = A(A+I)(A+2I)\cdots(A+(n-1)I);\quad n\geq 1; (A)_0 \equiv I. \tag{1.10}$$

2. Some recurrences relations for generalized Bateman's matrix polynomials

Suppose that the generalized Bateman's matrix polynomials is composed of two variables (e.g. [1]):

$$\begin{aligned}&\mathrm{B}_n^{(A_1,B_1;A_2,B_2)}(C_1,z;C_2,w)\\ &=\sum_{h=0}^{n}\sum_{k=0}^{n-h}\frac{(-nI)_{h+k}(I+A_1+B_1+nI)_h(I+A_2+B_2+nI)_k}{h!k!(I+A_1)_h(I+C_1)_h(I+A_2)_k(I+C_2)_k}z^hw^k,\end{aligned}\tag{2.1}$$

where $(A_1,B_1;A_2,B_2;C_1,C_2)$ are positive stable matrices in $\mathbb{C}^{n\times n}$, $z,w\in(-1,1)$.

Consider the partial differential operator:

$$D=\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial w}\right). \tag{2.2}$$

The conjugate relationships we can use in the following theorem can be written from the pochhammer relation as follows:

$$(-nI)_{h+k+1}=(-nI)\left[(-nI)+I\right]_{h+k}, \tag{2.3}$$

$$\begin{aligned}&(I+A_1+B_1+nI)_{h+1}\\ &=(I+A_1+B_1+nI)(2I+A_1+B_1+nI)_h\\ &=(I+A_1+B_1+nI)(3I+A_1+B_1+(n-1)I)_h,\end{aligned}\tag{2.4}$$

$$\begin{aligned}&(I+A_2+B_2+nI)_{k+1}\\ &=(I+A_2+B_2+nI)(2I+A_2+B_2+nI)_k\\ &=(I+A_2+B_2+nI)(3I+A_2+B_2+(n-1)I)_k,\end{aligned}\tag{2.5}$$

$$(I+A_1)_{h+1}=(I+A_1)(2I+A_1)_h, \tag{2.6}$$

$$(I+C_1)_{h+1}=(I+C_1)(2I+C_1)_h, \tag{2.7}$$

$$(I+A_2)_{k+1}=(I+A_2)(2I+A_2)_k, \tag{2.8}$$

$$(I+C_2)_{k+1}=(I+C_2)(2I+C_2)_k, \tag{2.9}$$

$$(I+A_1)_h=\frac{(I+A_1)}{(nI+A_1)}(2I+A_1)_h, \tag{2.10}$$

$$(I+A_2)_k = \frac{(I+A_2)}{(nI+A_2)}\,(2I+A_2)_k\,. \tag{2.11}$$

Theorem 2.1. *By acting the differential operator* (2.2) *for generalised Bateman's matrix polynomials* (2.1), *we obtain the following partial differential equation*:

$$\begin{aligned} & D\,\mathrm{B}_n^{(A_1,B_1;A_2,B_2)}(C_1,z;C_2,w) \\ & - \tfrac{(-nI)(nI+A_2)(I+A_1+B_1+nI)}{(I+A_1)(I+C_1)(I+A_2)}\mathrm{B}_{n-1}^{(I+A_1,I+B_1;I+A_2,B_2)}(I+C_1,z;C_2,w) \\ & - \tfrac{(-nI)(nI+A_1)(I+A_2+B_2+nI)}{(I+A_1)(I+A_2)(I+C_2)}\mathrm{B}_{n-1}^{(I+A_1,B_1;I+A_2,I+B_2)}(C_1,z;I+C_2,w) = 0. \end{aligned} \tag{2.12}$$

Proof. When we acting the differential operator (2.2) for generalized Bateman's matrix polynomials (2.1) and using the conjugate relations (2.3)-(2.11) we obtain the following equalities:

$$\begin{aligned} & D\mathrm{B}_n^{(A_1,B_1;A_2,B_2)}(C_1,z;C_2,w) \\ = & \left(\frac{\partial}{\partial z}+\frac{\partial}{\partial w}\right)\mathrm{B}_n^{(A_1,B_1;A_2,B_2)}(C_1,z;C_2,w) \\ = & \sum_{h=0}^{n}\sum_{k=0}^{n-h}\frac{h\,(-nI)_{h+k}\,(I+A_1+B_1+n)_h\,(I+A_2+B_2+n)_k}{h!k!\,(I+A_1)_h\,(I+C_1)_h\,(I+A_2)_k\,(I+C_2)_k}z^{h-1}w^k \\ & +\sum_{h=0}^{n}\sum_{k=0}^{n-h}\frac{k\,(-nI)_{h+k}\,(I+A_1+B_1+n)_h\,(I+A_2+B_2+n)_k}{h!k!\,(I+A_1)_h\,(I+C_1)_h\,(I+A_2)_k\,(I+C_2)_k}z^{h}w^{k-1} \\ = & \sum_{h=0}^{n}\sum_{k=0}^{n-h}\frac{(-nI)_{h+1+k}\,(I+A_1+B_1+n)_{h+1}\,(I+A_2+B_2+n)_k}{h!k!\,(I+A_1)_{h+1}\,(I+C_1)_{h+1}\,(I+A_2)_k\,(I+C_2)_k}z^{h}w^k \\ & +\sum_{h=0}^{n}\sum_{k=0}^{n-h}\frac{(-nI)_{h+k+1}\,(I+A_1+B_1+n)_h\,(I+A_2+B_2+n)_{k+1}}{h!k!\,(I+A_1)_h\,(I+C_1)_h\,(I+A_2)_{k+1}\,(I+C_2)_{k+1}}z^{h}w^k \\ = & \sum_{h=0}^{n}\sum_{k=0}^{n-h}\frac{(-nI)\,(-(n-1)I)_{h+k}\,(I+A_1+B_1+nI)\,(nI+A_2)\,(3I+A_1+B_1+(n-1)I)_h\,(2I+A_2+B_2+(n-1)I)_k}{h!k!\,(I+A_1)\,(I+A_2)\,(I+C_1)\,(2I+A_1)_h\,(2I+C_1)_h\,(2I+A_2)_k\,(I+C_2)_k}z^{h}w^k \\ & +\sum_{h=0}^{n}\sum_{k=0}^{n-h}\frac{(-nI)\,(-(n-1)I)_{h+k}\,(I+A_2+B_2+nI)\,(nI+A_1)\,(2I+A_1+B_1+(n-1)I)_h\,(3I+A_2+B_2+(n-1)I)_k}{h!k!\,(I+A_2)\,(I+A_1)\,(I+C_2)\,(2I+A_1)_h\,(I+C_1)_h\,(2I+A_2)_k\,(2I+C_2)_k}z^{h}w^k, \end{aligned}$$

i.e.,

$$\begin{aligned} & D\,\mathrm{B}_n^{(A_1,B_1;A_2,B_2)}(C_1,z;C_2,w) \\ = & \tfrac{(-nI)(nI+A_2)(I+A_1+B_1+nI)}{(I+A_1)(I+C_1)(I+A_2)}\mathrm{B}_{n-1}^{(I+A_1,I+B_1;I+A_2,B_2)}(I+C_1,z;C_2,w) \\ & + \tfrac{(-nI)(nI+A_1)(I+A_2+B_2+nI)}{(I+A_1)(I+A_2)(I+C_2)}\mathrm{B}_{n-1}^{(I+A_1,B_1;I+A_2,I+B_2)}(C_1,z;I+C_2,w)\,. \end{aligned}$$

Then we get the generalized Bateman's matrix polynomials (2.1) satisfying the partial differential equation (2.12). □

Now, if we put $A_1 = I$, $B_1 = I$, $A_2 = I$ and $B_2 = I$, respectively we get some recurrence relationships of the generalized Bateman's matrix polynomials as follows:

1.

$$D\,\mathrm{B}_n^{(I,B_1;A_2,B_2)}(C_1,z;C_2,w) - \tfrac{(-nI)(nI+A_2)(2I+B_1+nI)}{(2I)(I+C_1)(I+A_2)}\mathrm{B}_{n-1}^{(2I,I+B_1;I+A_2,B_2)}(I+C_1,z;C_2,w) - \tfrac{(-nI)((n+1)I)(I+A_2+B_2+nI)}{(2I)(I+A_2)(I+C_2)}\mathrm{B}_{n-1}^{(2I,B_1;I+A_2,I+B_2)}(C_1,z;I+C_2,w)=0.$$

2.

$$D\,\mathrm{B}_n^{(A_1,I;A_2,B_2)}(C_1,z;C_2,w) - \tfrac{(-nI)(nI+A_2)(2I+A_1+nI)}{(I+A_1)(I+C_1)(I+A_2)}\mathrm{B}_{n-1}^{(I+A_1,2I;I+A_2,B_2)}(I+C_1,z;C_2,w) - \tfrac{(-nI)(nI+A_1)(I+A_2+B_2+nI)}{(I+A_1)(I+A_2)(I+C_2)}\mathrm{B}_{n-1}^{(I+A_1,I;I+A_2,I+B_2)}(C_1,z;I+C_2,w)=0.$$

3.

$$D\,\mathrm{B}_n^{(A_1,B_1;I,B_2)}(C_1,z;C_2,w) - \tfrac{(-nI)((n+1)I)(I+A_1+B_1+nI)}{(2I)(I+A_1)(I+C_1)}\mathrm{B}_{n-1}^{(I+A_1,I+B_1;2I,B_2)}(I+C_1,z;C_2,w) - \tfrac{(-nI)(nI+A_1)(2I+B_2+nI)}{(I+A_1)(2I)(I+C_2)}\mathrm{B}_{n-1}^{(I+A_1,B_1;2I,I+B_2)}(C_1,z;I+C_2,w)=0.$$

4.

$$D\,\mathrm{B}_n^{(A_1,B_1;A_2,I)}(C_1,z;C_2,w) - \tfrac{(-nI)(nI+A_2)(I+A_1+B_1+nI)}{(I+A_1)(I+C_1)(I+A_2)}\mathrm{B}_{n-1}^{(I+A_1,I+B_1;I+A_2,I)}(I+C_1,z;C_2,w) - \tfrac{(-nI)(nI+A_1)(2I+A_2+nI)}{(I+A_1)(I+A_2)(I+C_2)}\mathrm{B}_{n-1}^{(I+A_1,B_1;I+A_2,2I)}(C_1,z;I+C_2,w)=0.$$

Also, if we put $A_1=A_2=I$ and $B_1=B_2=I$, respectively we get

5.

$$D\,\mathrm{B}_n^{(I,B_1;I,B_2)}(C_1,z;C_2,w) - \tfrac{(-nI)((n+1)I)(2I+B_1+nI)}{(2I)(I+C_1)(2I)}\mathrm{B}_{n-1}^{(2I,I+B_1;2I,B_2)}(I+C_1,z;C_2,w) - \tfrac{(-nI)((n+1)I)(2I+B_2+nI)}{(2I)(2I)(I+C_2)}\mathrm{B}_{n-1}^{(2I,B_1;2I,I+B_2)}(C_1,z;I+C_2,w)=0.$$

6.

$$D\,\mathrm{B}_n^{(A_1,I;A_2,I)}(C_1,z;C_2,w) - \tfrac{(-nI)(nI+A_2)(2I+A_1+nI)}{(I+A_1)(I+C_1)(I+A_2)}\mathrm{B}_{n-1}^{(I+A_1,2I;I+A_2,I)}(I+C_1,z;C_2,w) - \tfrac{(-nI)(nI+A_1)(2I+A_2+nI)}{(I+A_1)(I+A_2)(I+C_2)}\mathrm{B}_{n-1}^{(I+A_1,I;I+A_2,2I)}(C_1,z;I+C_2,w)=0.$$

Now, we take the another formula of generalized Bateman's matrix polynomials $B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,w)$ of two variables as given (e.g. [1]):

$$B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,w) = \sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(I+A_1+B_1+nI)_r\left(\frac{1}{2}(1+z_1)\right)_r(I+A_2+B_2+nI)_s\left(\frac{1}{2}(1+z_2)\right)_s}{r!s!(1+A_1)_r(C_1)_r(1+A_2)_s(C_2)_s}. \tag{2.13}$$

Again the above relation can be written by the double hypergeometric function as follows:

$$\begin{aligned}
&\mathrm{B}_n^{(A_1,B_1;A_2,B_2)}(C_1,z;C_2,w)\\
&= F^{1,2,2}_{0,2,2}\left[\begin{array}{lll} -n, I+A_1+B_1+nI, \frac{1}{2}(1+z_1); & I+A_2+B_2+nI, \frac{1}{2}(1+z_2); & \\ -, I+A_1, C_1 & ; I+A_2, C_2 & ; 1, 1 \end{array}\right],
\end{aligned}$$

where we have used a special case of the double hypergeometric function.

The relation (2.13) can be represented as follows:

$$\begin{aligned}
&B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,w)\\
&= \sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_r(I+A_1+B_1+nI)_r\big(\frac{1}{2}(1+z_1)\big)_r}{r!(1+A_1)_r(C_1)_r}B_n^{(A_2,B_2+r)}(C_2,w),
\end{aligned}$$

where $B_n^{(A_2,B_2+r)}(C_2,w)$ is the well-known generalized Bateman matrix polynomials of one variable.

The relationships between generalized Bateman's matrix polynomials of single variable and generalized Bateman's matrix polynomials of two variables are as follows (e.g. [1]):

$$B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,-1) = B_n^{(A_1,B_1)}(C_1,z), \tag{2.14}$$

$$B_n^{(A_1,B_1,A_2,B_2)}(C_1,-1,C_2,w) = B_n^{(A_2,B_2)}(C_2,w). \tag{2.15}$$

We established the following theorem by acting on the differential operator (2.2) for the generalized Bateman's matrix polynomials (2.13):

Theorem 2.2. *The generalized Bateman's matrix polynomials* (2.13) *and the differential operator* (2.2) *satisfies the following partial differential equation*:

$$\left[D - \sum_{h=1}^{r}\frac{1}{(h+z)} + \sum_{k=1}^{s}\frac{1}{(k+z)}\right]B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,w) = 0. \tag{2.16}$$

Proof. We can deduce this by using the differential operator (2.2) for the generalized Bateman's matrix polynomials of two variables (2.13) we get

$$\begin{aligned}
&D\,B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,w)\\
&= \left(\frac{\partial}{\partial z}+\frac{\partial}{\partial w}\right)\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(I+A_1+B_1+nI)_r\big(\frac{1}{2}(1+z)\big)_r(I+A_2+B_2+nI)_s\big(\frac{1}{2}(1+w)\big)_s}{r!s!(1+A_1)_r(C_1)_r(1+A_2)_s(C_2)_s}\\
&= \frac{\partial}{\partial z}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(I+A_1+B_1+nI)_r\big(\frac{1}{2}(1+z)\big)_r(I+A_2+B_2+nI)_s\big(\frac{1}{2}(1+w)\big)_s}{r!s!(1+A_1)_r(C_1)_r(1+A_2)_s(C_2)_s}\\
&\quad+\frac{\partial}{\partial w}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-n)_{r+s}(I+A_1+B_1+nI)_r\big(\frac{1}{2}(1+z)\big)_r(I+A_2+B_2+nI)_s\big(\frac{1}{2}(1+w)\big)_s}{r!s!(1+A_1)_r(C_1)_r(1+A_2)_s(C_2)_s}\\
&= \sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{h=1}^{r}\frac{1}{(h+z)}\frac{\big(-\frac{n}{2}\big)_{r+s}(I+A_1+B_1+nI)_r(1+z)_r(I+A_2+B_2+nI)_s(1+w)_s}{r!s!(1+A_1)_r(C_1)_r(1+A_2)_s(C_2)_s}
\end{aligned}$$

$$+\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{k=1}^{s}\frac{1}{(k+w)}\frac{\left(-\frac{n}{2}\right)_{r+s}(I+A_1+B_1+nI)_r(1+z)_r(I+A_2+B_2+nI)_s(1+w)_s}{r!s!(1+A_1)_r(C_1)_r(1+A_2)_s(C_2)_s}$$

$$=\sum_{h=1}^{r}\frac{1}{(h+z)}B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,w)+\sum_{k=1}^{s}\frac{1}{(k+w)}B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,w).$$

Therefore

$$D\ B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,w)$$

$$=\left[\sum_{h=1}^{r}\frac{1}{(h+z)}+\sum_{k=1}^{s}\frac{1}{(k+w)}\right]B_n^{(A_1,B_1,A_2,B_2)}(C_1,z,C_2,w).$$

Then the relation (2.16) is true. □

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