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PROJECTIONS AND SLICES OF MEASURES

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Abstract. We consider a generalization of the L^q-spectrum with respect to two Borel probability measures on $\mathbb{R}^n$ having the same compact support, and also study their behavior under orthogonal projections of measures onto an m-dimensional subspace. In particular, we try to improve the main result of Bahroun and Bhouri [4]. In addition, we are interested in studying the behavior of the generalized lower and upper L^q-spectrum with respect to two measures on "sliced" measures in an $(n-m)$-dimensional linear subspace. The results in this article establish relations with the L^q-spectrum with respect to two Borel probability measures and its projections and generalize some well-known results.

1. Introduction

The basic geometric properties of Hausdorff and packing dimensions [1–10, 12, 13, 16, 17, 19, 21, 23, 24, 26, 28, 31, 35–40, 42], as well as the dimension properties of intersections of sets and sections of measures [13, 14, 18, 20, 22, 26, 27, 29, 32, 34, 41], are well known. Recently there has been interest in the study of fractal dimensions of projection of sets and measures. The first significant work in this area was the article [25]. Marstrand proved that if E is a Borel subset of $\mathbb{R}^2$, then for orthogonal projection π_V onto the line V at angle θ to the x-axis

$$\dim_H\big(\pi_V(E)\big) = \min\big(\dim_H E, 1\big)$$

for almost all $\theta \in [0,\pi)$, where $\dim_H$ denotes the Hausdorff dimension. Later, this result was generalized for higher dimensions by Kaufman [23] and Mattila [26], who obtained similar results for the Hausdorff dimension of a measure. Let us mention that Falconer and Mattila [13] and Falconer and Howroyd [12] extended these results for the packing dimension of orthogonal projection onto m-dimensional subspaces of $\mathbb{R}^n$ of probability measure and for the packing dimension of the slices of measure by almost all $(n-m)$-planes V_a through point a.

Let μ be a Borel probability measure on a metric subspace of $\mathbb{R}^n$ with compact support. For $q \geq 0$ and $q \neq 1$ Hunt and Kaloshin [17] introduced the

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lower and upper q-dimensions of measure μ by

$$\underline{D}_\mu(q) = \liminf_{r\to 0} \frac{1}{(q-1)\log r} \log \int \mu(B(x,r))^{q-1} d\mu(x)$$

and

$$\overline{D}_\mu(q) = \limsup_{r\to 0} \frac{1}{(q-1)\log r} \log \int \mu(B(x,r))^{q-1} d\mu(x),$$

where $B(x,r)$ is the ball with center x and radius r, $r > 0$. If these dimensions coincide, then their common value is denoted $D_\mu(q)$ and called the q-dimension of μ. The q-dimension allows us to measure in certain cases the degree of singularity and in other ones the degree of regularity of measures [6, 15, 30, 33, 36]. Hunt and Kaloshin [17] showed that if $1 < q \le 2$, then the lower q-dimension $\underline{D}_\mu(q)$ equals

$$\underline{D}_{\mu_V}(q) = \min\big(m, \underline{D}_\mu(q)\big)$$

for almost all V, where μ_V is the image of μ under the orthogonal projection π_V onto $V \in G_{n,m}$. Recently, Järvenpää et al. [18] and also Falconer and O'Neil [14] reproved their result by studying certain appropriately defined convolution kernels. By these methods they also proved that for the upper q-dimension of projections of compactly supported Borel probability measure μ onto $V \in G_{n,m}$, $1 \le m < n$,

$$\overline{D}_{\mu_V}(q) = \overline{D}^m_\mu(q)$$

for $\gamma_{n,m}$-almost all V.

One of the interesting problems considered in the literature and related with the dimensions of projections [11,27,29,30,32] is the study of multifractal geometry of intersections of measures with lower dimensional subspaces, the so-called slices of measures. Falconer and O'Neil introduced [14] the generalized q-dimensions of slices of a measure by $(n-m)$-dimensional planes and proved that for all $V \in G_{n,n-m}$ and almost all a from the orthogonal complement $V^\perp$ of V

$$D_{\mu_{V_a}}(q) \le \max\Big(0, D_\mu(q) - m\Big).$$

Moveover, Falconer and Mattila [13] proved that if $\dim_H \mu > m$ for Borel probability measure μ on $\mathbb{R}^n$, then for almost all $a \in \mathbb{R}^n$ and $\gamma_{n,n-m}$-almost all $V_a = \{v + a : v \in V\}$, $V \in G_{n,m}$,

$$\dim_P \mu_{V_a} \ge \frac{(n-m)\dim_P \mu\Big(\dim_H \mu - m\Big)}{n \dim_H \mu - m \dim_P \mu},$$

$\dim_H \mu$ and $\dim_P \mu$ denotes the Hausdorff and the packing dimensions of the measure, respectively. We note that other studies of slices of probability measures were carried out in this direction [20, 22, 41], as well as measures of slices of specific sets, for example, self-similar sets [32] and dynamically defined sets [34], were considered.

Let μ and ν be two Borel probability measures on $\mathbb{R}^n$ with coincident compact supports. For $q \in \mathbb{R}^n$ Bhouri [7] proposed the following generalized lower and upper L^q-spectrum of measure μ with respect to ν

$$\underline{T}_{\mu,\nu}(q) = \liminf_{r\to 0} \frac{1}{\log r} \log \int \mu(B(x,r))^q d\nu(x)$$

and

$$\overline{T}_{\mu,\nu}(q) = \limsup_{r\to 0} \frac{1}{\log r} \log \int \mu(B(x,r))^q d\nu(x).$$

If $\underline{T}_{\mu,\nu}(q) = \overline{T}_{\mu,\nu}(q)$, their common value at q is denoted by $T_{\mu,\nu}(q)$ and called the generalized L^q-spectrum of μ relatively to ν. This quantity appears as a generalization of the q-spectral dimension $D_\mu(q)$. The behavior of such spectra under orthogonal projections is studied in [4, 7, 35]. As it turned out, this technique is very useful in studying the effect of one measure on another, both in theory and in applications.

As a continuation of these researches, we introduce a variation of the upper and lower L^q-spectrum defined in terms of a convolution with a certain kernel, according to the method proposed by Falconer and O'Neil [14]. In particular it allows us to see the effect of projection on the L^q-spectrum relatively to two measures. In the following, we give an example of measures μ and ν where the equality holds between the upper and lower bounds of the generalized L^q-spectral dimension of μ_V relatively to ν_V. These results extend the main results of Falconer and O'Neil in [14] and are more refined than those found in [4, 7]. In addition, we are interested in studying the behavior of generalized lower and upper L^q-spectrum relatively to two measures on $\mathbb{R}^n$ under "*sliced*" measures into $(n-m)$-dimensional linear subspace.

2. Preliminaries

Let m be an integer with $0 < m < n$ and $G_{n,m}$ stand for the Grassmannian manifold of all m-dimensional linear subspaces of $\mathbb{R}^n$ and we denote $\gamma_{n,m}$ the invariant Haar measure on $G_{n,m}$ such that $\gamma_{n,m}(G_{n,m}) = 1$. For $V \in G_{n,m}$ we define the projection map $\pi_V : \mathbb{R}^n \longrightarrow V$ as the usual orthogonal projection onto V. For a Borel probability measure μ on $\mathbb{R}^n$ supported on the compact set $\operatorname{supp}\mu$ and for $V \in G_{n,m}$ we define μ_V, the projection of μ onto V, by

$$\mu_V(A) = \mu(\pi_V^{-1}(A)) \quad \forall A \subseteq V.$$

Since μ has a compact support, then $\operatorname{supp}\mu_V = \pi_V(\operatorname{supp}\mu)$ for all $V \in G_{n,m}$. For any continuous function $f : V \longrightarrow \mathbb{R}$ we have

$$\int_V f d\mu_V = \int f(\pi_V(x)) d\mu(x)$$

whenever these integrals exist.

Throughout the paper, we assume that both μ and ν are compactly supported Borel probability measures with $\operatorname{supp}\mu = \operatorname{supp}\nu$ on $\mathbb{R}^n$. Recall the following theorem of Bahroun and Bhouri [4].

Theorem 2.1. *For $0 < m < n$ and $\gamma_{n,m}$-almost every $V \in G_{n,m}$*

1. *If $0 < q \leq 1$ and $\underline{T}_{\mu,\nu}(q) \leq mq$, then $\underline{T}_{\mu_V,\nu_V}(q) = \underline{T}_{\mu,\nu}(q)$.*
2. *If $q > 1$ and $\underline{T}_{\mu,\nu}(q) \leq m$, then $\underline{T}_{\mu_V,\nu_V}(q) = \underline{T}_{\mu,\nu}(q)$.*

Further, we also need an alternative characterization of the generalized upper L^q-spectrum with respect to measures μ and ν [7] obtained by convolving the measure ν with certain kernel given by $\min\{1, r^k|x-y|^{-k}\}$ for $x, y \in \mathbb{R}^n$, $r > 0$. For all $s \geq 0$, $q > 0$ and $k \in \mathbb{N}^*$

$$L^k_{s,q}(\mu,\nu) = \liminf_{r\to 0} r^{-s} \int \left(\int \min\{1, r^k|x-y|^{-k}\} d\mu(y) \right)^q d\nu(x)$$

and

$$\dim^k_q(\mu,\nu) = \sup\Big\{ s \geq 0 : \; L^k_{s,q}(\mu,\nu) < \infty \Big\}.$$

Proposition 2.2 ([7])**.** *For all $q > 0$, $\overline{T}_{\mu,\nu}(q) = \dim^n_q(\mu,\nu)$.*

Bhouri studied the behavior of the generalized upper L^q-spectrum relatively to two measures under orthogonal projections onto a lower dimensional linear subspaces. For $0 < m < n$ the following result was proved.

Theorem 2.3 ([7])**.** *Let $q > 0$. Then*

1. *For $0 < q \leq 1$, we have*
$$\overline{T}_{\mu_V,\nu_V}(q) = \dim^m_q(\mu,\nu) \text{ for } \gamma_{n,m}\text{-almost every } V \in G_{n,m}.$$
2. *For $q > 1$, we have*
$$\overline{T}_{\mu_V,\nu_V}(q) = \min\Big(mq, \dim^m_q(\mu,\nu) \Big) \text{ for } \gamma_{n,m}\text{-almost every } V \in G_{n,m}.$$

Remark 2.4. Let us notice that assertion 1 of the theorem is a generalization of the result of Järvenpää et al. [18], while the assertion 2 extends the result of Järvenpää et al. to the case $q > 1$, which is not considered in their paper.

3. Projection estimates for measures

3.1. Convolution properties

In this section we require an alternative characterization of the generalized upper and lower L^q-spectrum, defined on terms of the convolution. For $1 \leq m < n$ and $r > 0$ defined

$$\begin{array}{rccl} \phi^m_r : & \mathbb{R}^n & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \min\Big\{1, \; r^m|x|^{-m}\Big\}. \end{array}$$

Let $\mathcal{P}_n$ denote the set of all compactly supported Borel probability measures on $\mathbb{R}^n$. For $\mu \in \mathcal{P}_n$ and $V \in G_{n,m}$ we have

$$\mu * \phi^m_r(x) = \int \mu_V(B(x_V, r))dV = \int \min\Big\{1, \; r^m|x-y|^{-m}\Big\} d\mu(y).$$

So, converting into spherical coordinates and integrating by parts [12], we have

$$\mu * \phi_r^m(x) = mr^m \int_r^{+\infty} u^{-m-1}\mu(B(x,u))du. \tag{1}$$

We can use this approach for generalized L^q-spectrum with respect to measures μ and ν from $\mathcal{P}_n$, using appropriate definitions in terms of kernels. For $1 \le m < n$ and $q > 0$, we define

$$\underline{T}^m_{\mu,\nu}(q) = \liminf_{r\to 0} \frac{1}{\log r} \log \int \left(\int \min\left\{1,\ r^m|x-y|^{-m}\right\} d\mu(y) \right)^q d\nu(x)$$

and

$$\overline{T}^m_{\mu,\nu}(q) = \limsup_{r\to 0} \frac{1}{\log r} \log \int \left(\int \min\left\{1,\ r^m|x-y|^{-m}\right\} d\mu(y) \right)^q d\nu(x).$$

Lemma 3.1 ([14])**.** *Let $1 \le m < n$, $q > 0$, $\varepsilon > 0$ and $R > 1$. Then there are numbers $A, B > 0$ such that for all $\mu, \nu \in \mathcal{P}_n$ with* $\operatorname{supp}\mu = \operatorname{supp}\nu \subseteq B(0,R)$ *and* $0 < r < 1$

$$\begin{aligned} &A\, r^{mq+\varepsilon} \int_r^{\infty} u^{-mq-1} \int \mu(B(x,u))^q d\nu(x) du \\ \le& \int \left(\int \min\left\{1\ ,\ r^m|x-y|^{-m}\right\} d\mu(y) \right)^q d\nu(x) \\ \le&\ B\, r^{mq-\varepsilon} \int_r^{\infty} u^{-mq-1} \int \mu(B(x,u))^q d\nu(x) du. \end{aligned}$$

The next result is essentially a restatement of [14, Proposition 3.8]. We provide a proof for the reader's convenience.

Lemma 3.2. *For $q > 0$, we have*

$$\underline{T}^m_{\mu,\nu}(q) = \min\Big(mq, \underline{T}_{\mu,\nu}(q)\Big).$$

Proof. Recalling from [14, Proposition 2.3] that for all $x \in \mathbb{R}^n$ and $r > 0$

$$\int \min\left\{1,\ r^m|x-y|^{-m}\right\} d\mu(y) \ge \mu(B(x,r)),$$

it will be clear that for $q > 0$ we have

$$\underline{T}^m_{\mu,\nu}(q) \le \underline{T}_{\mu,\nu}(q) \qquad \text{and} \qquad \overline{T}^m_{\mu,\nu}(q) \le \overline{T}_{\mu,\nu}(q).$$

Also by using [14, Lemma 2.1] we have that for all $x \in \mathbb{R}^n$ and for any sufficiently small r,

$$cr^m \le \int \min\left\{1,\ r^m|x-y|^{-m}\right\} d\mu(y),$$

where $c > 0$ is independent of r. This leads to

$$\underline{T}^m_{\mu,\nu}(q) \le \overline{T}^m_{\mu,\nu}(q) \le mq.$$

In order to prove the other inequality, suppose that $\operatorname{supp}\mu = \operatorname{supp}\nu$ have diameter h. From Lemma 3.1 for $\varepsilon > 0$,

$$\int \Big(\mu * \phi_r^m(x)\Big)^q d\nu(x) \le B\, r^{mq-\varepsilon} \int_r^{+\infty} u^{-mq-1} \int \mu(B(x,u))^q d\nu(x) du.$$

If $t < \underline{T}_{\mu,\nu}(q)$, then

$$\int \mu(B(x,r))^q d\nu(x) \le c_1 r^t, \quad \forall r \le 2h,$$

where c_1 is independent of r, and

$$\int \mu(B(x,r))^q d\nu(x) = 1, \quad \forall r \ge 2h.$$

For $\varepsilon > 0$ and r is small enough,

$$\begin{aligned}
&\int \Big(\mu * \phi_r^m(x)\Big)^q d\nu(x) \\
&\le B\, r^{mq-\varepsilon} \int_r^{+\infty} u^{-mq-1} \int \mu(B(x,u))^q d\nu(x) du \\
&= B\, r^{mq-\varepsilon} \int_r^{2h} u^{-mq-1} \int \mu(B(x,u))^q d\nu(x) du \\
&\quad + B\, r^{mq-\varepsilon} \int_{2h}^{+\infty} u^{-mq-1} \int \mu(B(x,u))^q d\nu(x) du \\
&\le C_1 r^{mq-\varepsilon} \int_r^{2h} u^{-mq-1+t} du + C_2 r^{mq-\varepsilon} \int_{2h}^{+\infty} u^{-mq-1} du \\
&\le \begin{cases} C_3 r^{t-\varepsilon} & \text{if} \quad t < mq, \\ C_4 r^{mq-\varepsilon} & \text{if} \quad t \ge mq, \end{cases}
\end{aligned}$$

where C_i $(i = 1, \dots, 4)$ are independent of r. This gives that

$$\underline{T}^m_{\mu,\nu}(q) \ge \min(mq, t) \quad \text{for all} \quad t < \underline{T}_{\mu,\nu}(q).$$

Finally, we obtain

$$\underline{T}^m_{\mu,\nu}(q) \ge \min\Big(mq, \underline{T}_{\mu,\nu}(q)\Big). \qquad \square$$

Proposition 3.3.

(1) *For all sufficiently small r and $q > 0$, there exists c independent of r such that for all $V \in G_{n,m}$,*

$$\int \mu_V(B(x_V, r))^q d\nu_V(x_V) \ge c \int \Big(\int \min\{1,\ r^m |x-y|^{-m}\} d\mu(y)\Big)^q d\nu(x).$$

(2) *Let $0 < q \le 1$. For $\gamma_{n,m}$-almost all $V \in G_{n,m}$ and for all sufficiently small r,*

$$\int \mu_V(B(x_V, r))^q d\nu_V(x_V) \le C \int \Big(\int \min\{1,\ r^m |x-y|^{-m}\} d\mu(y)\Big)^q d\nu(x),$$

where C is independent of r.

Proof. We have

(1) The ideas needed to prove the statement can be found in the proof of Proposition 3.6 in [14] and Lemma 3.4 in [7].

(2) Follows immediately from Lemma 3.11 in [27], Jensen's inequality and Fubini's Theorem. □

The following results present alternative expressions of the L^q-spectrum in terms of the convolutions as well as general relations between the L^q-spectrum of measures and that of its orthogonal projections.

Corollary 3.4. *We have*

(1) *for all $q > 0$ and $V \in G_{n,m}$,*

$$\liminf_{r\longrightarrow 0} \frac{1}{\log r} \log \left(\frac{\displaystyle\int \left(\int \min\left\{1,\ r^m|x-y|^{-m}\right\} d\mu(y) \right)^q d\nu(x)}{\displaystyle\int \mu_V(B(x_V,r))^q d\nu_V(x_V)} \right) \geq 0;$$

(2) *for $0 < q \leq 1$ and $\gamma_{n,m}$-almost all $V \in G_{n,m}$,*

$$\lim_{r\to 0} \frac{1}{\log r} \log \left(\frac{\displaystyle\int \left(\int \min\left\{1,\ r^m|x-y|^{-m}\right\} d\mu(y) \right)^q d\nu(x)}{\displaystyle\int \mu_V(B(x_V,r))^q d\nu_V(x_V)} \right) = 0.$$

Theorem 3.5. *One has*

(1) *for all $q > 0$ and $V \in G_{n,m}$,*

$$\underline{T}_{\mu_V,\nu_V}(q) \leq \underline{T}^m_{\mu,\nu}(q) \quad \text{and} \quad \overline{T}_{\mu_V,\nu_V}(q) \leq \overline{T}^m_{\mu,\nu}(q);$$

(2) *for all $0 < q \leq 1$ and $\gamma_{n,m}$-almost all $V \in G_{n,m}$,*

$$\underline{T}_{\mu_V,\nu_V}(q) = \underline{T}^m_{\mu,\nu}(q) = \min\Big(mq, \underline{T}_{\mu,\nu}(q)\Big)$$

and

$$\overline{T}_{\mu_V,\nu_V}(q) = \overline{T}^m_{\mu,\nu}(q) = \dim^m_q(\mu,\nu);$$

(3) *for all $q > 1$ and $\gamma_{n,m}$-almost all $V \in G_{n,m}$,*

(a) *If $\underline{T}_{\mu,\nu}(q) \leq m$, then $\underline{T}_{\mu_V,\nu_V}(q) = \underline{T}^m_{\mu,\nu}(q) = \underline{T}_{\mu,\nu}(q)$.*

(b) *If $\overline{T}_{\mu,\nu}(q) \leq mq$, then $\dim^m_q(\mu,\nu) = \overline{T}_{\mu_V,\nu_V}(q) = \overline{T}^m_{\mu,\nu}(q)$.*

Proof. This follows from Theorems 2.1 and 2.3, Lemma 3.2, Proposition 3.3 and Corollary 3.4. □

Remark 3.6.

(1) Let us notice that assertions 1 and 2 are a generalization of the result of Falconer and O'Neil in [14]. The assertion 3 extends the result of Falconer and O'Neil to the case $q > 1$ untreated in their work.

(2) The assertion 2 improves the main result of Bahroun and Bhouri [4, Theorem 2.1(1)]. The results in Theorem 3.5 are more refined than those found in [4, 7].

3.2. Equality case

We give an example of measures μ and ν where the equality holds between the upper and lower bounds of the generalized L^q-spectral dimension of μ_V relatively to ν_V. Consider a compactly supported Borel probability measure μ on $\mathbb{R}^n$. For any integer s with $0 < m \leq s < n$, we define the s-energy of μ by

$$I_s(\mu) = \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s}.$$

Let ν be a compactly supported Borel probability measure satisfies the following condition, for a Borel set A in $\mathbb{R}^n$

$$\nu(A) \leq \mathcal{L}^n(A). \tag{2}$$

Theorem 3.7. *For $m \leq s < n$, suppose that $I_s(\mu) < \infty$ and ν satisfies (2). Then for $\gamma_{n,m}$-almost all $V \in G_{n,m}$ and all $q > 1$,*

(1) *if $2m < s$, we have*

$$T_{\mu_V,\nu_V}(q) = mq \quad \textit{for} \quad 1 < q < \infty;$$

(2) *if $m < s < 2m$, we have*

$$T_{\mu_V,\nu_V}(q) = mq \quad \textit{for } 1 < q < \frac{2m}{2m-s}$$

and

$$\tfrac{sq}{2} \leq \underline{T}_{\mu_V,\nu_V}(q) \leq \overline{T}_{\mu_V,\nu_V}(q) \leq mq \quad \textit{for} \quad q > \frac{2m}{2m-s}.$$

Before proving this theorem we need some preliminary results. Take $r > 0$ and denote by $\Theta(r)$ the set of r-mesh cubes C in $\mathbb{R}^n$, that is, cubes of the form $\prod_{j=1}^{n} \Big[k_j r,\ (k_j+1)r\Big[$, where $k_j \in \mathbb{Z}$.

Lemma 3.8. *For $q \geq 0$, we have*

$$\int \mu(B(x,\sqrt{n}r))^q d\nu(x) \geq \sum_{C\in\Theta(r)} \mu(C)^q \nu(C) \geq 3^{-nq} \int \mu(B(x,r))^q d\nu(x).$$

Proof. The proof for all these inequalities are very similar to those given for [14, Lemma 2.6]. □

Corollary 3.9. *For $q > 0$, we have*

$$\underline{T}_{\mu,\nu}(q) = \liminf_{r\to 0} \frac{1}{\log r} \log \sum_{C\in\Theta(r)} \mu(C)^q\nu(C),$$

$$\overline{T}_{\mu,\nu}(q) = \limsup_{r\to 0} \frac{1}{\log r} \log \sum_{C\in\Theta(r)} \mu(C)^q\nu(C).$$

For the measure μ on $\mathbb{R}^n$ and for $p > 1$, we say that $\mu \in L^p(\mathbb{R}^n)$ if there is a function $f \in L^p(\mathbb{R}^n)$ such that f is the Radon-Nikodym derivative of μ with respect to $\mathcal{L}^n$ for μ-a.e. x.

Lemma 3.10. *Fix $p > 1$. Suppose that $\mu \in L^p(\mathbb{R}^n)$ and ν satisfies* (2). *Then*

$$\underline{T}_{\mu,\nu}(q) \geq \begin{cases} \frac{nq}{p}(p-1), & \text{if} \quad q \geq p > 1, \\ nq, & \text{if} \quad 0 < q < p. \end{cases}$$

Proof. Let $f = \frac{d\mu}{d\mathcal{L}^n} \in L^p(\mathbb{R}^n)$. Using Hölder's inequality we obtain

$$\begin{aligned} \sum_{C\in\Theta(r)} \mu(C)^q \nu(C) &= \sum_{C\in\Theta(r)} \left[\left(\int_C f d\mathcal{L}^n \right)^q \nu(C) \right] \\ &\leq r^{nq(1-\frac{1}{p})} \sum_{C\in\Theta(r)} \left[\left(\int_C f^p d\mathcal{L}^n \right)^{\frac{q}{p}} \nu(C) \right] \\ &\leq \begin{cases} r^{nq(1-\frac{1}{p})} \left(\displaystyle\sum_{C\in\Theta(r)} \int_C f^p d\mathcal{L}^n \right)^{\frac{q}{p}} r^n, & \text{if} \quad q \geq p > 1, \\ c_1 r^{nq(1-\frac{1}{p})} \left(\displaystyle\sum_{C\in\Theta(r)} \int_C f^p d\mathcal{L}^n \right)^{\frac{q}{p}} r^{\frac{nq}{p}}, & \text{if} \quad p > q > 0, \end{cases} \\ &\leq \begin{cases} c_2 r^{n\left(\frac{q}{p}(p-1)+1\right)}, & \text{if} \quad q \geq p > 1, \\ c_3 r^{nq}, & \text{if} \quad p > q > 0, \end{cases} \end{aligned}$$

where the constants c_1 c_2, c_3 are independent of positive radius r. Taking lower limit gives the result. □

Proof of Theorem 3.7. It is a consequence of Theorem 2.3, Lemma 3.10 and [14, Proposition 3.11]. □

Remark 3.11.

(1) The results of Theorem 3.7 hold if we replace the assumptions $I_s(\mu) < \infty$ and ν satisfies (2) by μ is a Borel probability compactly supported measure on $\mathbb{R}^n$ with $\operatorname{supp}\mu = \operatorname{supp}\nu$, $I_s(\nu) < \infty$ and

$$\mu(A) \leq \nu(A) \quad \text{for all } A \subset \mathbb{R}^n.$$

(2) Due to Example 4.1 in [7], Theorem 3.7 and the above results are optimal and the results are the best possible one.

Question. Let $q \geq 0$, μ be a compactly supported Borel probability measure such that $I_s(\mu) < \infty$ for some $m \leq s < n$ and $\nu = \mu_q$ be a *Frostman like measure* (*Gibbs measures on conformal repellers*, see [7, Definition 5.1] and [5] for the definitions). Then, the following problem remains open:

$$T_{\mu_V,\nu_V}(q) = \underline{T}^m_{\mu,\nu}(q) = \overline{T}^m_{\mu,\nu}(q) = mq$$

for $\gamma_{n,m}$-almost all $V \in G_{n,m}$.

4. Slices of measures

In this section, we use convolution kernel ψ_r^m to study "*slices*" of measures on $\mathbb{R}^n$ for $1 \leq m \leq n-1$. These "*slices*" will be $(n-m)$-dimensional [26, 29]. Further, we denote by $\mathcal{B}_{n,m}$ the set of Borel probability measures μ on $\mathbb{R}^n$ with bounded support and satisfying the condition

$$\int |x-y|^{-m} d\mu(y) < \infty \tag{3}$$

for μ-almost all $x \in \mathbb{R}^n$. The condition (3) implies that the projected measure $\mu_{V^\perp}$ is absolutely continuous with respect to m-dimensional Lebesgue measure $\mathcal{L}^m_{V^\perp}$ on $V^\perp$ identified with $\mathbb{R}^m$ for $\gamma_{n,n-m}$-almost all V, where $V^\perp$ is the orthogonal complement of V. For $V \in G_{n,n-m}$ and $x \in \mathbb{R}^n$, we consider the translate V_x of V passing through x, defined by

$$V_x = \big\{\omega + x : \ \omega \in V\big\}.$$

Obviously, for $\mathcal{L}^m$-almost all $x \in \mathbb{R}^n$ there exists a Borel measure μ_{V_x} on V_x called the *slice* or section of μ by the m-plan V_x, such that

$$\int h d\mu_{V_x} = \lim_{r\to 0} \alpha(m)^{-1} r^{-m} \int_{V_x(r)} h d\mu$$

for all continuous function h with compact support, where $\alpha(m)$ is the volume of the m-dimensional unit ball and $V_x(r)$ is the r-neighborhood of V_x, defined as follows

$$V_x(r) = \big\{y, \ d(y, V_x) \leq r\big\}.$$

We recall that there exists $a \in V^\perp$ satisfying $V_a = V_x$. We define

$$\mu_{V_x} = \mu_{V_a}, \quad \text{if} \quad a = \pi_{V^\perp}(x).$$

Here $\pi_{V^\perp} : \mathbb{R}^n \to V^\perp$ is the orthogonal projection. We also recall the basic property of *slices* of measures [27]

$$\int_{a\in V^\perp} \int f d\mu_{V_a} d\mathcal{L}^m_{V^\perp}(a) = \int f d\mu \tag{4}$$

for all non-negative Borel function f on $\mathbb{R}^n$. Obviously,

$$\operatorname{supp} \mu_{V_a} \subseteq \operatorname{supp} \mu \cap V_a. \tag{5}$$

We modify the definitions of the generalization of the L^q-spectrum relatively to two measures, following Falconer and O'Neil [14], we use as the kernel

$$\psi_r^m(x) = \begin{cases} r^m |x|^{-m} & \text{if } |x| \leq r, \\ 0 & \text{if } |x| > r. \end{cases}$$

From (3), we have

$$\mu * \psi_r^m(x) = r^m \int_{y\in B(x,r)} |x-y|^{-m} d\mu(y) < \infty$$

for μ-almost all x. Using the equation (4), it is proved [13,27] that the following is true

$$\int \mu_{V_x}(B(x,r))dV \le c \int_{y\in B(x,2r)} |x-y|^{-m} d\mu(y)$$

for μ-almost all x, where c depends only on n and m. Moreover,

$$\int \mu_{V_x}(B(x,r))dV \le cr^{-m} \mu * \psi^m_{2r}(x). \tag{6}$$

Integration in parts [13] gives

$$\mu * \psi^m_r(x) \le c_1 r^m \int_0^{2r} u^{-m-1} \mu(B(x,u))du. \tag{7}$$

In similar way, for $q > 0$ and $1 \le m < n$ we define a modified generalized upper and lower L^q-spectrum relatively to two compactly supported Borel probability measures μ and ν on $\mathbb{R}^n$, using the function ψ^m_r, as follows:

$$\underline{T}^{m_*}_{\mu,\nu}(q) = \liminf_{r\to 0} \frac{1}{\log r} \log \int \Big(\mu * \psi^m_r(x)\Big)^q d\nu(x)$$

and

$$\overline{T}^{m_*}_{\mu,\nu}(q) = \limsup_{r\to 0} \frac{1}{\log r} \log \int \Big(\mu * \psi^m_r(x)\Big)^q d\nu(x).$$

The following theorem gives an important relation between the generalization of the L^q-spectrum relatively to original measures and its *slices.*

Theorem 4.1. *Let $\mu, \nu \in \mathcal{B}_{n,m}$ and $0 < q \le 1$. If $\underline{T}_{\mu,\nu}(q) > mq$, then for $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ and $\mathcal{L}^m_{V^\perp}$-almost all $a \in V^\perp$, we have*

(1) $\underline{T}_{\mu_{V_a},\nu_{V_a}}(q) \ge \underline{T}^{m_*}_{\mu,\nu}(q) - mq = \underline{T}_{\mu,\nu}(q) - mq.$

(2) $\overline{T}_{\mu_{V_a},\nu_{V_a}}(q) \ge \overline{T}^{m_*}_{\mu,\nu}(q) - mq.$

Proof. It is an easy consequence of the following lemmas. □

Lemma 4.2 ([14])**.** *Let $m > 0$, $q > 0$, $R > 1$ and $\varepsilon > 0$. There exist $A > 0$ and $B > 0$ such that, for all non decreasing function $f : [0,\infty) \to [0,\infty)$ with $f(u)$ constant for $u \ge R$, we have*

$$A\, r^\varepsilon \int_r^\infty u^{-mq-1} f(u)^q du \le \Big(\int_r^\infty u^{-m-1} f(u) du\Big)^q \le B\, r^{-\varepsilon} \int_r^\infty u^{-mq-1} f(u)^q du$$

for all $0 < r \le 1$.

Lemma 4.3. *Let $q > 0$ and μ, ν be two compactly supported Borel probability measures on $\mathbb{R}^n$. If $\underline{T}_{\mu,\nu}(q) > mq$, then*

$$\underline{T}^{m_*}_{\mu,\nu}(q) = \underline{T}_{\mu,\nu}(q).$$

Proof. Since $\mu * \psi_r^m(x) \geq \mu(B(x,r))$, it is clear that, for $q > 0$, we have

$$\underline{T}_{\mu,\nu}^{m_*}(q) \leq \underline{T}_{\mu,\nu}(q).$$

For positive q, using inequality (7) and Lemma 4.2, we have

$$\begin{aligned}\int \Big(\mu * \psi_r^m(x)\Big)^q d\nu(x) &\leq c_1 r^{mq} \int \left(\int_0^{2r} u^{-m-1}\mu(B(x,u))du\right)^q d\nu(x)\\ &\leq c_2 r^{mq-\varepsilon} \int\int_0^{2r} u^{-mq-1}\mu(B(x,u))^q du d\nu(x),\end{aligned}$$

where $\varepsilon > 0$. If $\underline{T}_{\mu,\nu}(q) > t > mq$, we have for sufficiently small r,

$$\int_0^{2r} \mu(B(x,u))^q d\nu(x) \leq c_3 u^t.$$

Thus,

$$\int \Big(\mu * \psi_r^m(x)\Big)^q d\nu(x) \leq c_4 r^{mq-\varepsilon} \int_0^{2r} u^{-mq-1+t} du \leq c_5 r^{t-\varepsilon},$$

where c_i $(i = 1, \dots, 5)$ are independent of r, implies that $\underline{T}_{\mu,\nu}^{m_*}(q) \geq t$ for all $t < \underline{T}_{\mu,\nu}(q)$. □

Lemma 4.4. *Let $\mu, \nu \in \mathcal{B}_{n,m}$ and $0 < q < 1$. Given $\varepsilon > 0$, for $\gamma_{n,n-m}$-almost all V in $G_{n,n-m}$ and $\mathcal{L}_{V^\perp}^m$-almost all $a \in V^\perp$, we have*

$$\int_{x\in V_a} \mu_{V_a}(B(x,r))^q d\nu_{V_a}(x) \leq r^{-mq-\varepsilon} \int \Big(\mu * \psi_{2r}^m(x)\Big)^q d\nu(x). \tag{8}$$

Proof. Using (4), Fubini's theorem, Hölder's inequality and (6), for sufficiently small r, we obtain

$$\begin{aligned}&\int_V \left(\int_{a\in V^\perp}\int_{x\in V_a} \mu_{V_a}(B(x,r))^q d\nu_{V_a}(x) d\mathcal{L}_{V^\perp}^m(a)\right) dV\\ \leq\ &\int_V \int_{x\in\mathbb{R}^n} \mu_{V_x}(B(x,r))^q d\nu(x) dV\\ \leq\ &\int_{x\in\mathbb{R}^n} \left(\int_V \mu_{V_x}(B(x,r))^q dV\right) d\nu(x)\\ \leq\ &\int_{x\in\mathbb{R}^n} \left(\int_V \mu_{V_x}(B(x,r)) dV\right)^q d\nu(x)\\ \leq\ &C r^{-mq} \int_{x\in\mathbb{R}^n} \Big(\mu * \psi_{2r}^m(x)\Big)^q d\nu(x).\end{aligned}$$

Inequality (8) follows from applying the Borel-Cantelli lemma. □

Remark 4.5. The results of Theorem 4.1 hold if we replace the hypothesis $\mu \in \mathcal{B}_{n,m}$ by μ is a Borel probability compactly supported measure on $\mathbb{R}^n$ with $\operatorname{supp}\mu = \operatorname{supp}\nu$ and

$$\mu(A) \leq \nu(A) \quad \text{for all } A \subseteq \mathbb{R}^n.$$

The following example is constructed in a similar way as in [13, Example 5.2]. It shows that, for $q > 0$, the behavior of the generalized upper L^q-spectrum relatively to μ and ν under *slices* measures is different from the generalized upper L^q-spectrum relatively to μ and ν.

Example 4.6. Let $m < d < D < n$. There exists a Borel probability compactly supported measure μ on $\mathbb{R}^n$ such that the following properties hold:

(1) For some positive constant M,
$$Mr^D \le \mu(B(x,r)) \le M^{-1} r^d$$
for all $x \in \operatorname{supp}\mu$ and $0 < r \le 1$.

(2) There exist sequences $(r_i)_i$ and $(R_i)_i$ of positive real numbers going to zero such that
$$\mu(B(x, \sqrt{n} r_i)) = r_i^d \text{ and } \mu(B(x, \tfrac{R_i}{2})) = R_i^D \quad \text{for all } x \in \operatorname{supp}\mu,$$

(3) For $\gamma_{n,n-m}$-almost all $V \in G_{n,n-m}$ and for all $x \in \mathbb{R}^n$,
$$\overline{\dim}_B(\operatorname{supp}\mu \cap V_x) \le \frac{(n-m)D(d-m)}{nd - mD},$$
where $\overline{\dim}_B$ is the box-counting dimension (see [27]).

Now, for $m = 1$, $d = \frac{4}{3}$, $D = \frac{5}{3}$ and $n = 2$, we take a Borel probability compactly supported measure ν on $\mathbb{R}^2$ with $\operatorname{supp}\mu = \operatorname{supp}\nu$ and
$$\mu(B(x,r)) \le \nu(B(x,r)) \quad \text{for all } x \in \operatorname{supp}\mu \quad \text{and } r > 0.$$
For $q > 0$ and thanks to assertions (1) and (2), it is easy to show that
$$\underline{T}_{\mu,\nu}(q) = \frac{4q}{3} \quad \text{and} \quad \overline{T}_{\mu,\nu}(q) = \frac{5q}{3}.$$
Hence, for all $V \in G_{2,1}$ and all $a \in V^\perp$, we have
$$\overline{T}_{\mu_{V_a},\nu_{V_a}}(q) \le \overline{T}_{\mu_{V_a},\mu_{V_a}}(q) \le q\overline{\dim}_B(\operatorname{supp}\mu_{V_a})$$
for more details, see [18]. Moreover, by (5) and (3) we have for $\gamma_{2,1}$-almost all $V \in G_{2,1}$ and $\mathcal{L}^1_{V^\perp}$-almost all $a \in V^\perp$,
$$\overline{T}_{\mu_{V_a},\nu_{V_a}}(q) \le q\overline{\dim}_B(\operatorname{supp}\mu \cap V_x) < \overline{T}_{\mu,\nu}(q).$$

Remark 4.7. The results developed by Falconer and O'Neil [14] are obtained as a special case of the multifractal Theorems by setting $\mu = \nu$.

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