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THE COMPOSITION OF HURWITZ-LERCH ZETA FUNCTION WITH PATHWAY INTEGRAL OPERATOR

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Abstract. The aim of the present investigation is to establish the composition formulas for the pathway fractional integral operator connected with Hurwitz-Lerch zeta function and extended Wright-Bessel function. Some interesting special cases have also been discussed.

1. Introduction

The series representation of Hurwitz-Lerch zeta function and its integral form are given as follows (see [5, p. 27]):

$$\Phi(z,\xi,b)=\sum_{k=0}^{\infty}\frac{z^k}{(k+b)^{\xi}}, \tag{1.1}$$

$$(b\in\mathbb{C}\backslash\mathbb{Z}_0^-;\ \xi\in\mathbb{C}\text{ when }|z|<1;\Re(\xi)>1\text{ for }|z|=1).$$

Also, the integral representation of $\Phi(z,\xi,b)$ is given by

$$\Phi(z,\xi,b)=\frac{1}{\Gamma(\xi)}\int_0^{\infty}\frac{t^{\xi-1}e^{-bt}}{1-ze^{-t}}dt=\frac{1}{\Gamma(\xi)}\int_0^{\infty}\frac{t^{\xi-1}e^{-(b-1)t}}{e^t-z}dt, \tag{1.2}$$

$$(\Re(\xi)>0,\ \Re(b)>0\text{ for }|z|\leq 1(z\neq 1);\ \Re(\xi)>1\text{ for }z=1).$$

Goyal and Laddha [7] defined the Hurwitz-Lerch zeta function and its integral representation in the following form:

$$\Phi_v^*(z,\xi,b)=\sum_{k=0}^{\infty}\frac{(v)_k}{k!}\frac{z^k}{(k+b)^{\xi}}, \tag{1.3}$$

$$(v\in\mathbb{C},b\in\mathbb{C}\backslash\mathbb{Z}_0^-,\xi\in\mathbb{C}\text{ for }|z|<1;\ \Re(\xi-v)>1\text{ for }|z|=1),$$

and

$$\Phi_v^*(z,\xi,b)=\frac{1}{\Gamma(\xi)}\int_0^{\infty}\frac{t^{\xi-1}e^{-bt}}{(1-ze^{-t})^v}dt=\frac{1}{\Gamma(\xi)}\int_0^{\infty}\frac{t^{\xi-1}e^{-(b-v)t}}{(e^t-z)^v}dt, \tag{1.4}$$

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$$(\Re(\xi) > 0, \Re(b) > 0 \text{ for } |z| \leq 1 (z \neq 1); \Re(\xi) > 1 \text{ for } z = 1).$$

Garg et al. [6] defined an extension of (1.1) and (1.3) and its integral representation as:

$$\Phi_{\gamma,\vartheta;v}(z,\xi,b) = \sum_{k=0}^{\infty} \frac{(\gamma)_k(\vartheta)_k}{(v)_k k!} \frac{z^k}{(k+b)^{\xi}}, \tag{1.5}$$

$$(\gamma, \vartheta, v \in \mathbb{C}, b \in \mathbb{C}\backslash\mathbb{Z}_0^-, \xi \in \mathbb{C} \text{ for } |z| < 1;\ \Re(\xi + v - \gamma - \vartheta) > 1 \text{ for } |z| = 1).$$

$$\Phi_{\gamma,\vartheta;v}(z,\xi,b) = \frac{1}{\Gamma(\xi)} \int_0^{\infty} t^{\xi-1} e^{-bt} {}_2F_1(\gamma, \vartheta; v; ze^{-t}) dt, \tag{1.6}$$

$$(\Re(\xi) > 0, \Re(b) > 0 \text{ for } |z| \leq 1 (z \neq 1);\ \Re(\xi) > 1 \text{ for } z = 1).$$

Parmar and Raina [11] introduced a new extension of Hurwitz-Lerch zeta function and its integral representation as follows:

$$\Phi_{\gamma,\vartheta;v}(z,\xi,b;p) = \sum_{k=0}^{\infty} \frac{(\gamma)_k \mathfrak{B}(\vartheta + k, v - \vartheta; p)}{\mathfrak{B}(\vartheta, v - \vartheta) k!} \frac{z^k}{(k+b)^{\xi}}, \tag{1.7}$$

$(p \geq 0, \gamma, \vartheta, v \in \mathbb{C}, b \in \mathbb{C}\backslash\mathbb{Z}_0^-, \xi \in \mathbb{C} \text{ for } |z| < 1, \Re(\xi + v - \gamma - \vartheta) > 1 \text{ for } |z| = 1)$,
where the extended beta function $\mathfrak{B}(r, s; p)$ is defined by [3]:

$$\mathfrak{B}(r,s;p) = \mathfrak{B}_p(r,s) = \int_0^1 x^{r-1}(1-x)^{s-1} e^{-\frac{p}{x(1-x)}} dx, \tag{1.8}$$

and $\Re(p) > 0, \Re(r) > 0, \Re(s) > 0$, respectively.

Shadab et al. [15] introduced a new and modified extension of beta function as follows:

$$\mathfrak{B}_p^{\lambda}(r,s) = \mathfrak{B}(r,s;p,\lambda) = \int_0^1 x^{r-1}(1-x)^{s-1} E_{\lambda}\left(-\frac{p}{x(1-x)}\right) dx, \tag{1.9}$$

where $\Re(r) > 0, \Re(s) > 0$ and $E_{\lambda}(\cdot)$ is Mittag-Leffler function defined by

$$E_{\lambda}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\lambda k + 1)}. \tag{1.10}$$

A new extension of Hurwitz-Lerch zeta function [13] in term of extended beta function (1.9) is defined as

$$\begin{aligned} \Phi_{\gamma,\vartheta;\nu}[z,\xi,b;p,\psi] &= \Phi^{\psi}_{\gamma,\vartheta;\nu}[z,\xi,b;p] \\ &= \sum_{k=0}^{\infty} \frac{(\gamma)_k \mathfrak{B}_p^{\psi}(\vartheta + k, \nu - \vartheta)}{\mathfrak{B}(\vartheta, \nu - \vartheta)\ k!} \frac{z^k}{(k+b)^{\xi}}, \end{aligned} \tag{1.11}$$

$$(p \geq 0, \psi > 0, \gamma, \vartheta, \nu \in \mathbb{C}, b \in \mathbb{C}\backslash\mathbb{Z}_0^-, \xi \in \mathbb{C} \text{ for } |z| < 1,$$
$$\Re(\xi + \nu - \gamma - \vartheta) > 1 \text{ for } |z| = 1).$$

The extended Wright-Bessel (Bessel-Maitland) function [2] is defined by

$$J_{\mu,\lambda}^{\delta,\gamma,\eta}(z,p)=\sum_{k=0}^{\infty}\frac{\mathfrak{B}_p(\gamma+k\lambda,\eta-\gamma)}{\mathfrak{B}(\gamma,\eta-\gamma)}\frac{(\eta)_{k\lambda}(-z)^k}{\Gamma(\mu+\delta k+1)k!},\tag{1.12}$$

where $p\geq 0,\mu,\delta,\gamma,\eta\in\mathbb{C};\Re(p)>0,\Re(\delta)\geq 0,\Re(\mu)>-1,\Re(\gamma)>0,\Re(\eta)>0$. Recently, pathway fractional integral (PFI) operator introduced by Nair [10], which is associated with the pathway model defined by Mathai [8] and Mathai and Haubold [9]:

$$\left(\mathcal{P}_{0+}^{\varsigma,\varepsilon}f\right)(x)=x^{\varsigma}\int_0^{\left[\frac{x}{\mathrm{a}(1-\varepsilon)}\right]}\left(1-\frac{\mathrm{a}(1-\varepsilon)\xi}{x}\right)^{\frac{\varsigma}{(1-\varepsilon)}}f(\xi)d\xi,\tag{1.13}$$

where Lebesgue measurable function $f\in\mathcal{L}(a,b)$ for real or complex term valued function, $\varsigma\in\mathbb{C}$, $\Re(\varsigma)>0,\mathrm{a}>0$ and $\varepsilon<1$ (ε is a pathway parameter).

The pathway model for a real scalar ε and scalar random variables is represented by the probability density function (p.d.f.) in following manner:

$$f(x)=\frac{c}{|x|^{1-v}}\left[1-\mathrm{a}(1-\varepsilon)|x|^{\delta}\right]^{\frac{\beta}{(1-\varepsilon)}},\tag{1.14}$$

where $x\in(-\infty,\infty);\delta>0;\beta\geq 0;1-\mathrm{a}(1-\varepsilon)|x|^{\delta}>0;v>0$ and ε, c denote the pathway parameter, normalizing constant, respectively.

Additionally, for $\varepsilon\in\Re$, the normalizing constants are expressed in following way:

$$c=\begin{cases}\frac{1}{2}\frac{\delta[\mathrm{a}(1-\varepsilon)]^{\frac{v}{\delta}}\Gamma\left(\frac{v}{\delta}+\frac{\beta}{1-\varepsilon}+1\right)}{\Gamma\left(\frac{v}{\delta}\right)\Gamma\left(\frac{\beta}{1-\varepsilon}+1\right)} & (\varepsilon<1),\\ \frac{1}{2}\frac{\delta[\mathrm{a}(1-\varepsilon)]^{\frac{v}{\delta}}\Gamma\left(\frac{\beta}{\varepsilon-1}\right)}{\Gamma\left(\frac{v}{\delta}\right)\Gamma\left(\frac{\beta}{\varepsilon-1}-\frac{v}{\delta}\right)} & \left(\frac{1}{\varepsilon-1}-\frac{v}{\delta}>0,\varepsilon>1\right),\\ \frac{1}{2}\frac{\delta[\mathrm{a}\beta]^{\frac{v}{\delta}}}{\Gamma\left(\frac{v}{\delta}\right)}, & (\varepsilon\to 1).\end{cases}$$

It is noted that if $\varepsilon<1$, finite range density with $1-\mathrm{a}(1-\varepsilon)|x|^{\delta}>0$, and (1.14) can be considered a member of the extended generalized type-1 beta family. Also the triangular density, the uniform density, the extended type-1 beta density and various other probability density functions, are precise special cases of the pathway density function defined in (1.14) for $\varepsilon<1$.

For example, if $\varepsilon>1$ and by setting $(1-\varepsilon)=-(\varepsilon-1)$ in (1.13) yields

$$\left(\mathcal{P}_{0+}^{\varsigma,\varepsilon}f\right)(x)=x^{\varsigma}\int_0^{\left[\frac{x}{-\mathrm{a}(\varepsilon-1)}\right]}\left(1+\frac{\mathrm{a}(\varepsilon-1)\xi}{x}\right)^{\frac{\varsigma}{-(\varepsilon-1)}}f(\xi)d\xi\tag{1.15}$$

and

$$f(x)=\frac{c}{|x|^{1-v}}\left[1+\mathrm{a}(\varepsilon-1)|x|^{\delta}\right]^{\frac{\beta}{-(\varepsilon-1)}},\tag{1.16}$$

provided that $x\in(-\infty,\infty);\delta>0;\beta\geq 0;\varepsilon>1$ characterizes the extended generalized type-2 beta model for real x. The specific cases of the density function (1.16) includes the type-2 beta density function, the p density function, the student t density function. For $\varepsilon \to 1$, (1.13) diminishes to the Laplace integral transform.

In similar way if $\varepsilon = 0$, a $= 1$ and ς take the place of $\varsigma - 1$, then (1.13) diminishes to the familiar Riemann-Liouville (R-L) fractional integral operator $\mathcal{I}_{0+}^{\varsigma} f$ (e.g., [14]).

$$\left(\mathcal{P}_{0+}^{\varsigma-1,0} f\right)(x) = \Gamma(\varsigma)\left(\mathcal{I}_{0+}^{\varsigma} f\right)(x), \quad (\Re(\varsigma) > 1). \tag{1.17}$$

The PFI operator (1.13) leads to numerous interesting illustrations such as fractional calculus associated to probability density functions and their significant in statistical theory. Now a day, many researcher working on PFI formulae associated with various special function (see [1, 4]).

The generalized Wright hypergeometric function is given by the series [16]:

$$ {}_r\Psi_s(z) = {}_r\Psi_s\left[\begin{array}{c} (a_i,\, c_i)_{1,r}; \\ (b_j,\, d_j)_{1,s}; \end{array} z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{r} \Gamma(a_i + c_i k)\, z^k}{\prod_{j=1}^{s} \Gamma(b_j + d_j k)\, k!}. \tag{1.18}$$

where $a_i,\, b_j \in \mathbb{C}$ and $c_i,\, d_j \in \Re$, $(c_i,\, d_j \neq 0;\, i = 1, 2, \ldots, r; j = 1, 2, \ldots, s)$.

The present investigation for findings includes the idea of the Hadamard product that can be used to decomposing a newly emerged function into two known functions.

Presume that $f(z) := \sum_{q=0}^{\infty} a_q z^q$ and $g(z) := \sum_{q=0}^{\infty} b_q z^q$ are two power series whose convergence radii are given respectively by R_f and R_g. The power series is then described as Hadamard product (see [12]) as

$$(f * g)(z) := \sum_{q=0}^{\infty} a_q b_q z^q. \tag{1.19}$$

The radius of convergence R of the Hadamard product series $(f * g)(z)$ satisfies $R_f \cdot R_g \leq R$. If, in specific, up an entire function is defined by one of the power series, then the Hadamard product series also defines a whole function.

Here, our objective is to study the Hurwitz-Lerch zeta function and extended Wright-Bessel function, which is connected with the PFI operator (1.13) to present their integral formulas, results in terms of the generalized Wright hypergeometric function and Hadamard product of the Hurwitz-Lerch zeta function and extended Wright-Bessel function. Suitable connections of some particular cases are also pointed out.

2. Main results

In this section, we establish the PFI formula involving the Hurwitz-Lerch zeta function and extended Wright-Bessel function which is stated in Theorems 2.1, 2.5 and 2.7, 2.11 respectively.

Theorem 2.1. *For* $\sigma, \gamma, \vartheta, \upsilon, \varsigma, \xi \in \mathbb{C}$, $\Re(\sigma) > 0$, $\Re(\varsigma) > 0$; $\Re\left(\frac{\varsigma}{1-\varepsilon}\right) > -1$, $\varepsilon < 1$, $p \geq 0$, $\psi > 0$, $b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ *and* $w \in \Re$. *Then following formula holds true*:

$$\mathcal{P}_{0+}^{\varsigma,\varepsilon}[\tau^{\sigma-1}\Phi_{\gamma,\vartheta;\nu}^{\psi}\{w\tau,\xi,b;p\}](t) = \frac{t^{\varsigma+\sigma}\ \Gamma\left(1+\frac{\varsigma}{1-\varepsilon}\right)}{[\alpha(1-\varepsilon)]^{\sigma}} \times\ \Phi_{\gamma,\vartheta;\nu}^{\psi}\left[\frac{wt}{\alpha(1-\varepsilon)},\xi,b;p\right] * {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ \left(\sigma+\frac{\varsigma}{1-\varepsilon}+1,1\right);\end{array}\ \frac{wt}{\alpha(1-\varepsilon)}\right]. \tag{2.1}$$

Proof. We indicate the RHS of the equation (2.1) by $\Im_1$ and invoking equation (1.11) and (1.13), we have

$$\Im_1 = t^{\varsigma}\int_0^{\left[\frac{t}{\alpha(1-\varepsilon)}\right]}\tau^{\sigma-1}\left[1-\frac{\alpha(1-\varepsilon)\tau}{t}\right]^{\frac{\varsigma}{1-\varepsilon}}\sum_{k=0}^{\infty}\frac{(\gamma)_k\ \mathfrak{B}_p^{\psi}(\vartheta+k,\nu-\vartheta)}{\mathfrak{B}(\vartheta,\nu-\vartheta)\ \ k!}\frac{(w\tau)^k}{(k+b)^{\xi}}d\tau.$$

By changing the order of summation and integration, we get

$$\Im_1 = t^{\varsigma}\sum_{k=0}^{\infty}\frac{(\gamma)_k\mathfrak{B}_p^{\psi}(\vartheta+k,\nu-\vartheta)}{\mathfrak{B}(\vartheta,\nu-\vartheta)\ \ k!}\frac{w^k}{(k+b)^{\xi}} \times\int_0^{\left[\frac{t}{\alpha(1-\varepsilon)}\right]}\tau^{\sigma+k-1}\left[1-\frac{\alpha(1-\varepsilon)\tau}{t}\right]^{\frac{\varsigma}{1-\varepsilon}}d\tau.$$

Using the substitution $x = \frac{\alpha(1-\varepsilon)\tau}{t}$, we can change the limit of integration in to the following

$$\Im_1 = \frac{t^{\varsigma+\sigma}}{[\alpha(1-\varepsilon)]^{\sigma}}\sum_{k=0}^{\infty}\frac{(\gamma)_k\ \mathfrak{B}_p^{\psi}(\vartheta+k,\nu-\vartheta)}{\mathfrak{B}(\vartheta,\nu-\vartheta)\ \ k!}\frac{w^k}{(k+b)^{\xi}}\left(\frac{t}{\alpha(1-\varepsilon)}\right)^k \times\int_0^1 x^{\sigma+k-1}\left[1-x\right]^{\frac{\varsigma}{1-\varepsilon}}dx.$$

Now, by calculating the inner integral and use the beta function formula, we are seeing it.

$$\Im_1 = \frac{t^{\varsigma+\sigma}}{[\alpha(1-\varepsilon)]^{\sigma}}\sum_{k=0}^{\infty}\frac{(\gamma)_k\ \mathfrak{B}_p^{\psi}(\vartheta+k,\nu-\vartheta)}{\mathfrak{B}(\vartheta,\nu-\vartheta)\ (k+b)^{\xi}\ \ k!}\frac{\Gamma\left(1+\frac{\varsigma}{1-\varepsilon}\right)\Gamma(\sigma+k)}{\Gamma\left(\sigma+k+\frac{\varsigma}{1-\varepsilon}+1\right)}\left(\frac{wt}{\alpha(1-\varepsilon)}\right)^k.$$

By applying the Hadmard product (1.19) in above term, which in the view of (1.11) and (1.18), yields the desired formula (2.1). □

Here we present some corollaries as special cases of Theorem 2.1.

Corollary 2.2. *If $p = 0$ and $\psi = 1$ in Theorem 2.1, we have*

$$\begin{aligned}\mathcal{P}_{0+}^{\varsigma,\varepsilon}[\tau^{\sigma-1}\Phi_{\gamma,\vartheta;\nu}(w\tau,\xi,b)](t) &= \frac{t^{\varsigma+\sigma}\ \Gamma\left(1+\frac{\varsigma}{1-\varepsilon}\right)}{[\alpha(1-\varepsilon)]^{\sigma}} \\ \times\ \Phi_{\gamma,\vartheta;\nu}\left[\frac{wt}{\alpha(1-\varepsilon)},\xi,b\right] &* {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ \left(\sigma+\frac{\varsigma}{1-\varepsilon}+1,1\right);\end{array}\ \frac{wt}{\alpha(1-\varepsilon)}\right].\end{aligned} \tag{2.2}$$

Corollary 2.3. *If $\alpha = 1, \varepsilon = 0, \varsigma = \varsigma - 1$ in Theorem 2.1, we obtain*

$$\begin{aligned}&\mathcal{I}_{0+}^{\varsigma}[\tau^{\sigma-1}\Phi_{\gamma,\vartheta;\nu}^{\psi}\{w\tau,\xi,b;p\}](t)\\ &= t^{\varsigma+\sigma-1}\,\Phi_{\gamma,\vartheta;\nu}^{\psi}\,[wt,\xi,b;p] * {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ (\sigma+\varsigma,1)\,;\end{array}\ wt\right].\end{aligned} \tag{2.3}$$

Corollary 2.4. *If $p = 0$ and $\psi = 1$ in Corollary 2.3, we get*

$$\begin{aligned}&\mathcal{I}_{0+}^{\varsigma}[\tau^{\sigma-1}\Phi_{\gamma,\vartheta;\nu}\{w\tau,\xi,b\}](t)\\ &= t^{\varsigma+\sigma-1}\,\Phi_{\gamma,\vartheta;\nu}\,[wt,\xi,b] * {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ (\sigma+\varsigma,1)\,;\end{array}\ wt\right].\end{aligned} \tag{2.4}$$

Theorem 2.5. *For $\sigma,\gamma,\vartheta,v,\varsigma,\xi \in \mathbb{C}$, $\Re(\sigma) > 0$, $\Re(\varsigma) > 0$; $\Re\left(\frac{\varsigma}{1-\varepsilon}\right) > -1$, $\varepsilon > 1$, $p \geq 0$, $\psi > 0$, $b \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $w \in \Re$, we have*

$$\begin{aligned}\mathcal{P}_{0+}^{\varsigma,\varepsilon}[\tau^{\sigma-1}\Phi_{\gamma,\vartheta;\nu}^{\psi}\{w\tau,\xi,b;p\}](t) &= \frac{t^{\varsigma+\sigma}\ \Gamma\left(1-\frac{\varsigma}{\varepsilon-1}\right)}{[-\alpha(\varepsilon-1)]^{\sigma}} \\ \times\ \Phi_{\gamma,\vartheta;\nu}^{\psi}\left[\frac{wt}{-\alpha(\varepsilon-1)},\xi,b;p\right] &* {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ \left(\sigma-\frac{\varsigma}{\varepsilon-1}+1,1\right);\end{array}\ \frac{wt}{-\alpha(\varepsilon-1)}\right].\end{aligned} \tag{2.5}$$

Proof. We denote, for convenience, the RHS of the equation (2.5) by $\Im_2$, and invoking equation (1.11) and (1.15), we have

$$\Im_2 = t^{\varsigma}\int_0^{\left[\frac{t}{-\alpha(\varepsilon-1)}\right]}\tau^{\sigma-1}\left[1+\frac{\alpha(\varepsilon-1)\tau}{t}\right]^{\frac{\varsigma}{-(\varepsilon-1)}}\sum_{k=0}^{\infty}\frac{(\gamma)_k\mathfrak{B}_p^{\psi}(\vartheta+k,\nu-\vartheta)}{\mathfrak{B}(\vartheta,\nu-\vartheta)\ \ k!}\frac{(w\tau)^k}{(k+b)^{\xi}}d\tau.$$

Now changing the order of integration and summation, we get

$$\begin{aligned}\Im_2 = t^{\varsigma}&\sum_{k=0}^{\infty}\frac{(\gamma)_k\,\mathfrak{B}_p^{\psi}(\vartheta+k,\nu-\vartheta)}{\mathfrak{B}(\vartheta,\nu-\vartheta)\ \ k!}\frac{(w)^k}{(k+b)^{\xi}}\\ &\times\int_0^{\left[\frac{t}{-\alpha(\varepsilon-1)}\right]}\tau^{\sigma+k-1}\left[1+\frac{\alpha(\varepsilon-1)\tau}{t}\right]^{\frac{\varsigma}{-(\varepsilon-1)}}d\tau.\end{aligned}$$

By setting $y = \frac{-\alpha(\varepsilon-1)\tau}{t}$, we can change the limit of integration in to the following

$$\Im_2 = \frac{t^{\varsigma+\sigma}}{[-\alpha(\varepsilon-1)]^{\sigma}}\sum_{k=0}^{\infty}\frac{(\gamma)_k\mathfrak{B}_p^{\psi}(\vartheta+k,\nu-\vartheta)}{\mathfrak{B}(\vartheta,\nu-\vartheta)\ \ k!}\frac{w^k}{(k+b)^{\xi}}\left(\frac{t}{-\alpha(\varepsilon-1)}\right)^k$$

$$\times \int_0^1 y^{\sigma+k-1}\,[1-y]^{\frac{\varsigma}{-(\varepsilon-1)}}\,dy.$$

Now, by calculating the inner integral and use the beta function formula, we are seeing it.

$$\Im_2 = \frac{t^{\varsigma+\sigma}}{[-\alpha(\varepsilon-1)]^{\sigma}} \sum_{k=0}^{\infty} \frac{(\gamma)_k\ \mathfrak{B}_p^{\psi}(\vartheta+k,\nu-\vartheta)}{\mathfrak{B}(\vartheta,\nu-\vartheta)\ (k+b)^{\xi}\ \ k!} \frac{\Gamma\left(1-\frac{\varsigma}{\varepsilon-1}\right)\Gamma(\sigma+k)}{\Gamma\left(\sigma+k-\frac{\varsigma}{\varepsilon-1}+1\right)} \left(\frac{wt}{-\alpha(\varepsilon-1)}\right)^{k}.$$

By applying the Hadmard product (1.19) in above term, which in the view of (1.11) and (1.18), yields the desired formula (2.5). □

Corollary 2.6. *If $p=0$ and $\psi=1$ in Theorem 2.5, we have*

$$\begin{aligned}\mathcal{P}_{0+}^{\varsigma,\varepsilon}[\tau^{\sigma-1}\Phi_{\gamma,\vartheta;\nu}\{w\tau,\xi,b\}](t) &= \frac{t^{\varsigma+\sigma}\ \Gamma\left(1-\frac{\varsigma}{\varepsilon-1}\right)}{[-\alpha(\varepsilon-1)]^{\sigma}} \\ &\times\ \Phi_{\gamma,\vartheta;\nu}\left[\frac{wt}{-\alpha(\varepsilon-1)},\xi,b\right] * {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ \left(\sigma-\frac{\varsigma}{\varepsilon-1}+1,1\right);\end{array}\ \frac{wt}{-\alpha(\varepsilon-1)}\right].\end{aligned} \tag{2.6}$$

Theorem 2.7. *For $\sigma,\gamma,\eta,\delta,\mu,\varsigma\in\mathbb{C}$, $(\Re(p),\Re(\sigma),\Re(\varsigma),\Re(\gamma),\Re(\eta))>0$, $\Re(\delta)\geq 0$; $\Re\left(\frac{\varsigma}{1-\varepsilon}\right)>-1$, $\varepsilon<1$, $\Re(\mu)>-1$, $p\geq 0$ and $w\in\Re$, we have*

$$\begin{aligned}\mathcal{P}_{0+}^{\varsigma,\varepsilon}[\tau^{\sigma-1}J_{\mu,\lambda}^{\delta,\gamma,\eta}(w\tau,p)](t) &= \frac{t^{\varsigma+\sigma}\ \Gamma\left(1+\frac{\varsigma}{1-\varepsilon}\right)}{[\alpha(1-\varepsilon)]^{\sigma}} \\ &\times\ J_{\mu,\lambda}^{\delta,\gamma,\eta}\left[\frac{wt}{\alpha(1-\varepsilon)},p\right] * {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ \left(\sigma+\frac{\varsigma}{1-\varepsilon}+1,1\right);\end{array}\ \frac{-wt}{\alpha(1-\varepsilon)}\right].\end{aligned} \tag{2.7}$$

Proof. Taking right-hand side of equation (2.7) with denoted $\Im_3$, we have

$$\Im_3 = t^{\varsigma}\int_0^{\left[\frac{t}{\alpha(1-\varepsilon)}\right]} \tau^{\sigma-1}\left[1-\frac{\alpha(1-\varepsilon)\tau}{t}\right]^{\frac{\varsigma}{1-\varepsilon}} \sum_{k=0}^{\infty}\frac{\mathfrak{B}_p(\gamma+k\lambda,\eta-\gamma)(\eta)_{k\lambda}(-w\tau)^k}{\mathfrak{B}(\gamma,\eta-\gamma)\Gamma(\mu+\delta k+1)\ \ k!}d\tau.$$

By altering the order of summation and integration, we get

$$\begin{aligned}\Im_3 = t^{\varsigma}\sum_{k=0}^{\infty}&\frac{\mathfrak{B}_p(\gamma+k\lambda,\eta-\gamma)(\eta)_{k\lambda}(-w)^k}{\mathfrak{B}(\gamma,\eta-\gamma)\Gamma(\mu+\delta k+1)\ \ k!} \\ &\times\int_0^{\left[\frac{t}{\alpha(1-\varepsilon)}\right]}\tau^{\sigma+k-1}\left[1-\frac{\alpha(1-\varepsilon)\tau}{t}\right]^{\frac{\varsigma}{1-\varepsilon}}\ d\tau.\end{aligned}$$

Simplification of integral, we have

$$\Im_3 = \frac{t^{\varsigma+\sigma}}{[\alpha(1-\varepsilon)]^{\sigma}}\ \sum_{k=0}^{\infty}\frac{\mathfrak{B}_p(\gamma+k\lambda,\eta-\gamma)(\eta)_{k\lambda}}{\mathfrak{B}(\gamma,\eta-\gamma)\Gamma(\mu+\delta k+1)\ k!}$$

$$\times \frac{\Gamma(\sigma+k)\,\Gamma\left(\frac{\varsigma}{1-\varepsilon}+1\right)}{\Gamma\left(\sigma+k+\frac{\varsigma}{1-\varepsilon}+1\right)}\left(\frac{-wt}{\alpha(1-\varepsilon)}\right)^{k}.$$

By applying the Hadmard product (1.19) in above term, which in the view of (1.12) and (1.18), yields the desired formula (2.7). □

Here we mentioned some special cases as corollaries given below.

Corollary 2.8. *If $p = 0$ in Theorem 2.7, then it becomes in the following extended Wright-Bessel function as*

$$\mathcal{P}^{\varsigma,\varepsilon}_{0+}[\tau^{\sigma-1}J^{\delta,\gamma,\eta}_{\mu,\lambda}(w\tau)](t) = \frac{t^{\varsigma+\sigma}\,\Gamma\left(1+\frac{\varsigma}{1-\varepsilon}\right)}{[\alpha(1-\varepsilon)]^{\sigma}} \times J^{\delta,\gamma,\eta}_{\mu,\lambda}\left[\frac{wt}{\alpha(1-\varepsilon)}\right] * {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ \left(\sigma+\frac{\varsigma}{1-\varepsilon}+1,1\right);\end{array}\ \frac{-wt}{\alpha(1-\varepsilon)}\right]. \tag{2.8}$$

Corollary 2.9. *If $\alpha = 1, \varepsilon = 0, \varsigma = \varsigma - 1$ in Theorem 2.7, we obtain*

$$\mathcal{I}^{\varsigma}_{0+}[\tau^{\sigma-1}J^{\delta,\gamma,\eta}_{\mu,\lambda}(w\tau,p)](t) = t^{\varsigma+\sigma-1}\,J^{\delta,\gamma,\eta}_{\mu,\lambda}(wt,p) * {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ (\sigma+\varsigma,1);\end{array}\ -wt\right]. \tag{2.9}$$

Corollary 2.10. *If we put $p = 0$ in Corollary 2.9, we get*

$$\mathcal{I}^{\varsigma}_{0+}[\tau^{\sigma-1}J^{\delta,\gamma,\eta}_{\mu,\lambda}(w\tau)](t) = t^{\varsigma+\sigma-1}\,J^{\delta,\gamma,\eta}_{\mu,\lambda}(wt) * {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ (\sigma+\varsigma,1);\end{array}\ -wt\right]. \tag{2.10}$$

Theorem 2.11. *For $\sigma,\gamma,\eta,\delta,\mu,\varsigma \in \mathbb{C}$, $(\Re(p),\Re(\sigma),\Re(\varsigma),\Re(\gamma),\Re(\eta)) > 0$, $\Re(\delta) \geq 0$; $\Re\left(\frac{\varsigma}{1-\varepsilon}\right) > -1$, $\varepsilon > 1$, $\Re(\mu) > -1$, $p \geq 0$ and $w \in \Re$, we have*

$$\mathcal{P}^{\varsigma,\varepsilon}_{0+}[\tau^{\sigma-1}J^{\delta,\gamma,\eta}_{\mu,\lambda}(w\tau,p)](t) = \frac{t^{\varsigma+\sigma}\,\Gamma\left(1-\frac{\varsigma}{\varepsilon-1}\right)}{[-\alpha(\varepsilon-1)]^{\sigma}} \times J^{\delta,\gamma,\eta}_{\mu,\lambda}\left[\frac{wt}{-\alpha(\varepsilon-1)},p\right] * {}_2\Psi_1\left[\begin{array}{c}(\sigma,1),(1,1);\\ \left(\sigma-\frac{\varsigma}{\varepsilon-1}+1,1\right);\end{array}\ \frac{wt}{\alpha(\varepsilon-1)}\right]. \tag{2.11}$$

Proof. Taking right-hand side of equation (2.11) with denoted $\Im_4$, we have

$$\Im_4 = t^{\varsigma}\int_0^{\left[\frac{t}{-\alpha(\varepsilon-1)}\right]} \tau^{\sigma-1}\left[1+\frac{\alpha(\varepsilon-1)\tau}{t}\right]^{\frac{\varsigma}{-(\varepsilon-1)}} \times \sum_{k=0}^{\infty}\frac{\mathfrak{B}_p(\gamma+k\lambda,\eta-\gamma)(\eta)_{k\lambda}(-w\tau)^k}{\mathfrak{B}(\gamma,\eta-\gamma)\Gamma(\mu+\delta k+1)\ \ k!}d\tau.$$

By changing the order of summation and integration, we get

$$= \frac{t^{\varsigma+\sigma}}{[-\alpha(\varepsilon-1)]^{\sigma}} \sum_{k=0}^{\infty} \frac{\mathfrak{B}_p(\gamma+k\lambda,\eta-\gamma)(\eta)_{k\lambda}(-w)^k}{\mathfrak{B}(\gamma,\eta-\gamma)\Gamma(\mu+\delta k+1)\ \ k!} \left(\frac{t}{-\alpha(\varepsilon-1)}\right)^k \times \int_0^1 y^{\sigma+k-1}\,[1-y]^{\frac{\varsigma}{-(\varepsilon-1)}}\,dy.$$

Now, by calculating the inner integral and use the beta function formula, taking the Hadmard product (1.19), which in the interpretation of (1.12) and (1.18), gives the desired formula (2.11). □

Corollary 2.12. *If $p=0$ in Theorem 2.11, then it becomes in the following extended Wright-Bessel function as*

$$\mathcal{P}_{0+}^{\varsigma,\varepsilon}[\tau^{\sigma-1}J_{\mu,\lambda}^{\delta,\gamma,\eta}(w\tau)](t) = \frac{t^{\varsigma+\sigma}\ \Gamma\left(1-\frac{\varsigma}{\varepsilon-1}\right)}{[-\alpha(\varepsilon-1)]^{\sigma}} \times\ J_{\mu,\lambda}^{\delta,\gamma,\eta}\left[\frac{wt}{-\alpha(\varepsilon-1)}\right] * {}_2\Psi_1\left[\begin{matrix}(\sigma,1),(1,1);\\ \left(\sigma-\frac{\varsigma}{\varepsilon-1}+1,1\right);\end{matrix}\ \frac{wt}{\alpha(\varepsilon-1)}\right]. \tag{2.12}$$

3. Conclusion

In this investigation, we present the composition formulae for the pathway fractional integral operator associated with Hurwitz-Lerch zeta function and extended Wright-Bessel function. The reported findings are important and can contribute to give the number of other forms for pathway fractional integral operator associated with various kind of special functions.

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