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Where's the Procedural Fluency?: U.S. Fifth Graders' Demonstration of the Standard Multiplication Algorithm

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For elementary school children, learning the standard multiplication algorithm with accuracy, clarity, consistency, and efficiency is a daunting task. Nonetheless, what should be our expectation in procedural fluency, for example, in finding the product of 25 and 37 among fifth grade students? Collectively, has the mathematics education community emphasized the value of conceptual understanding to the detriment of procedural fluency? In addition to examining these questions, we survey multiplication algorithms throughout history and in textbooks and reconceptualize the standard multiplication algorithm by using a new tool called the Multiplication Aid Template.

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I. INTRODUCTION

At present, all modern societies have adopted the use of the Hindu-Arabic numerals and their use in the place-value, decimal number system. For example, travelers, throughout the world, will concur that Gate 123 at an airport represents "Gate One Hundred, Two Tens, and Three Ones." We could also venture and verify that the nearby gates might be Gate 121, Gate 122, Gate 124, and Gate 125. In short, we expect a systematized, *standard* approach to numbering so that the information is useful.

What constitutes a *standard* in a particular context? How do we unequivocally state a well-defined *standard* for all to embrace? In regard to mathematics education standards, in 1980, the National Council of Teachers of Mathematics (NCTM) published *An Agenda*

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for Action. In the subsequent years, the visionary document provided the impetus for *Curriculum and Evaluation Standards* (1989), *Professional Teaching Standards* (1991), and *Assessment Standards* (1995). In 2000, NCTM's *Principles and Standards for School Mathematics* further articulated:

> The **Principles** are statements reflecting basic precepts that are fundamental to a high-quality mathematics education. The document elaborates the underlying assumptions, values, and evidence on which these Principles are founded. The **Standards** are descriptions of what mathematics instruction should enable students to know and do. Together, the **Principles** and **Standards** constitute a vision to guide educators as they strive for the continual improvement of mathematics education in classrooms, schools, and educational systems. (NCTM, 2000, p. 2)

2010 ushered in the era of *The Common Core State Standards for Mathematics* (*CCSSM*). Educators, who work with preservice teachers in the United States, are familiar with the Council for the Accreditation of Educator Preparation (CAEP) Standards. Typically, organizations, such as the NCTM, have the purview to articulate subject-specific standards. Afterall, the NCTM is the largest mathematics education organization in North America. To actualize the focus, rigor, and cohesiveness of the *CCSSM*, the National Governors Association and Council of Chief State School Officers (NGA & CCSSO) provided both the funding and the mandate.

To counter the often stated "a mile wide and an inch deep mathematics curriculum," the mathematics education community at large has underscored the need to develop children's *conceptual understanding* of mathematics. Nonetheless, what should be our expectation in *procedural fluency*, for example, in finding the product of 25 and 37 among fifth grade students? Collectively, has the community emphasized the value of conceptual understanding to the detriment of procedural fluency? In this paper, we (1) survey multiplication algorithms throughout history and in textbooks, (2) gauge procedural fluency among U.S. fifth graders in multiplying 25 and 37, and (3) reconceptualize the standard multiplication algorithm.

II. MULTIPLICATION ALGORITHMS THROUGHOUT HISTORY AND IN TEXTBOOKS

In this section, we first survey several multiplication algorithms that were utilized throughout history. Then, we shift our focus toward how textbooks present the standard multiplication algorithm for elementary school students in the United States and South Korea. Finally, we examine how college textbooks elaborate on the standard multiplication algorithm for preservice teachers.

1. MULTIPLICATION ALGORITHMS THROUGHOUT HISTORY

In *Number Stories of Long Ago* (1919), David Eugene Smith documents the myriad multiplication algorithms throughout history. For instance, in Figure 1, Fibonacci (a.k.a. Leonardo of Pisa), in 1202, multiplied 49 and 8 in the following manner (Smith, 1919, p. 66):

$$\begin{array}{r} 392 \\ 8 \\ 49 \end{array}$$

Figure 1. Fibonacci's work on 49 × 8

Compared to a contemporary multiplication algorithm, we note the following differences:

1. The product (392) appears above the multiplicand (49) and the multiplier (8).
2. There is an omission of the horizontal line segment that separates the multiplicand and multiplier to the product.
3. There is an omission of a symbol, such as ×, to convey the multiplication operation.
4. Finally, there is no "work" to reveal the utilized algorithm. It is plausible that the multiplicand and multiplier are small enough to deduce the product mentally.

About three centuries later, Cuthbert Tonstall, a theologian and mathematician during the 16th-century England, provided the below work (Figure 2) in multiplying two multi-digit whole numbers (Smith, 1919, p. 67):

$$\begin{array}{r} 60503 \\ 4020 \\ \hline 00000 \\ 121006 \\ 00000 \\ 242012 \\ \hline 243222060 \end{array}$$

Figure 2. Tonstall's work on 60503 × 4020

The Tonstall’s algorithm addresses the three differences between the Fibonacci’s and a contemporary multiplication algorithm. Specifically, we observe: (1) The product appears below the multiplicand and multiplier; (2) the horizontal line segments separate the multiplicand and multiplier from the partial products as well as the product; (3) a reader can easily comprehend the detailed, partial-products work. Even though the context would have been obvious, we do not see a symbol for the multiplication operation.

In addition, we discern:

1. The value, 00000, represents the product of 60503 and 0. If traveling back in time were possible, one could surely inquire, “Why the need for so many zeros?”
2. It is notable that, in the year 1522, we see the detailed work showing the leftward shifts of the partial products to account for the place values of the multiplier (4020).

The below unique multiplication algorithms (Figures 3 through 5) appeared in 1478 (Smith, 1919, pp. 70-71). In Figure 3, to multiply 934 by 314, each digit of 314 shifts diagonally left and downward with the units value (4) appearing on the top right, followed by the tens value (10) on the second line, and finally, the hundreds value (300) on the third line. Similar to the Tonstall’s algorithm, we observe the leftward shifts of the partial products (3736, $934\underline{0}$, and $2802\underline{00}$) with the implied, underlined 0s. The work aligns quite nicely due to the aid of the “forward slashes” (⧄).

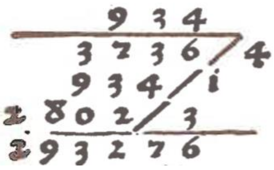


Figure 3. Fifteenth century algorithm on 934 × 314 (a)

In Figure 4 below, we observe the multiplication work comprising the same factors. Can you discern the algorithm? Pause for a few minutes and decipher the algorithm.

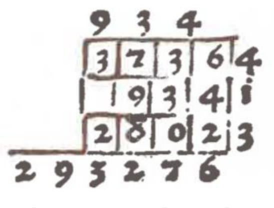


Figure 4. Fifteenth century algorithm on 934 × 314 (b)

Comparable to the above Figure 3 work, we discernibly notice the multiplicand (934), the multiplier (314), the modified partial products (3736, 934, and 2802), and the product (293276). In particular, we highlight the following: There is a precise, grid arrangement that corresponds to the modified partial products between the multiplicand (934) and each digit of the multiplier in reverse order (i.e., 4, 1, 3 instead of 3, 1, 4). Moreover, it appears the grid work of Figure 4, while aligned to the product (293276), is not aligned to the multiplicand (934). In other words, the digits, 9, 3, and 4, are not aligned with the digits, 3, 7, 3, and 6. At this point, one might still wonder, “How do we actually obtain the product from the displayed work?” The answer may lie in the next algorithm.

Figure 5 represents what is commonly known as the lattice method. First, we can clearly see the same multiplicand (934) horizontally on the top and the same multiplier (314) vertically to the right. Contrary to the two previous examples, 314 is written in the reverse order, i.e., the hundreds value (300) appears on the top, followed by the tens value (10) on the second line, and finally, the units value (4) on the third line. Next, this particular lattice (or the grid construction) has nine partial products. For example, since 4 × 3 = 12 and 3 × 3 = 9, these products are represented as 1 ⁄ 2 and 0 ⁄ 9 respectively. After determining the nine products, the remaining task is to combine the values to find the product. Adding the diagonal values between the forward slashes, we have the following:

6 (the first value from the bottom right)
7 (the sum of 2 + 1 + 4 is 7)
2 (the sum of 6 + 1 + 3 + 0 + 2 is 12; after carrying the 1 ten into the next diagonal group, the resulting value is 2)
3 (the sum of 3 + 9 + 0 + 9 + 1 is 22; after carrying the 2 tens into the next diagonal group and combining the previously carried value of 1, the resulting value is 3)
9 (the sum of 0 + 7 + 0 is 7; after combining the previously carried value of 2, the resulting value is 9)
2 (the last value from the top left)

Writing the above values in reverse order (or the values wrapped along the left side to the bottom), we obtain the product, 293276. It is worthwhile to devote some time to understand why the lattice method produced the correct answer.

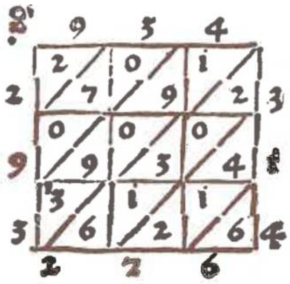


Figure 5. Fifteenth century algorithm on 934 × 314 (c)

Now, let us return to the Figure 4 work. Having examined the above lattice method, we could conjecture that to obtain the correct product, 293276, we begin with the upper-right value, 6. Next, from 6 and if we consider the set of diagonally right and downward values (3 and 4), we obtain the sum of 7. Continuing this scheme, we can deduce the rest of the values:

2 (the sum of 7 + 3 + 2 is 12; after carrying the 1 ten into the next diagonal group, the resulting value is 2)

3 (the sum of 3 + 9 + 0 is 12; after carrying the 1 ten into the next diagonal group and combining the previously carried value of 1, the resulting value is 3)

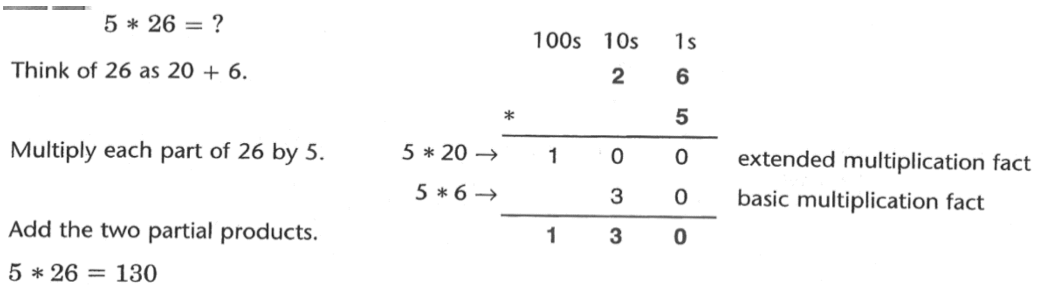
9 (after combining 8 and the previously carried value of 1, the resulting value is 9)

2 (the last value from the bottom left)

Again, writing the above values in reverse order, we obtain the product, 293276. We note that the articulated algorithm could have been the plausible approach back in the 15th century.

2. THE STANDARD MULTIPLICATION ALGORITHM IN U.S. TEXTBOOKS

Everyday Mathematics is one of the most popular elementary school mathematics textbook series in the United States. In 2012, *Everyday Mathematics* published its first Common Core State Standard Edition. In the 4th grade *Student Reference Book* (Bell et al., 2012a, p. 18), we see the following example (Figure 6):



5 * 26 = ?

Think of 26 as 20 + 6.

Multiply each part of 26 by 5.

Add the two partial products.

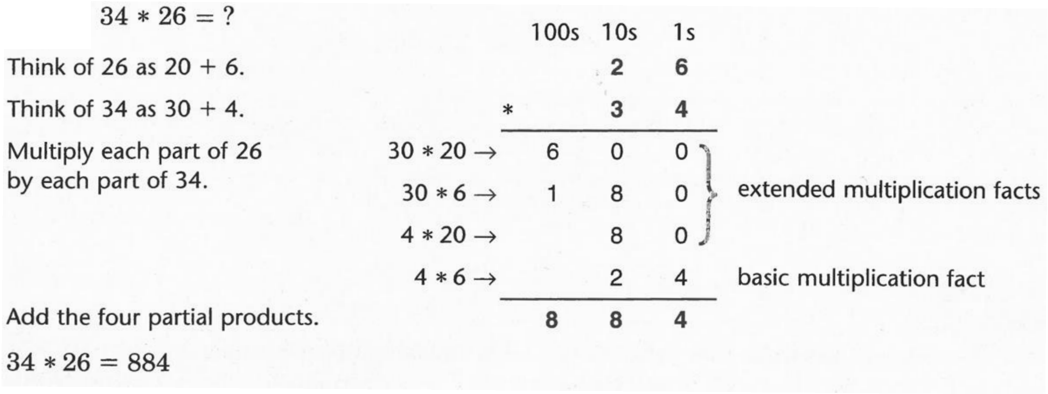
5 * 26 = 130

	100s	10s	1s	
		2	6	
*			5	
5 * 20 →	1	0	0	extended multiplication fact
5 * 6 →		3	0	basic multiplication fact
	1	3	0	

Figure 6. *Everyday Mathematics'* partial-products method (a)

Using the * symbol to represent the multiplication operation, the example provides the steps to finding the product of 26 and 5. After stating, "Think of 26 as 20 + 6," we see the partial-products method: First, obtain 100 from 5 * 20 and then, 30 from 5 * 6. The order in multiplying 5 by 20 first and then 5 by 6 appears to be atypical. Learning the partial-products method itself is valuable and also serves as a precursor to the standard multiplication algorithm. However, the order in obtaining the partial products (100 and 30) may not easily translate into the sequential steps of the standard multiplication algorithm.

Another example (Figure 7) in the *Student Reference Book* (Bell et al., 2012a, p. 18) appears to confirm that given 26 * 34, the objective is to multiply the higher value (or the tens value (30)) of the multiplier with 20 first and then 6. The remaining steps involve multiplying the units value (4) of the multiplier with 20 first and then 6. The textbook refers to the multiplication of two single-digit whole numbers as "basic multiplication fact," while the multiplications involving a number with powers of 10 as "extended multiplication facts."



34 * 26 = ?

Think of 26 as 20 + 6.

Think of 34 as 30 + 4.

Multiply each part of 26 by each part of 34.

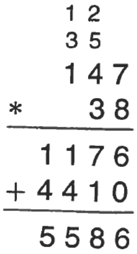
Add the four partial products.

34 * 26 = 884

	100s	10s	1s	
		2	6	
*		3	4	
30 * 20 →	6	0	0	extended multiplication facts
30 * 6 →	1	8	0	
4 * 20 →		8	0	
4 * 6 →		2	4	basic multiplication fact
	8	8	4	

Figure 7. *Everyday Mathematics'* partial-products method (b)

Regarding the standard multiplication algorithm, the *Teacher's Lesson Guide* (Bell et al., 2012b, p. A23) provides the below example (Figure 8):



```
   1 2
   3 5
   1 4 7
*    3 8
---------
  1 1 7 6
+ 4 4 1 0
---------
  5 5 8 6
```

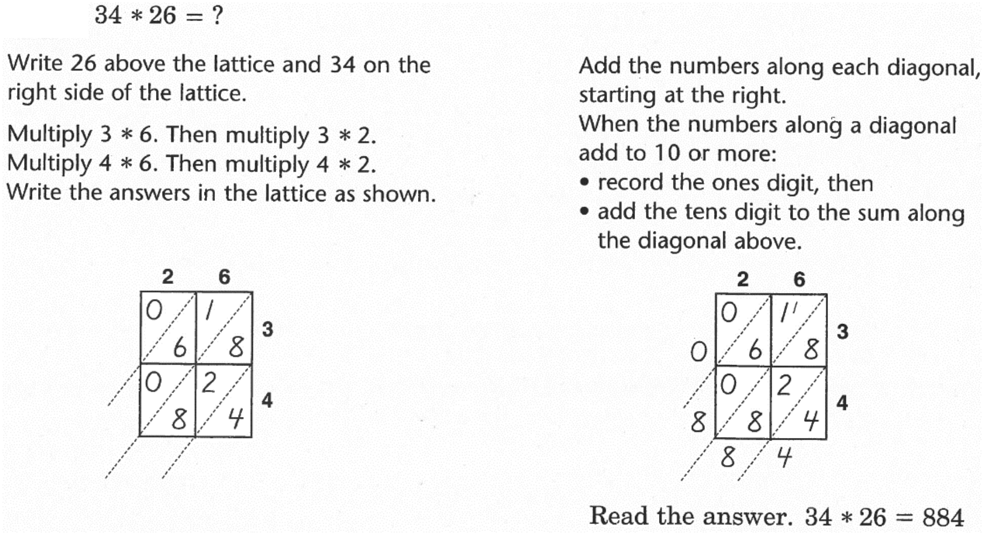
Figure 8. *Everyday Mathematics'* traditional multiplication algorithm

The *Teacher's Lesson Guide* refers to this algorithm as "U.S. traditional multiplication." To determine 147 * 30, the work comprises the carried value of 200 from 7 * 30. Yet, the "2" is placed in the tens place, and there lies an inherent contradiction. While this has been the "traditional" approach, it should no longer be the "standard" method. Incredulously, the authors (p. A23) acknowledge:

> Many people, when asked why the "2" carried from "3 * 7" is written in the 10s place, will explain that it stands for "2 tens." But this "2" really means "2 hundreds" since the "3" is really "3 tens." U.S. traditional multiplication is efficient—though not as efficient as a calculator—but it is not, despite its familiarity, conceptually transparent.

In short, they readily admit that placing the carried value, 200, in the tens place "is not ... conceptually transparent." However, there is no effort to rectify this. In the last section of this paper, we shall resolve the mathematical inconsistency.

The Figure 9 example is also found in the *Everyday Mathematics' Student Reference Book* (Bell et al., 2012a, p. 19). Employing the familiar lattice method to find the same product, the authors missed an opportunity: They could have guided students to make sense of the connection between the lattice and the partial-products methods. When students connect mathematical ideas, their understanding is deeper and more lasting, and they come to view mathematics as a coherent whole" (NCTM, 2000, p. 4). Isn't learning richer to have the conceptual understanding behind the procedures? Contrarily, we see the very prescriptive steps without any justification.



34 * 26 = ?

Write 26 above the lattice and 34 on the right side of the lattice.

Multiply 3 * 6. Then multiply 3 * 2.
Multiply 4 * 6. Then multiply 4 * 2.
Write the answers in the lattice as shown.

Add the numbers along each diagonal, starting at the right.
When the numbers along a diagonal add to 10 or more:

- record the ones digit, then
- add the tens digit to the sum along the diagonal above.

Read the answer. 34 * 26 = 884

Figure 9. *Everyday Mathematics'* lattice method

3. THE STANDARD MULTIPLICATION ALGORITHM IN SOUTH KOREAN TEXTBOOKS

Unlike the United States, South Korea has had the nationalized mathematics education standards since 1955. The First National Curriculum, from 1955 to 1963, had the theme, "Real Life," followed by "Mathematics Structure," "New Math," "Back to Basics," "Problem Solving," "Problem Solving and Informational Society," "Learner," etc. Until recently, the country had also developed and disseminated the nationwide-adopted mathematics textbooks for all elementary school students.

In *Mathematics in Grade 3, Volume 2: Teacher's Guide* (Korean Ministry of Education, 2014a, p. 15), we have the below example (Figure 10) demonstrating the standard multiplication algorithm to multiply 216 by 2:

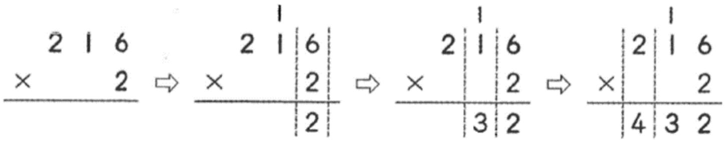


Figure 10. Korean textbook's standard multiplication algorithm (a)

Left to right, there is a relevant progression showing the multiplication between 2 and the factors, 6, 10, and 200. The vertical line segments, as well as the use of color numbers, offer an added focus. The algorithm clearly displays the carried value, 10, from 6 × 2.

Another work sample (Figure 11) from the *Teacher's Guide* (Korean Ministry of Education, 2014a, p. 144) represents an effective guided learning. First, students can inspect the partial-products and the standard multiplication algorithm work involving 128 × 2. Then, given a comparable expression, 317 × 2, they need to emulate the modeled work.

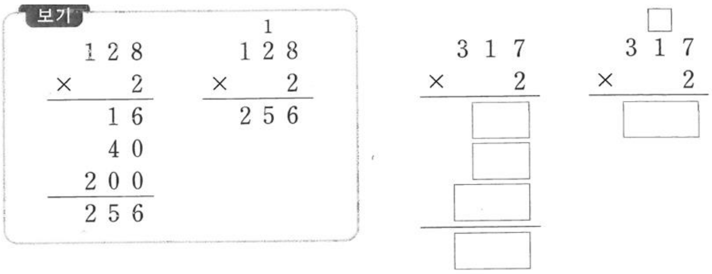


Figure 11. Korean textbook's standard multiplication algorithm (b)

Another example (Figure 12) from the *Teacher's Guide* (Korean Ministry of Education, 2014a, p. 23) provides the steps to finding the product of 52 × 13. The use of the vertical line segments to encourage students to align their work is applicable. It is quite odd that the authors felt a need to omit the "0" from the partial-product, 520, in stage three. Actually, we would prefer—and it makes more sense—to leave the value, 520, as is.

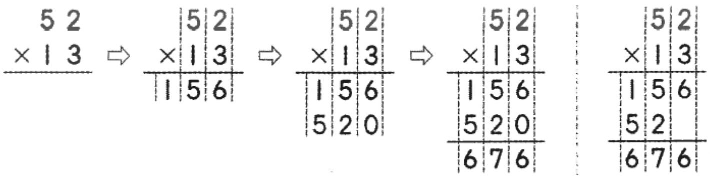


Figure 12. Korean textbook's standard multiplication algorithm (c)

Unlike the above Figure 10 work, our final example (Figure 13) from the *Teacher's Guide* (Korean Ministry of Education, 2014a, p. 25) omits any carried values. This may have been an oversight since it is too much to demand from third-grade students to compute these values mentally.

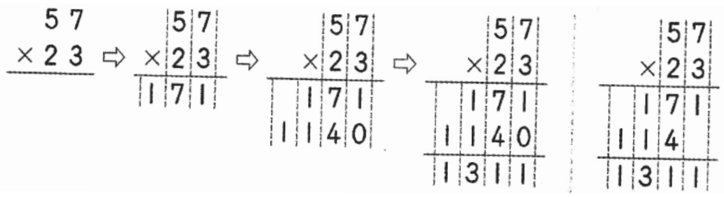


Figure 13. Korean textbook's standard multiplication algorithm (d)

The Figure 14 example is from *Mathematics in Grade 4, Volume 1: Teacher's Guide* (Korean Ministry of Education, 2014a, p. 196). Similar to the above Figure 12 example, there is no emphasis in showing the carried values. Most likely, the task to introduce and reinforce the standard multiplication algorithm is left for teachers.

$$\begin{array}{r} 4\,2\,3 \\ \times\quad 5\,6 \\ \hline 2\,5\,3\,8 \end{array} \;\Rightarrow\; \begin{array}{r} 4\,2\,3 \\ \times\quad 5\,6 \\ \hline 2\,5\,3\,8 \\ 2\,1\,1\,5 \end{array} \;\Rightarrow\; \begin{array}{r} 4\,2\,3 \\ \times\quad 5\,6 \\ \hline 2\,5\,3\,8 \\ 2\,1\,1\,5 \\ \hline 2\,3\,6\,8\,8 \end{array}$$

Figure 14. Korean textbook's standard multiplication algorithm (e)

On page 71 of the same *Teacher's Guide* (Korean Ministry of Education, 2014b), there is the detailed explanation (Figure 15) to find the product of 374 and 58 with the use of the lattice method. One the left, the table values correspond to 374 times 1 through 9 in the lattice form. On the right, there is an emphasis that the diagonal values correspond to the place values.

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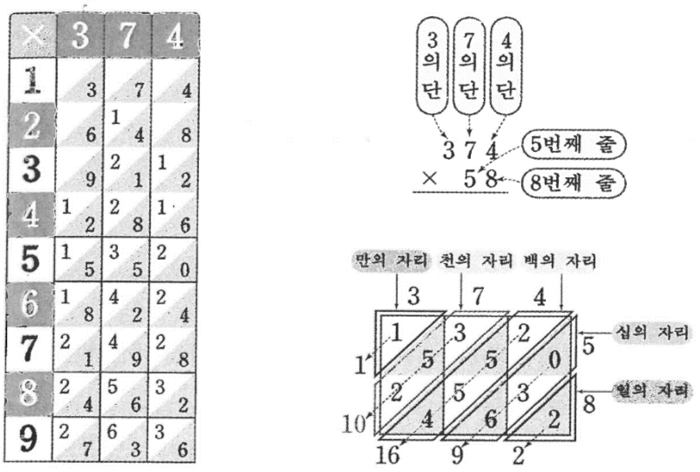


×	3	7	4
1	3	7	4
2	6	1 4	8
3	9	2 1	1 2
4	1 2	2 8	1 6
5	1 5	3 5	2 0
6	1 8	4 2	2 4
7	2 1	4 9	2 8
8	2 4	5 6	3 2
9	2 7	6 3	3 6

10000+10000+1600+90+2=21692

따라서 374×58=21692입니다.

Figure 15. Korean textbook's lattice method

While the U.S. lattice model (Figure 9) carries over tens from each place value, the South Korean lattice model does not carry over. Instead, the lattice model tabulates the partial products, for example, 10000, 10000, 1600, 90, and 2, from each place value.

4. THE STANDARD MULTIPLICATION ALGORITHM IN TEXTBOOKS FOR PRESERVICE TEACHERS

In this section, we examine how two textbooks for prospective elementary school teachers present the standard multiplication algorithm.

A Problem Solving Approach to Mathematics for Elementary School Teachers (2020) by Billstein, Boschmans, Libeskind, and Lott is a popular selection among the U.S. teacher education programs. In the below work sample (Figure 16), the authors demonstrate how to obtain the partial products of 23 × 14 with the use of the base-ten blocks (p. 153).

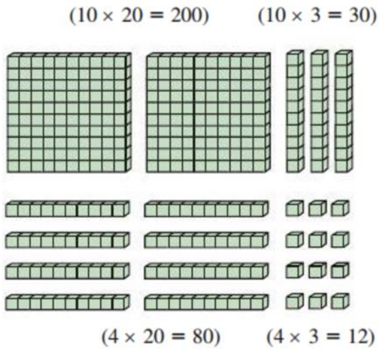


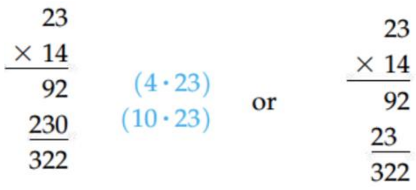
Figure 16. Billstein’s partial-products method (a)

In Figure 17, the authors demonstrate procedurally the partial-products method and de-emphasize the order of the partial products. If one were to adopt the order of 200, 30, 80, and 12, then this algorithm will not align well with the standard multiplication algorithm.

23			23	
× 14			× 14	
200	(10× 20)		12	(4 × 3)
30	(10× 3)	**or**	80	(4 × 20)
80	(4 × 20)		30	(10 × 3)
+ 12	(4 × 3)		+ 200	(10 × 20)
322	(200 + 30 + 80 + 12)		322	(12 + 80 + 30 + 200)

Figure 17. Billstein’s partial-products method (b)

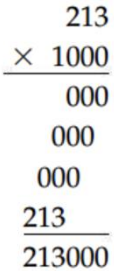
The authors further claim that the above work “leads to an algorithm for multiplication” (Figure 18, p. 153). It appears the standard multiplication algorithm is simply indicated to show how it is connected to the partial-products method. There is no effort to expound upon the steps to finding the product by using the standard multiplication algorithm. In particular, we do not see the carried value, 10, from 3 × 4. Moreover, there is no engagement with any complex examples that require more detailed work. Most likely, the task to expand upon the standard multiplication algorithm is left for professors.



$$\begin{array}{r} 23 \\ \times\ 14 \\ \hline 92 \\ \underline{230} \\ 322 \end{array} \quad \begin{array}{l} \\ \\ (4 \cdot 23) \\ (10 \cdot 23) \\ \\ \end{array} \quad \text{or} \quad \begin{array}{r} 23 \\ \times\ 14 \\ \hline 92 \\ \underline{23} \\ 322 \end{array}$$

Figure 18. Billstein's standard multiplication algorithm (a)

A notable example from the textbook includes the Tonstall's approach to finding the product (Figure 19, p. 153).



$$\begin{array}{r} 213 \\ \times\ 1000 \\ \hline 000 \\ 000 \\ 000 \\ \underline{213} \\ 213000 \end{array}$$

Figure 19. Billstein's standard multiplication algorithm (b)

Beckmann's *Mathematics for Elementary Teachers with Activities* (2018) may be the most used textbook to reconceptualize mathematical topics for prospective elementary school teachers. To help construct the reasoning behind why, for example, multiplying 34 by 10 will shift 3 tens to 3 hundreds and 4 ones to 4 tens, the author provides the below illustration (Figure 20, p. 150). This is comparable to the *Everyday Mathematics'* reference to the "extended multiplication facts." In addition, students need to recall this important concept when applying the standard multiplication algorithm.

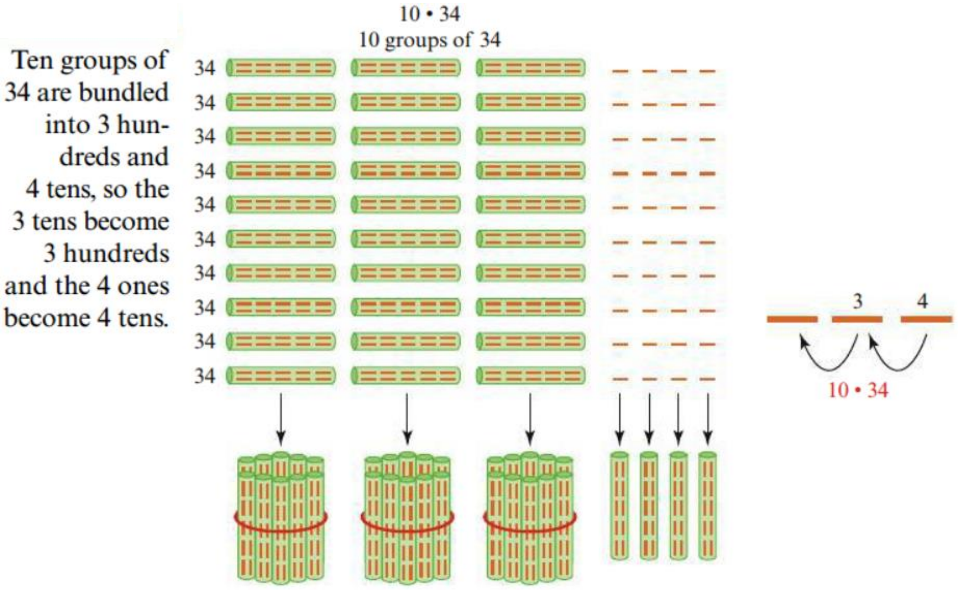
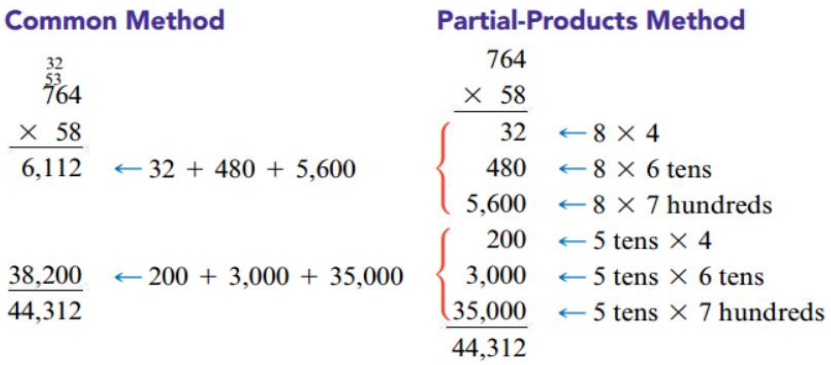


Figure 20. Beckmann's multiplication by 10 illustration

Similar to Billstein and his colleagues' textbook (2020) and the previously examined elementary school mathematics textbooks, Beckmann (2018) presents the partial-products method and connects it to the standard multiplication algorithm. Specifically, Figure 21 (p. 186) provides the six partial products from 764 × 58. By combining these products strategically, i.e., 32 + 480 + 5600 = 6122 and 200 + 3000 + 35000 = 38200, she demonstrates the connection between the algorithms. Beckmann prefers to call the standard multiplication algorithm the "Common Method."



Common Method

$$\begin{array}{r l} {}^{32}_{53}764 & \\ \times\ 58 & \\ \hline 6{,}112 & \leftarrow 32 + 480 + 5{,}600 \\ & \\ 38{,}200 & \leftarrow 200 + 3{,}000 + 35{,}000 \\ \hline 44{,}312 & \end{array}$$

Partial-Products Method

$$\begin{array}{r l} 764 & \\ \times\ 58 & \\ \hline 32 & \leftarrow 8 \times 4 \\ 480 & \leftarrow 8 \times 6 \text{ tens} \\ 5{,}600 & \leftarrow 8 \times 7 \text{ hundreds} \\ 200 & \leftarrow 5 \text{ tens} \times 4 \\ 3{,}000 & \leftarrow 5 \text{ tens} \times 6 \text{ tens} \\ 35{,}000 & \leftarrow 5 \text{ tens} \times 7 \text{ hundreds} \\ \hline 44{,}312 & \end{array}$$

Figure 21. Beckmann's linking between the partial-products method and the standard multiplication algorithm

We would like to point out that Beckmann displays the four carried values above and near the digits 7 and 6 of the multiplicand, 764. Without any further clarification, she appears to presume that preservice teachers will understand their purpose.

III. PROCEDURAL FLUENCY OF 25 × 37 AMONG U.S. FIFTH GRADERS

Having surveyed a historical progression of multiplication algorithms and a review of the algorithms in the textbooks, we now attend to elementary school students' understanding of the multiplication algorithm. Student work samples will provide us a more tangible picture of their procedural understanding and misunderstanding.

By 2014, the Commonwealth of Pennsylvania rebranded a version of the *CCSSM* and implemented *The Pennsylvania Core Standards in Mathematics (PCSM).* Like many states, the reason for removing the word, "Common," was to disassociate from the perceived "nationalized" mathematics curriculum and to retain the Commonwealth's identity and autonomy. Despite the different titles, the *PCSM* mirrors the *CCSSM.*

Recall the task: Find the answer to 25 × 37. Do the sampled Pennsylvania fifth graders demonstrate a great deal of congruence in their work? Does the overall work display a ubiquitous approach that could be deemed as the "standard" multiplication algorithm? Before answering these questions, we first examine the progression of the *CCSSM* in the context of developing students' procedural fluency and conceptual understanding of multiplication. Specifically, we underscore below the relevant standards and illustrative examples found in the grades 2 through 5.

Grade 2: Work with equal groups of objects to gain foundations for multiplication. (For example, "*by pairing objects or counting them by 2s; write an equation to express an even number as a sum of two equal addends*") (NGA & CCSSO, 2010, p. 19).

Grade 3: Represent and solve problems involving multiplication and division. (For example, "*describe a context in which a total number of objects can be expressed as 5 × 7.*")

Understand properties of multiplication and the relationship between multiplication and division. (For example, "*if 6 × 4 = 24 is known, then 4 × 6 = 24 is also known. (Commutative Property of Multiplication)*")

Multiply and divide within 100. (For example, "*knowing that 8 × 5 = 40, one knows 40 ÷ 5 = 8...* ***By the end of Grade 3, know from memory all products of two one-digit numbers*** [emphasis added]") (NGA & CCSSO, 2010, p. 23).

Grade 4: Use the four operations with whole numbers to solve problems. (For example, "*interpret 35 = 5 × 7 as a statement that 35 is 5 times as many as 7 and 7 times as many as 5*.")

Use place value understanding and properties of operations to perform multi-digit arithmetic. (For example, "*fluently add and subtract multi-digit whole numbers using the standard algorithm*") (NGA & CCSSO, 2010, p. 29).

Grade 5: **Understand the place value system.** (emphasis added) (For example, "*explain patterns in the number of zeros of the product when multiplying a number by powers of 10*.")

Perform operations with multi-digit whole numbers and with decimals to hundredths. (For example, "***fluently multiply multi-digit whole numbers using the standard algorithm*** [emphasis added]") (NGA & CCSSO, 2010, p. 35).

In the *CCSSM*, developing the concept of multiplication traverses four grades. As the above Clusters indicate, the Common Core underscores students to make sense of the concept of multiplication before finalizing with the "standard algorithm." In particular, having completed grade 3, students are expected "*to know from memory all products of two one-digit numbers*." Moreover, having completed grade 5, students are expected to "*fluently multiply multi-digit whole numbers using the standard algorithm*." (Note that we need to stress that the *CCSSM* is not a curriculum. The standards do not explicitly articulate *how* to teach and learn mathematics. In the United States, it can be daunting to formulate a set of cohesive standards at the state level, to interpret them accurately at the district level, to adopt the standards-aligned textbooks and resources at the school level, and to teach in the spirit of the standards at the classroom level.)

It is essential to emphasize that the data from the 12 fifth graders was collected toward the end of the academic year and come from one school district in western Pennsylvania. All students had had their mathematics learning articulated by the *PCSM*-aligned curriculum. For the last two years, the students had had the same mathematics teachers. More poignantly, the students had had very similar mathematical teaching and learning experiences.

Procedural fluency presupposes an accurate and efficient way to find the solution to 25 × 37. Among the twelve fifth grade students, we might expect one or two to make errors that result in wrong answers. Unexpectedly and dishearteningly, we discovered that five out of twelve students could not determine the product (925). First, we shall examine the detailed work relating to the correct solution. Then, our focus turns to the detailed work relating to the incorrect work. Throughout our analysis, we shall explore: Are there any patterns to the students' work?

Two students produced the below work (Figure 22):

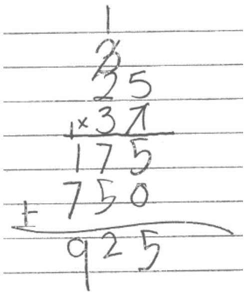


Figure 22. Students' correct work using the standard multiplication algorithm (a)

We see the product, 175, from 25 × 7. The students indicated carrying 30 from 5 × 7 by placing 3 above 2 in the tens place value. Unconventionally, the students crossed out 7, perhaps, to signify having used the factor. Interestingly, the students also crossed out the carried value, 30, because it had been used to find the product of 25 x 7. Next, we see the product, 750, from 25 × 30. Similarly, the students indicated carrying "10" from 5 × 3 by placing 1 above 3 in the tens place value. This multiplication algorithm also includes the use of the × symbol for the multiplication operation and the + symbol to denote the addition of the partial products (175 and 750). The students inserted the horizontal line segments to separate the multiplicand and multiplier to the partial products and also to the correct product (925). Finally, a "small 1," *above* the hundreds place value of 175, is noticeable and indicates carrying 100 from 70 + 50.

The below student work (Figure 23) is virtually identical to the above work in Figure 22. The only difference lies in the "small 1" appearing *next to* the hundreds place value of 175.

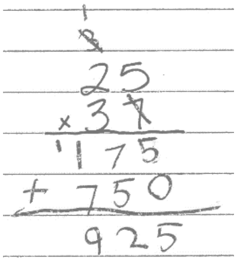


Figure 23. Student's correct work using the standard multiplication algorithm (b)

Compared to the previous work samples, the below work (Figure 24) provides a slight variation: 7 is not crossed out.

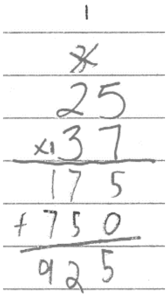


Figure 24. Student's correct work using the standard multiplication algorithm (c)

In the Figure 25 work, the student did not cross out 7 or 3. Furthermore, she circled the three carried values to highlight their significance. Two of the carried values are indicated as +1 and +3, not simply 1 and 3. One could also make a case that there is the + symbol accompanying the "small 1."

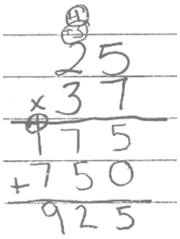


Figure 25. Student's correct work using the standard multiplication algorithm (d)

In the Figure 26 work, the student did not cross out 7 or 3. While adding the partial products of 175 and 750, he omitted the "small 1," the carried value.

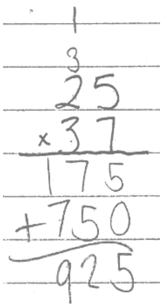


Figure 26. Student's correct work using the standard multiplication algorithm (e)

The Figure 27 work below rounds out the seven students' approaches toward the correct solution. This particular student decided to display only two of the three carried values and emphasized the "small 1" by circling it and attaching the + symbol. She may have erased the carried value, 3, and wrote the other carried value, 1, in its place. Like several other students, she did not cross out 7. Unlike the previous work samples, there is less care into aligning the digits to correspond to their place values. For example, the 7 tens and 5 tens from the partial products are not right below the 2 tens and 3 tens. A plausible explanation could be that while one of the authors wrote the original values (25 and 37) and their spacing, the student's own spacing is based on her typical work.

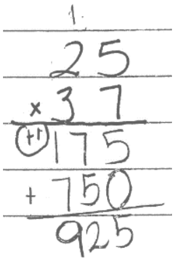


Figure 27. Student's correct work using the standard multiplication algorithm (f)

From the correct solutions, we noticed that the students flexibly accommodated the standard algorithm of multiplication. When students understand the concept behind the algorithm, they are able to customize the algorithm and efficiently use it.

To contrast with the above multiplication algorithms that produced the correct solution, we shall now examine five incorrect work samples. In particular, we want to make sense of the students' shortcomings and to explore some practical ways to overcome them.

The below Figure 28 work is essentially identical to the previous Figure 25 work. Despite getting the correct product, 35, from 5 × 7, the student incorrectly concluded 165 for 25 × 7. Hence, compared to 925, his answer differs by 10.

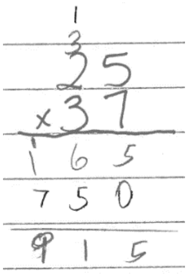


Figure 28. Student's incorrect work using the standard multiplication algorithm (a)

In Figure 29, the student erroneously equated 25 × 7 as 75. Upon a closer examination, readers will concur that the "small 1" represents the carried value of 100 from 70 + 50. Thus, instead of 175, by using 75 as the partial product of 25 × 7, her answer differs by 100.

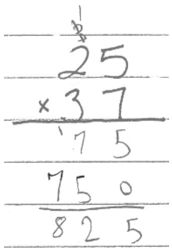


Figure 29. Student's incorrect work using the standard multiplication algorithm (b)

In the below Figure 30 work, the student incorrectly concluded that 25 × 30 as 650. This error was made despite his indication that 5 × 30 as 150 for we can clearly see that the carried value (100) is in the "hundreds" place at the top. We note that this is the first work sample that carried the value to the hundreds place. All prior work placed 1 in the tens place value. On the other hand, we cannot conclude with certainty whether the intention was to place the carried value in the hundreds place. The notation could simply indicate "the carried value."

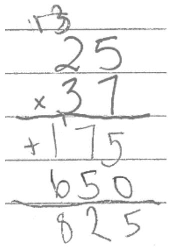


Figure 30. Student's incorrect work using the standard multiplication algorithm (c)

In Figure 31, comparable to the above Figure 30 work, the carried value, 100, from 5 × 30, is located closer to the hundreds place. Moreover, there are multiple errors in concluding that 25 × 7 as 160: It appears the student determined 5 × 7 as 30. This error is plausible if he counted by the multiples of 5, such as, 5, 10, 15, 20, 25, 30, but miscounting 7 times. Regardless of the underlying error, he has clearly not met the standard: *By the end of Grade 3, know from memory all products of two one-digit numbers* (NGA & CCSSO, 2010, p. 23). To compound his misunderstanding, he deduced 20 × 7 + 30 as 160. Furthermore, he stopped with the partial products and did not pursue the final answer. This work sample does not model the Common Core's Standards for Mathematical Practice 1:

Make sense of problems and *persevere in solving the problem* (emphasis added, NGA & CCSSO, 2010, p. 6).

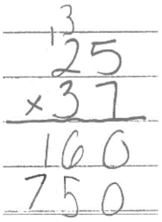


Figure 31. Student's incorrect work using the standard multiplication algorithm (d)

There is much to unpack in the final incorrect work sample (Figure 32). First, in the center, we recognize a common strategy in using a two-by-two grid work (so called "the area model for multiplication") to determine the partial products from 20 × 7, 5 × 7, 20 × 30, and 5 × 30. The student completed this portion rather satisfactorily. To the right, we see her effort to add the partial products (600, 150, 140, 35). While finding the correct sum would lead to the correct answer, she concluded 895. Most likely, she overlooked the value, 30.

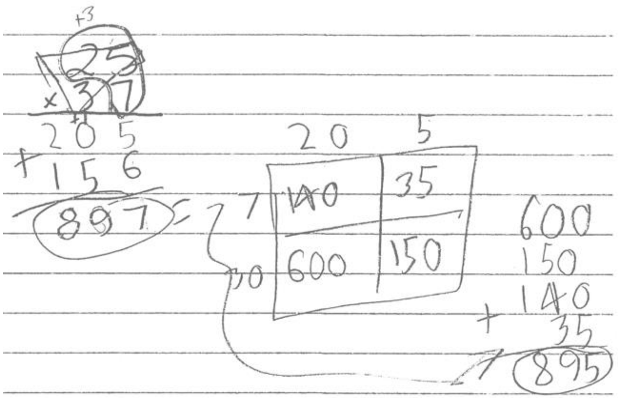


Figure 32. Student's incorrect work using the standard multiplication algorithm (e)

Next, disassociating from all preceding work, she provided additional work around the originally posed question. On the left, she indicated the groupings of (25 and 7) and (25 and 3). To find 25 × 7, she carried the correct value, 30, and yet, wrote down 205. This may be due to adding 140 (the product of 20 and 7), 30 (the carried value), and 35 (the product of 5 and 7). To find 25 × 3, she may have multiplied 5 and 3 to get 15, and then multiplied 2 and 3 to get 6. Rearranging these partial products in a creative manner, she wrote 156. Furthermore, we observe that she found the sum of 205 and 156 as 897. Upon a closer examination, we can see some indication of erased value, 461, that most likely corresponds to the sum value of 205 and 156. Recognizing the contradictory values between this sum and 895, she opted to copy—and yet miscopied—895 (the value on the right) as 897. We further conjecture that having tried to find the product in two different ways that resulted in two different answers, she placed her trust in the work that led to the value, 895. After all, she could have easily verified the four partial products. Along the way, she also dismissed the obvious contradiction: The sum of 205 and 156 is not equal to 897. Finally, she circled and equated both values, 895 and 897, as a flippant way to say, "I got the same answer." Regardless of the underlying errors, she has clearly not met the standard: *Fluently multiply multi-digit whole numbers using the standard algorithm*.

If she had understood the connections between the partial products obtained by using the algorithm and by using the area model, she would have verified that 25 × 7 (from the algorithm) equals 140 + 30 (from the area model) and 25 × 30 (from the algorithm) equals 600 + 150 (from the area model). As educators, we would all welcome an opportunity to interview the student in order to understand her thinking processes more completely.

The first three incorrect work samples, represented by Figures 28 through 30, involve "careless" errors. Nevertheless, we, the educators, are responsible to model and guide our students to minimize them. The latter two work samples (Figures 31 & 32) have major conceptual shortcomings.

Ideally, the teacher will need to work with the students individually to identify their misconceptions, to construct the correct meaning of multiplication, and to connect their conceptual understanding to the standard multiplication algorithm. This is a daunting task since as fifth graders, they have reinforced the incorrect reasoning for some years. Mathematics educators at all levels have to do better. A learning gap, such as this, will have long-term, negative implications.

In the next section, we introduce an effective and efficient way to teach and learn the multiplication algorithm. By emphasizing the place value, this envisioned approach may curtail operational errors in multiplication.

IV. ARTICULATING AN IDEAL STANDARD MULTIPLICATION ALGORITHM

It would be a challenging task to conceptualize a standard multiplication algorithm—the one that educators throughout the world would adopt. However, at the school level, at the district level, at the state level, or even at the country level, educators should articulate the algorithm explicitly and intentionally. Below, we explore a more conceptually based approach to teaching the standard multiplication algorithm.

In South Korea and the United States, children practice writing the basic *Hangul* characters (Figure 33) and the English alphabets (Figure 34) in blocks. Similar to riding a bicycle with training wheels, the blocks provide the guidance to emulate, refine, and perfect the respective cultures' written symbols. In time, the teacher will wean off the use of the blocks.

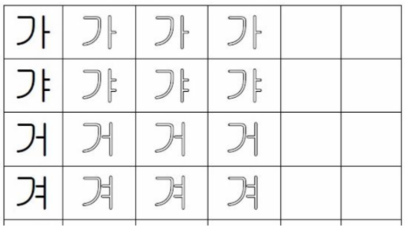


Figure 33. *Hangul* characters in blocks

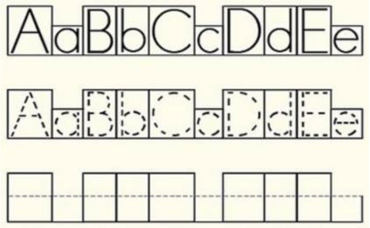
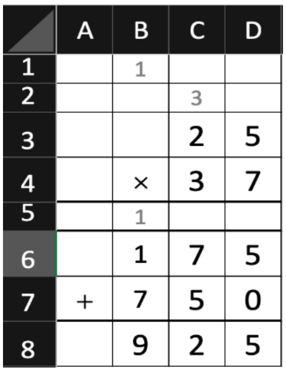


Figure 34. Alphabets in blocks

To foster students' understanding about numbers and their basic operations, we need to develop and reinforce the important concept of place value. Despite researchers and educators' efforts to document systematic ways to develop students' place value concept (e.g., Fuson, 1986; Fuson & Briars, 1990; Kamii, 1989), there is not a "standard" or an "optimized" way to teach place value in multiplication. Toward the envisioned standard multiplication algorithm, our first suggestion is to use the grids. We shall see that the below Figure 35 work in *Excel* provides more than just neat work. In detailed, comprehensive steps, we shall develop the algorithm by recalling the previously examined task: Find the answer to 25 × 37.



	A	B	C	D
1		1		
2			3	
3			2	5
4		×	3	7
5		1		
6		1	7	5
7	+	7	5	0
8		9	2	5

Figure 35. Standard multiplication algorithm work sample in grids

The figure includes the letters (A through D) to represent the columns and the numbers (1 through 8) to represent the rows. Using the grids, we know precisely that column D represents the ones place values, column C represents the tens place values, and column B represents the hundreds place values.

Next, in row 6, we see the familiar partial product, 175. To find this value, we use the carried value, 30. Identical to most of the student work samples, we place only 3 in C2 since it is in the tens place. Note that we decreased the size and lightened the shade of 3, as well as the other carried values, for contrast.

In determining the partial product, 750, we place the carried value, 100, in B1. First, unlike many student work samples, it is more appropriate to place 1 in column B than in column C since 1 actually represents one hundred. Second, we write any carried values from 25 × 7 in row 2 and any carried values from 25 × 30 in row 1—hence, the carried values, 30 and 100, are in C2 and B1, respectively. In short, rather than crossing out any carried values, like 3, students should be able to monitor their computations more efficiently and verify their overall work more accurately by using multiple rows.

This standard multiplication algorithm also uses the × symbol for the multiplication operation and the + symbol to denote the addition of the partial products. Moreover, a horizontal line segment separates the partial products to the product. Finally, another unique feature of the algorithm entails the use of the double horizontal line segments in row 5. Rather than encouraging students to place the "small 1" (the carried value of 100 from 70 + 50) in the vicinity of 1 in 175 (for example, see Figure 22), we designate the space to document and align their work. Similar to the other carried values in rows 1 and 2, we do not need much gap between the segments—just enough to denote the carried values when adding the partial products.

The below grid papers (Figure 36) have the sufficient space to show all work involving multiplications of up to two three-digit whole numbers. We shall call this tool, the Multiplication Aid Template (MAT).

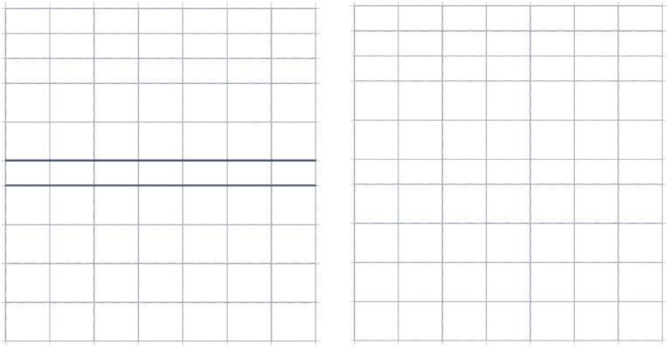


Figure 36. The multiplication aid template (MAT)

Initially, students would use the left grid paper to accentuate the double horizontal line segments and then transition to the right grid paper to provide all detailed work. Note that we can extend the MAT horizontally or vertically as needed.

In Figure 37, using the MAT, we carried out the following steps to find the product of 72 × 58.

1. Starting from the rightmost column (G), copy 72, 58, and × above the double horizontal line segments in row 6. Initially, the segments should be three cells long (E to G), and teachers need to model drawing these segments along the guidelines.
2. Computing 72 × 8, clearly indicate the carried value, 10, in F3.
3. Computing 72 × 50, clearly indicate the carried value, 100, in E2.
4. Finally, indicate the sum of the partial products (576 and 3600) with the + symbol, extend the double horizontal line segments into one more cell (D) to include the carried value, 1000, in D6 and conclude the product is 4176 with another appropriately long horizontal line segment.

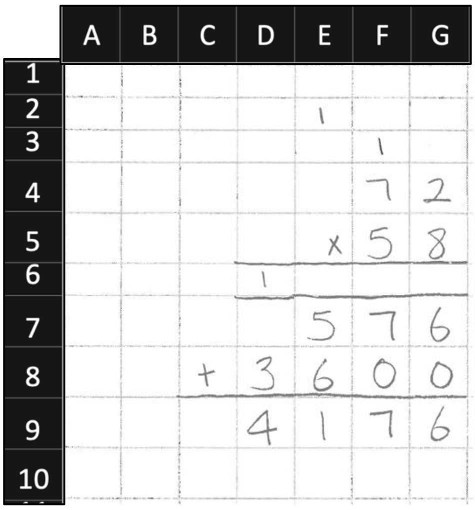


Figure 37. Standard multiplication algorithm work sample using the MAT (a)

Figure 38 involve more complex multiplication work samples with the MAT. Eventually, the teacher will wean off the use of the MAT entirely.

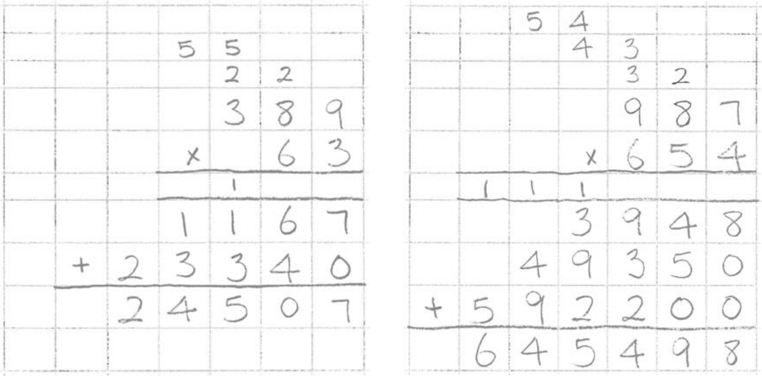


Figure 38. Standard multiplication algorithm work samples using the MAT (b)

V. CONCLUDING REMARKS

In recent decades, mathematics educators and researchers have focused more on conceptual understanding. To be clear, we are not de-emphasizing the importance of conceptual understanding in this paper. Rather, we underscore the concepts within the multiplication algorithm that has been regarded solely as a procedure.

Multiplication algorithms should promote computational accuracy and efficiency. The historical examples indicate the varied ways that people have pondered about this. In the students' work samples, one glaring shortcoming we highlight is the prevalent misplacements of the carried values. We advocate using the Multiplication Aid Template (MAT) to indicate any carried value by aligning to its place value. In fact, the MAT provides the needed template to align all numbers according to their place values. A simple and yet useful tool, the MAT facilitates students to produce work based on their justifiable reasoning. In short, employing the standard multiplication algorithm, we need to emphasize that procedural fluency involves much more than just simple recall of multiplication facts. To make sense of the algorithm, students need to connect the procedures with their conceptual reasoning.

Our examination of the school mathematics textbooks from the United States and South Korea, as well as the textbooks for preservice teachers, conveys that there is a lack of explicit instructions in teaching the standard multiplication algorithm. Not only is there no broadly accepted "standard," but it is left for the classroom teachers and university professors to devise their own algorithms. This approach results in lack of clarity and consistency. The MAT may be one way to overcome this issue.

To have synergy for change, educators need time and care for dialogues. This could be a focused professional development or a discussion in the hallway, but our profession demands that we exchange our ideas to formulate improved ideals in how we teach and how our children learn mathematics. It is our hope that this paper becomes an impetus for productive dialogues. Notably, we suggest further study on teachers' and students' effective use of the MAT tool while teaching and learning the standard multiplication algorithm.

REFERENCES

Beckmann, S. (2018). *Mathematics for elementary teachers with activities (5th ed.).* Pearson Education, Inc.

Bell, M., Bell, J., Bretzlauf, J., Dillard, Flanders, J., Hartfield, R., Issacs, A., Leslie, D., McBride, J., Pitvorec, K., & Saecker, P. (2012a). *Everyday mathematics: Student reference book (grade 4).* Chicago: The McGraw-Hill Companies, Inc.

Bell, M., Bretzlauf, J., Dillard, A., Hartfield, R., Issacs, A., Maxcy, R., McBride, J., Pitvorec, K., Saecker, P., Balfanz, R., Carroll, W., & Sconiers, S. (2012b). *Everyday mathematics: Teacher's lesson guide: vol. 1 (grade 4).* Chicago: The McGraw-Hill Companies, Inc.

Billstein, R., Boschmans, B., Libeskind, S., & Lott, J. (2020). *A Problem solving approach to mathematics for elementary school teachers* (13th ed.) Pearson Education, Inc.

Fuson, K. C. (1986). Roles of representation and verbalization in the teaching of multi-digit addition and subtraction. *European Journal of Psychology of Education, 1*(2), 35-56.

Fuson, K. C., & Briars, D. J. (1990). Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. *Journal for Research in Mathematics Education, 21*(3), 180–206.

Kamii, C. (1989). *Young children continue to reinvent arithmetic, 2nd grade. New York: Teachers College Press.* [This book has been replaced by a 2nd edition published in 2004.]

Korean Ministry of Education (2014a). *Mathematics in Grade 3, Volume 2: Teacher's Guide.*

Korean Ministry of Education (2014b). *Mathematics in Grade 4, Volume 1: Teacher's Guide.*

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional teaching standards.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Executive Summary Principles and Standards for School Mathematics.* Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for mathematics initiatives.* Washington, DC: Authors.

Smith, D. E. (1919). *Number stories of long ago.* Boston: Ginn and Company.