Nonlinear Functional Analysis and Applications
Vol. 26, No. 4 (2021), pp. 663-684
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.04.02
http://nfaa.kyungnam.ac.kr/journal-nfaa

ON GENERALIZED (α, β)-NONEXPANSIVE MAPPINGS IN BANACH SPACES WITH APPLICATIONS

F. Akutsah[1] and O. K. Narain[2]

[1]School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal, Durban, South Africa
e-mail: 216040405@stu.ukzn.ac.za, akutsah@gmail.com

[2]School of Mathematics, Statistics and Computer Science
University of KwaZulu-Natal, Durban, South Africa
e-mail: naraino@ukzn.ac.za

Abstract. In this paper, we present some fixed point results for a general class of nonexpansive mappings in the framework of Banach space and also proposed a new iterative scheme for approximating the fixed point of this class of mappings in the frame work of uniformly convex Banach spaces. Furthermore, we establish some basic properties and convergence results for our new class of mappings in uniformly convex Banach spaces. Finally, we present an application to nonlinear integral equation and also, a numerical example to illustrate our main result and then display the efficiency of the proposed algorithm compared to different iterative algorithms in the literature with different choices of parameters and initial guesses. The results obtained in this paper improve, extend and unify some related results in the literature.

1. Introduction

The concept of fixed points theory and its application has proven to be a vital tool in the study of nonlinear functional analysis and it is a very useful tool in establishing the existence and uniqueness theorems for nonlinear

[0]Received January 4, 2021. Revised March 29, 2021. Accepted April 1, 2021.
[0]2010 Mathematics Subject Classification: 47H09, 47H10, 37C25.
[0]Keywords: Generalized (α, β)-nonexpansive type 1 mapping, generalized (α, β)-nonexpansive type 2 mapping, fixed point, iterative scheme, strong and weak convergence theorems.
[0]Corresponding author: F. Akutsah(216040405@stu.ukzn.ac.za).

ordinary, partial and random differential and integral equations in different abstract spaces. Due to its applicability, authors generalize the well celebrated Banach contraction theorem [2] by establishing fixed point results for nonlinear mappings which are more general than the Banach contraction. We recall the following. Let C be a nonempty subset of a Banach space X and $T : C \to C$ a self-mapping. A point $x \in X$ is said to be a fixed point of T if $Tx = x$.

Definition 1.1. ([5, 9, 10, 16, 18, 19]) A mapping $T : C \to C$ is said to be

(1) nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;
(2) mean nonexpansive, if there exist $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1$ such that $\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$, for all $x, y \in C$;
(3) satisfy condition (C), if $\frac{1}{2}\|Tx-x\| \leq \|x-y\| \Rightarrow \|Tx-Ty\| \leq \|x-y\|$; for all $x, y \in C$;
(4) satisfy condition (C_λ), if $\lambda\|Tx-x\| \leq \|x-y\| \Rightarrow \|Tx-Ty\| \leq \|x-y\|$, for all $x, y \in C$;
(5) generalized mean nonexpansive mapping if there exist $\alpha, \beta, \lambda \in [0, 1)$, with $\alpha + \beta < 1$ such that for all $x, y \in C$, $\lambda\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$;
(6) α-nonexpansive mapping if there exists $\alpha < 1$ such that for all $x, y \in C$, $\|Tx - Ty\|^2 \leq \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2$;
(7) quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T.

It is worth mentioning that nonexpansive mappings are continuous on their domains but mean nonexpansive, generalized mean nonexpansive, mappings satisfying condition (C), condition (C_λ) need not be continuous. Due to this fact, these mappings are more fascinating and applicable compare to nonexpansive mappings.

Question 1: Now a natural question that arises is, does a class of mapping exist, that contains mean nonexpansive, generalized mean nonexpansive, mappings satisfying condition (C), condition (C_λ), α-nonexpansive mappings and other nonexpansive type mappings are in existence in the literature?

In 1965, Browder [3] proved that the class of nonexpansive self mappings on a closed and bounded subset of a uniformly convex Banach space has a fixed point. Thereafter, researchers introduced different iterative schemes to approximate fixed points of nonlinear mappings in different abstract spaces. In this area of research, developing a faster and more efficient iterative algorithms for approximating the fixed points of nonlinear mappings still remain an open question and active area of research.

In 2011, Phuengrattana and Suantai [13] introduced SP-iterative process, as follows:

Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = (1-\gamma_n)y_n + \gamma_n T y_n, \ n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}, \{\beta\}$ and $\{\gamma\}$ are sequences in $[0, 1]$. They proved that their iterative process converges faster than all of Picard, Mann [8], Ishikawa [7], Noor [11], Abass et al. [1], processes and some other existing ones in literature.

In 2020, Chuadchawna et al. in [4] introduced an iterative process called generalized M-iteration in the framework of hyperbolic spaces. For the sake of completeness, we give the corresponding definition of generalized M-iteration in the frame work of normed space as follows:

Let C be a convex subset of a normed space E and $T : C \to C$ be any nonlinear mapping. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = \beta_n z_n + (1-\beta_n) T z_n, \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n) T y_n, \ n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. They established some fixed point results in the framework of hyperbolic spaces. They also stated it clearly that for $\beta_n = \gamma_n = 0$, then iterative process (1.2) becomes M-iteration [17]. More so, they claim the the generalized M-iteration converges faster than the M-iteration introduced in [17] and they gave a numerical example to justify this claim.

Remark 1.2. We note that if $\alpha = \beta_n = \gamma_n = \frac{1}{2}$, then the iterative processes (1.2) and (1.1) are the same.

Question 2: Now a natural question arises that can we introduce an iterative algorithm that converges faster than (1.1), (1.2) and a host of other iterative algorithms in the literature?

Motivated by the above research work and the ongoing research in this direction, we provide an affirmative answer to the above questions raised, in this work by introducing a new class of mapping, namely, generalized (α, β)-nonexpansive mappings type and a new iterative scheme whose rate of convergence is faster than existing iterative algorithms in the literature. In addition, we establish convergence results for these proposed iterative algorithms. Finally, we apply our result to an integral equations. The results obtained in this paper improved, extend and unify some related results in the literature.

2. Preliminaries

We give some definitions and results that will be used in the sequel.

Let X be a Banach space with dimension greater than or equal to 2. The function $\delta_X(\epsilon) : (0,2] \to [0,1]$ defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1; \|y\| = 1, \epsilon = \|x-y\|\right\}$$

is called the modulus of convexity of X. If $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$, then X is called uniformly convex. Let X be a Banach space, X^* its dual and $S(X) = \{x \in X : \|x\| = 1\}$. We have that the value of $f \in X^*$ at $x \in X$ is defined by $\langle x, f\rangle$.

Definition 2.1. ([6], [16])

(1) The multivalued mapping $J : X \to 2^{X^*}$ defined by

$$J(x) = \left\{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\right\}$$

is called the normalized duality mapping.

(2) A Banach space X is smooth if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in S(X)$. In this case, the norm of X is called Gateaux differentiable. It is known that J is single valued if X is smooth.

(3) A Banach space X is Frechet differentiable norm, if for each $x \in S(X)$ the limit above exists and is attained uniformly for $y \in S(X)$. In this case, we have that for all $x, h \in X$,

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x+h\|^2 \leq \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|),$$

where $J(x)$ is the Frechet derivative of the functional $\frac{1}{2}\|\cdot\|$ at $x \in X$ and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\downarrow 0} \frac{b(t)}{t} = 0$.

(4) A Banach space X is said to have Opial property [12] if for every weakly convergent sequence $\{x_n\}$ in X with weak limit y, we have

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - z\|, \ \ \forall z \in X$$

with $y \neq z$.

(5) Let C be a closed convex and bounded subset of X and $T : C \to C$ be a nonexpansive mapping. Then there exists a sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Such $\{x_n\}$ is called an almost fixed point sequence for T.

Definition 2.2. ([5], [16]) Let C be a nonempty subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. For all $x, y \in X$.

(1) An asymptotic radius of $\{x_n\}$ at x is defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|;$$

(2) An asymptotic radius of $\{x_n\}$ relative to C is defined by

$$r(C, \{x_n\}) = \inf\{r, (x, \{x_n\}) : x \in C\};$$

(3) An asymptotic center of $\{x_n\}$ relative to C is defined by

$$A(C, \{x_n\}) = \{r(x, \{x_n\}) = r(C, \{x_n\}) : x \in C\}.$$

We note that $A(C, \{x_n\})$ is nonempty and more so, if X is uniformly convex, then $A(C, \{x_n\})$ has exactly one point (see [6]).

Lemma 2.3. ([14]) *Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq c$, $\limsup_{n\to\infty} \|y_n\| \leq c$ and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = c$ holds for some $c \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 2.4. ([15]) Let C be a subset of a normed space X. A mapping $T : C \to C$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ and that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T))$ denotes distance from x to $F(T)$.

3. Main results

3.1. **Generalized (α, β)-Nonexpansive Mappings.** In this section, we introduce the notion of generalized (α, β)-nonexpansive mappings and give some basic properties for this class of mappings.

Definition 3.1. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be generalized (α, β)-nonexpansive type 1 if there exist $\alpha, \beta, \lambda \in [0, 1)$, with $\alpha \leq \beta$ and $\alpha + \beta < 1$ such that for all $x, y \in C$, $\lambda\|Tx - x\| \leq \|x - y\|$, then

$$\|Tx - Ty\| \leq \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|.$$

Remark 3.2. It is easy to see that the following statements are true.

(1) If $\alpha = \beta = 0$ and $\lambda = \frac{1}{2}$, then the generalized (α, β)-nonexpansive type 1 mapping satisfying the condition (C).
(2) If $\alpha = \beta = 0$ and $\lambda \in [0, 1)$, then the generalized (α, β)-nonexpansive type 1 mapping satisfying condition (C_λ).

Definition 3.3. Let C be a nonempty subset of a Banach space X. A mapping $T : C \to C$ is said to be generalized (α, β)-nonexpansive type 2 if there exist $\alpha, \beta, \lambda \in [0, 1)$, with $\alpha + \beta < 1$ such that for all $x, y \in C$, $\lambda\|Tx - x\| \leq \|x - y\|$ then

$$\|Tx - Ty\| \leq \max\left\{P(x, y), Q(x, y)\right\}, \tag{3.1}$$

where

$$P(x, y) = \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|$$

and

$$Q(x, y) = \alpha\|x - Tx\| + \beta\|y - Ty\| + (1 - (\alpha + \beta))\|x - y\|.$$

Proposition 3.4. *We know that the following statements from the definitions.*

(1) *Every nonexpansive mapping is a generalized (α, β)-nonexpansive type 1 mapping.*
(2) *Every mean nonexpansive mapping is a generalized (α, β)-nonexpansive type 1 mapping.*
(3) *All mappings satisfying condition (C) is an (α, β)-nonexpansive type 1 mapping.*
(4) *All mappings satisfying condition (C_λ) is an (α, β)-nonexpansive type 1 mapping.*

The following example shows that the converse of these statements are not always true.

Example 3.5. Let $C = \{(0, 0), (1, 0), (3, 0)\}$ be a subset of $\mathbb{R}^2$ with norm $\|\cdot\|$ on C defined $\|(x_1, x_2)\| = |x_1| + |x_2|$. Then $(C, \|\cdot\|)$ is a Banach space. Define a mapping $T : C \to C$ by

$$T(x) = \begin{cases} (0, 0), & \text{if } x \in \{(0, 0), (1, 0)\}, \\ (1, 0), & \text{if } x = (3, 0). \end{cases} \tag{3.2}$$

For $\lambda = \frac{1}{10}, \alpha = \frac{1}{2}$, and $\beta = \frac{1}{3}$, we consider the following cases.

Case I: For $x = (0, 0)$ and $y = (0, 0)$. It is easy to see that T is a generalized $(\frac{1}{2}, \frac{1}{3})$-nonexpansive type 1 mapping.

Case II a: For $x = (0, 0)$ and $y = (1, 0)$. We have that

$$\frac{1}{10}\|(0, 0) - (0, 0)\| = 0 < 1 = \|x - y\|$$

and

$$\|Tx - Ty\| = 0 < \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.$$

Case II b:

For $x = (1,0)$ and $y = (0,0)$. We have that

$$\frac{1}{10}\|(0,0) - (0,0)\| = 0 < 1 = \|x - y\|$$

and

$$\|Tx - Ty\| = 0 < \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.$$

Case III a: For $x = (0,0)$ and $y = (3,0)$. We have that

$$\frac{1}{10}\|(0,0) - (0,0)\| = 0 < 3 = \|x - y\|$$

and

$$\begin{aligned}\|Tx - Ty\| &= |(0,0) - (1,0)| = 1\\ &< \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.\end{aligned}$$

Case III b:

For $x = (3,0)$ and $y = (0,0)$. We have that

$$\frac{1}{10}\|(3,0) - (1,0)\| = \frac{1}{5} < 3 = \|x - y\|$$

and

$$\begin{aligned}\|Tx - Ty\| &= |(1,0) - (0,0)| = 1\\ &< \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.\end{aligned}$$

Case IV a: For $x = (1,0)$ and $y = (3,0)$. We have that

$$\frac{1}{10}\|(1,0) - (0,0)\| = \frac{1}{10} < 2 = \|x - y\|,$$

and

$$\begin{aligned}\|Tx - Ty\| &= |(0,0) - (1,0)| = 1\\ &< \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.\end{aligned}$$

Case IV b:

For $x = (3,0)$ and $y = (1,0)$. We have that

$$\frac{1}{10}\|(3,0) - (1,0)\| = \frac{1}{5} < 2 = \|x - y\|,$$

and

$$\begin{aligned}\|Tx - Ty\| &= |(1,0) - (0,0)| = 1 \\ &< \frac{1}{2}\|y - Tx\| + \frac{1}{3}\|x - Ty\| + \frac{1}{6}\|x - y\|.\end{aligned}$$

Case V: For $x = y = (3,0)$. We have

$$\frac{1}{10}\|(3,0) - (1,0)\| = \frac{1}{5} > 0 = \|x - y\|.$$

Also, $x = y = (1,0)$. We have

$$\frac{1}{10}\|(1,0) - (0,0)\| = \frac{1}{10} > 0 = \|x - y\|,$$

so, we have nothing to show.

Thus, we have that T is a generalized $(\frac{1}{2}, \frac{1}{3})$-nonexpansive type 1 mapping.

Now, we establish that T is not a mean nonexpansive, generalized mean non-expansive, mappings satisfying condition (C), condition (C_λ) and α-nonexpansive mappings.

Indeed, we suppose that T is a mean nonexpansive mapping, so therefore, there exists nonnegative real numbers α and β, with $\alpha + \beta \leq 1$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\| + \beta\|x - Ty\|$$

for all $x, y \in C$. Now, consider $x = (0,0)$ and $y = (1,0)$, we then have that

$$\begin{aligned}\|Tx - Ty\| &= 0 \\ &\leq \alpha\|x - y\| + \beta\|x - Ty\| \\ &= \alpha.\end{aligned}$$

Thus, we obtain that $\alpha \leq 1$ and $\beta = 0$. So therefore, T is a nonexpansive mapping, which is a contradiction.

Proposition 3.6. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping with $F(T) \neq \emptyset$. Then T is quasi-nonexapansive.*

Proof. Let $x \in F(T)$ and $y \in C$,

$$\lambda\|Tx - x\| = 0 \leq \|x - y\|.$$

So, we have

$$\begin{aligned}\|x - Ty\| &= \|Tx - Ty\| \\ &\leq \alpha\|y - Tx\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\| \\ &= \alpha\|y - x\| + \beta\|x - Ty\| + (1 - (\alpha + \beta))\|x - y\|,\end{aligned}$$

it implies that

$$(1-\beta)\|x-Ty\| \leq (1-\beta)\|x-y\|.$$

That is,

$$\|x-Ty\| \leq \|x-y\|.$$

This means that T is quasi-nonexpanisve. $\square$

Theorem 3.7. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to some $y \in C$. We show that $y \in F(T)$. Since

$$\lambda\|Tx_n - x_n\| = 0 \leq \|x_n - y\|,$$

so, we have

$$\begin{aligned}\|x_n - Ty\| &= \|Tx_n - Ty\| \\ &\leq \alpha\|y - Tx_n\| + \beta\|x_n - Ty\| + (1-(\alpha+\beta))\|x_n - y\|,\end{aligned}$$

it implies that

$$\|x_n - Ty\| \leq \|x_n - y\|.$$

Since $\lim_{n\to\infty}\|x_n - y\| = 0$, we obtain

$$\lim_{n\to\infty}\|x_n - Ty\| = 0.$$

As such, we have that $Ty = y$. Hence, $F(T)$ is closed. Now suppose that X is strictly convex and C is convex. We show that $F(T)$ is convex. Let $x, y \in F(T), z \in C$ with $x \neq y$. Since

$$\lambda\|x - Tx\| = 0 \leq \|x - z\|,$$

we obtain

$$\|x - Tz\| = \|Tx - Tz\| \leq \alpha\|z - Tx\| + \beta\|x - Tz\| + (1-(\alpha+\beta))\|x - z\|,$$

it implies that

$$\|x - Tz\| \leq \|x - z\|. \tag{3.3}$$

Using similar argument, we have

$$\|y - Tz\| \leq \|y - z\|. \tag{3.4}$$

Let $z = \gamma x + (1-\gamma)y \in C$, for $\gamma \in [0,1]$. Then from (3.3) and (3.4), we obtain

$$\begin{aligned}\|x-y\| &\leq \|x-Tz\| + \|Tz-y\| \\ &\leq \|x-z\| + \|z-y\| \\ &= \|x-(\gamma x+(1-\gamma)y)\| + \|(\gamma x+(1-\beta)y-y\| \\ &\leq (1-\gamma)\|x-x\| + \gamma\|x-y\| + (1-\gamma)\|x-y\| + \gamma\|y-y\| \\ &= \|x-y\|.\end{aligned} \tag{3.5}$$

Using the fact that X is strictly convex, there exists $\mu \in [0,1]$ such that $Tz = \mu x + (1-\mu)y$. Now

$$(1-\mu)\|x-y\| = \|Tx-Tz\| \leq \|x-z\| = (1-\gamma)\|x-y\| \tag{3.6}$$

and

$$\mu\|x-y\| = \|Ty-Tz\| \leq \|x-z\| = \gamma\|x-y\|. \tag{3.7}$$

From the above inequalities (3.6) and (3.7), we have $1-\mu \leq 1-\gamma$ and $\mu \leq \gamma$, this implies that $\mu = \gamma$. Thus, $z \in F(T)$, which implies that $F(T)$ is convex. □

In view of Proposition 3.4, we have the following corollaries.

Corollary 3.8. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 3.9. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mean nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 3.10. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mapping satisfying condition (C). Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 3.11. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a mapping satisfying condition (C_λ). Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Corollary 3.12. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized mean nonexpansive mapping. Then $F(T)$ is closed. Furthermore, if X is strictly convex and C is convex, then $F(T)$ is convex.*

Lemma 3.13. *Let C be a nonempty subset of a Banach space X. Suppose that $T : C \to C$ is a generalized (α, β)-nonexpansive type 1 mapping on C. Then for all $x, y \in C$ and for $\gamma \in [0, 1)$, we have the following:*

(1) $\|T^2x - Tx\| < \|Tx - x\|$.
(2) *Either* $\frac{\gamma}{2}\|x - Tx\| \leq \|x - y\|$ *or* $\frac{\gamma}{2}\|Tx - T^2x\| \leq \|Tx - y\|$.
(3) *Either* $\|Tx - Ty\| \leq \alpha\|Tx - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\|$ *or* $\|T^2x - Ty\| \leq \alpha\|T^2x - y\| + \beta\|Ty - Tx\| + (1 - (\alpha + \beta))\|Tx - y\|$.

Proof. (1) For all $x \in C$, we have that $\lambda\|Tx - x\| \leq \|Tx - x\|$, which implies that

$$\begin{aligned}\|T^2x - Tx\| &= \|T(Tx) - Tx\| \\ &\leq \alpha\|T(Tx) - x\| + \beta\|Tx - Tx\| + (1 - (\alpha + \beta))\|Tx - x\| \\ &= \alpha\|T(Tx) - x\| + (1 - (\alpha + \beta))\|Tx - x\| \\ &\leq \alpha[\|T(Tx) - Tx\| + \|Tx - x\|] + (1 - (\alpha + \beta))\|Tx - x\| \\ &= \alpha\|T^2x - Tx\| + (1 - \beta))\|Tx - x\|,\end{aligned}$$

this implies that

$$\|T^2x - Tx\| \leq \frac{1 - \beta}{1 - \alpha}\|Tx - x\| < \|Tx - x\|.$$

(2) Suppose, on the contrary $\frac{\gamma}{2}\|x - Tx\| > \|x - y\|$ and $\frac{\gamma}{2}\|Tx - T^2x\| > \|Tx - y\|$, for some $x, y \in C$. Now, using (1), observe that

$$\begin{aligned}\|x - Tx\| &\leq \|x - y\| + \|y - Tx\| \\ &< \frac{\gamma}{2}\|x - Tx\| + \frac{\gamma}{2}\|Tx - T^2x\| \\ &< \frac{\gamma}{2}\|x - Tx\| + \frac{\gamma}{2}\|x - Tx\| \\ &= \gamma\|x - Tx\| \\ &< \|x - Tx\|,\end{aligned}$$

which is a contradiction. Thus, we obtain the desired result.
(3) The proof here follows from (2). Thus, we omit it. □

Lemma 3.14. *Let C be a nonempty subset of a Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping. Then for all $x, y \in C$,*

$$\|x - Ty\| \leq \frac{(2 + \alpha + \beta)}{(1 - \beta)}\|x - Tx\| + \|x - y\|.$$

Proof. From Lemma 3.13, we have that for all $x, y \in C$,

$$\|Tx - Ty\| \leq \alpha\|Tx - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\|$$

or

$$\|T^2x - Ty\| \leq \alpha\|T^2x - y\| + \beta\|Ty - Tx\| + (1 - (\alpha + \beta))\|Tx - y\|.$$

Considering $\|Tx - Ty\| \leq \alpha\|Tx - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\|$, we obtain that

$$\begin{aligned}
&\|x - Ty\| \\
&\leq \|x - Tx\| + \|Tx - Ty\| \\
&\leq \|x - Tx\| + \alpha\|Tx - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\| \\
&\leq \|x - Tx\| + \alpha\|Tx - x\| + \alpha\|x - y\| + \beta\|Ty - x\| + (1 - (\alpha + \beta))\|x - y\| \\
&= (1 + \alpha)\|x - Tx\| + \beta\|Ty - x\| + (1 - \beta)\|x - y\|,
\end{aligned}$$

it implies that

$$\begin{aligned}
\|x - Ty\| &\leq \frac{(1 + \alpha)}{(1 - \beta)}\|x - Tx\| + \|x - y\| \\
&\leq \frac{(2 + \alpha + \beta)}{(1 - \beta)}\|x - Tx\| + \|x - y\|.
\end{aligned}$$

Also, considering $\|T^2x - Ty\| \leq \alpha\|T^2x - y\| + \beta\|Ty - Tx\| + (1 - (\alpha + \beta))\|Tx - y\|$, using (1) of Lemma 3.13, we obtain that

$$\begin{aligned}
&\|x - Ty\| \\
&\leq \|x - Tx\| + \|Tx - T^2x\| + \|T^2x - Ty\| \\
&< \|x - Tx\| + \|x - Tx\| + \alpha\|T^2x - y\| + \beta\|Ty - Tx\| \\
&\quad + (1 - (\alpha + \beta))\|Tx - y\| \\
&\leq 2\|x - Tx\| + \alpha\|T^2x - Tx\| + \alpha\|Tx - y\| + \beta\|Ty - x\| + \beta\|x - Tx\| \\
&\quad + (1 - (\alpha + \beta))\|Tx - y\| \\
&< 2\|x - Tx\| + \alpha\|x - Tx\| + \alpha\|Tx - y\| + \beta\|Ty - x\| + \beta\|x - Tx\| \\
&\quad + (1 - (\alpha + \beta))\|Tx - y\| \\
&= (2 + \alpha + \beta)\|x - Tx\| + \beta\|Ty - x\| + (1 - \beta)\|x - y\|,
\end{aligned}$$

it implies that

$$\|x - Ty\| \leq \frac{(2 + \alpha + \beta)}{(1 - \beta)}\|x - Tx\| + \|x - y\|.$$

Thus in both cases, we obtain the desired result. $\square$

Theorem 3.15. *Let C be a nonempty closed subset of a Banach space X with Opial property and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping with $\lambda = \frac{\gamma}{2}, \gamma \in [0, 1)$. If $\{x_n\}$ converges weakly to x and*

$\lim_{n\to\infty} \|Tx_n - x_n\| = 0$, *then* $Tx = x$. *That is* $I - T$ *is demiclosed at zero, where* I *is the identity mapping on* X.

Proof. By Lemma 3.13

$$\lambda\|x_n - Tx_n\| \le \|x_n - x\|.$$

Thus by the definition of generalized (α, β)-nonexpansive type 1 mapping T,

$$\|Tx_n - Tx\| \le \alpha\|Tx_n - x\| + \beta\|Tx - x_n\| + (1 - (\alpha + \beta))\|x_n - x\|.$$

Now, observe that

$$\begin{aligned}
&\|x_n - Tx\| \\
&\le \|x_n - Tx_n\| + \|Tx_n - Tx\| \\
&\le \|x_n - Tx_n\| + \alpha\|Tx_n - x\| + \beta\|Tx - x_n\| + (1 - (\alpha + \beta))\|x_n - x\| \\
&\le \|x_n - Tx_n\| + \alpha\|Tx_n - x_n\| + \alpha\|x_n - x\| + \beta\|Tx - x_n\| \\
&\quad + (1 - (\alpha + \beta))\|x_n - x\| \\
&= (1 + \alpha)\|x_n - Tx_n\| + \beta\|Tx - x_n\| + (1 - \beta))\|x_n - x\|,
\end{aligned}$$

it implies that

$$\|x_n - Tx\| \le \frac{1 + \alpha}{(1 - \beta)}\|x_n - Tx_n\| + \|x_n - x\|.$$

Using our hypothesis, we have that

$$\liminf_{n\to\infty} \|x_n - Tx\| \le \liminf_{n\to\infty} \|x_n - x\|. \tag{3.8}$$

Using our hypothesis that $\{x_n\}$ converges weakly to x and Opial property, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - Tx\|,$$

which contradicts (3.8). Thus, we have that $Tx = x$. □

Theorem 3.16. *Let* C *be a nonempty compact subset of a Banach space* X *and* $T : C \to C$ *be a generalized* (α, β)*-nonexpansive type 1 mapping with* $\lambda = \frac{\gamma}{2}, \gamma \in [0, 1)$. *Then* T *has a fixed point in* C *if and only if* T *admits an almost fixed point sequence.*

Proof. The proof follows a similar approach as in Theorem 3.15, and thus, we omit it. □

3.2. **Convergence Results.** In this section, we established some convergence results of a new three steps iterative algorithm generated by the generalized (α, β)-nonexpansive type 1 mapping in a uniformly convex Banach space. We define our iterative process as follows:

For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\alpha_n)Tz_n + \alpha_n T^2 z_n, \\ x_{n+1} = T[(1-\beta_n)T^2 z_n + \beta_n T^2 y_n], \ n \geq 0, \end{cases} \tag{3.9}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.

Lemma 3.17. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by* (3.9)*. Then, we have the followings:*

(i) $\{x_n\}$ *is bounded;*
(ii) $\lim_{n\to\infty} \|x_n - x^*\|$ *exists for all $x^* \in F(T)$.*

Proof. Let $x^* \in F(T)$, using (3.9) and Proposition 3.6, we obtain

$$\begin{aligned} \|z_n - x^*\| &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n \|Tx_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{3.10}$$

Also, using (3.9), (3.10) and Proposition 3.6, we obtain

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\alpha_n)Tz_n + \alpha_n T^2 z_n - x^*\| \\ &\leq (1-\alpha_n)\|Tz_n - x^*\| + \alpha_n \|T(Tz_n) - x^*\| \\ &\leq (1-\alpha_n)\|z_n - x^*\| + \alpha_n \|Tz_n - x^*\| \\ &\leq (1-\alpha_n)\|z_n - x^*\| + \alpha_n \|z_n - x^*\| \\ &= \|z_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{3.11}$$

Lastly, using (3.9), (3.11) and Proposition 3.6, we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|T[(1-\beta_n)T^2z_n+\beta_nT^2y_n]-x^*\| \\
&\le (1-\beta_n)\|T^2z_n-x^*\|+\beta_n\|T^2y_n-x^*\| \\
&= (1-\beta_n)\|T(Tz_n)-x^*\|+\beta_n\|T(Ty_n)-x^*\| \\
&\le (1-\beta_n)\|Tz_n-x^*\|+\beta_n\|Ty_n-x^*\| \\
&\le (1-\beta_n)\|z_n-x^*\|+\beta_n\|y_n-x^*\| \\
&\le (1-\beta_n)\|x_n-x^*\|+\beta_n\|x_n-x^*\| \qquad (3.12) \\
&= \|x_n-x^*\|.
\end{aligned}$$

This shows that $\{\|x_n-x^*\|\}$ is bounded and non-increasing for all $x^*\in F(T)$. Thus, $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-x^*\|$ exists. □

Lemma 3.18. *Let C be a nonempty closed and convex subset of a uniformly convex Banach space X and $T: C\to C$ be a generalized (α,β)-nonexpansive type 1 mapping with $F(T)\neq\emptyset$. Suppose that $\{x_n\}$ is defined by* (3.9). *Then* $\lim_{n\to\infty}\|Tx_n-x_n\|=0$.

Proof. Let $x^*\in F(T)$. It follows from Lemma 3.17 that $\{x_n\}$ is bounded and $\lim_{n\to\infty}\|x_n-x^*\|$ exists for all $x^*\in F(T)$. Suppose that $\lim_{n\to\infty}\|x_n-x^*\|=c$. From (3.10), we obtain that $\|z_n-x^*\|\le\|x_n-x^*\|$. Taking limsup of both sides, we have

$$\limsup_{n\to\infty}\|z_n-x^*\|\le c. \tag{3.13}$$

In addition, using Proposition 3.6, we obtain that $\|Tx_n-x^*\|\le\|x_n-x^*\|$, and that

$$\limsup_{n\to\infty}\|Tx_n-x^*\|\le c. \tag{3.14}$$

From (3.12), we have

$$\|x_{n+1}-x^*\|\le(1-\beta_n)\|z_n-x^*\|+\beta_n\|x_n-x^*\|.$$

Taking the $\liminf_{n\to\infty}$ of both sides and rearranging the inequalities, we have

$$c\le(1-\beta_n)\limsup_{n\to\infty}\|z_n-c\|+\beta_nc,$$

that is,

$$c\le\liminf_{n\to\infty}\|z_n-x^*\|. \tag{3.15}$$

From (3.13) and (3.15), we obtain that $\lim_{n\to\infty}\|z_n-x^*\|=c$. That is,

$$\lim_{n\to\infty}\|(1-\gamma_n)x_n+\gamma_nTx_n-x^*\|=c.$$

Thus, by Lemma 2.3, we have $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. □

Theorem 3.19. *Let X be a uniformly convex Banach space which satisfies the Opial condition and C be a nonempty closed convex subset of X. Let $T : C \to C$ be a generalized (α, β)-nonexpansive type 1 mapping such that $\lambda = \frac{\gamma}{2} \in [0, \frac{1}{2}]$ with $F(T) \neq \emptyset$ and $\{x_n\}$ be a sequence defined by* (3.9)*. Then, $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. It has been established in Lemma 3.17 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and that $\{x_n\}$ is bounded. Now, since X is uniformly convex, we can find a subsequence say $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly in C. We now establish that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Let x and y be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Theorem 3.18, we have that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed with respect to zero by Theorem 3.15, we therefore have that $Tx = x$. Using a similar approach, we can show that $y = Ty$. It follows from Lemma 3.17 that $\lim_{n\to\infty} \|x_n - y\|$ exists. Now, suppose that $x \neq y$, then by the Opial condition,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x\| &= \lim_{k\to\infty} \|x_{n_k} - x\| \\
&< \lim_{k\to\infty} \|x_{n_k} - y\| \\
&= \lim_{n\to\infty} \|x_n - y\| \\
&= \lim_{j\to\infty} \|x_{n_j} - y\| \\
&< \lim_{j\to\infty} \|x_{n_j} - x\| \\
&= \lim_{n\to\infty} \|x_n - x\|.
\end{aligned}$$

This is a contradiction. So $x = y$. Hence, $\{x_n\}$ converges weakly to a fixed point of $F(T)$ and this completes the proof. □

Theorem 3.20. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a generalized (α, β)-nonexpansive type 1 mapping on C, $\{x_n\}$ be defined by* (3.9) *and $F(T) \neq \emptyset$. Then, $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0,$$

where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

Proof. Let $\{x_n\}$ converges to x^* a fixed point of T. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, x^*)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. It follows from Lemma 3.17 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and that $\lim_{n\to\infty} d(x_n, F(T))$ exists for all

$x^* \in F(T)$. By our hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Suppose $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{u_k\}$ is a sequence in $F(T)$ such that for all $n \in \mathbb{N}$,

$$\|x_{n_k} - u_k\| < \frac{1}{2^k},$$

it follows from (3.12) that $\|x_{n+1} - u_k\| \leq \|x_n - u_k\| < \frac{1}{2^k}$, hence

$$\begin{aligned}\|u_{k+1} - u_k\| &\leq \|u_{k+1} - x_{n+1}\| + \|x_{n+1} - u_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}.\end{aligned}$$

Thus, we have that $\{u_k\}$ is a Cauchy sequence in $F(T)$. Also, by Theorem 3.7, we have that $F(T)$ is closed. Thus $\{u_k\}$ is a convergent sequence in $F(T)$. Now, suppose that $\{u_k\}$ converges to $p \in F(T)$. Therefore, since

$$\|x_{n_k} - p\| \leq \|x_{n_k} - u_k\| + \|u_k - p\| \to 0 \text{ as } k \to \infty,$$

we obtain that $\lim\limits_{k\to\infty} \|x_{n_k} - p\| = 0$ and so $\{x_{n_k}\}$ converges strongly to $p \in F(T)$. Since $\lim\limits_{n\to\infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p. □

Theorem 3.21. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let T be a generalized (α, β)-nonexpansive type 1 mapping, $\{x_n\}$ be defined by* (3.9) *and $F(T) \neq \emptyset$. Let T satisfy condition (I). Then, $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Using Lemma 3.17 and Theorem 3.18, we obtain that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Using the fact that for all $x \in C$,

$$\begin{aligned}0 &\leq \lim_{n\to\infty} f(d(x_n, F(T)) \\ &\leq \lim_{n\to\infty} \|x_n - Tx_n\| \\ &= 0\end{aligned}$$

and that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0.$$

Since, f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$, it then follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus using Theorem 3.20, we obtain that $\{x_n\}$ converges strongly to $p \in F(T)$. □

4. Application to nonlinear integral equation

In this section, we present an application of our result to nonlinear integral equation of the form:

$$x(t) = g(t) + \gamma \int_a^b M(t,s)h(t,x(s))ds, \tag{4.1}$$

where $\gamma \in (0,1], M : [a,b] \times [a,b] \to \mathbb{R}^+$, $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$ with ordered relation $\leq$ in X defined as for $x, y \in X, x \leq y$ if and only if $x(s) \leq y(s)$ for all $s \in [a,b]$. We defined $\| \cdot \| : X \times X \to [0,\infty)$ by $\|x - y\| = \sup_{s\in[a,b]} |x(s) - y(s)|$.

Theorem 4.1. *Let $X = C([a,b],\mathbb{R})$ and $T : X \to X$ the operator given by*

$$Tx(t) = g(t) + \gamma \int_a^b M(t,s)h(t,x(s))ds$$

for all $t, s \in [a,b]$, where $\gamma \in [0,1], M : [a,b] \times [a,b] \to \mathbb{R}^+$, $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $h : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$. Furthermore, suppose the following conditions hold:

(1) *there exists a continuous mapping $\upsilon : X \times X \to [0,\infty)$ such that*

$$|h(s,x(s)) - h(s,y(s))| \leq \upsilon(x,y)|x(s) - y(s)|$$

for all $s \in [a,b]$ and $x, y \in X$.

(2) *there exists $\omega \in [0,1]$, such that*

$$\int_a^b M(t,s)\upsilon(x,y) \leq \omega.$$

Then the integral equation (4.1) *has a solution.*

Proof. Without loss of generality, we suppose that $x \leq y$, so that

$$\sup\{|y(s) - x(s)| : s \in [a,b]\} \geq \sup\{|Tx(s) - x(s)| : s \in [a,b]\},$$

which implies that

$$\lambda\|Tx - x\| \leq \|Tx - x\| \leq \|y - x\|,$$

where $\lambda \in [0, 1)$. Thus, we have that

$$\begin{aligned}
&|Ty(s) - Tx(s)| \\
&= \left\| g(t) + \gamma \int_a^b M(t,s)h(t,y(s)) - g(t) - \gamma \int_a^b M(t,s)h(t,x(s))ds \right\| \\
&\leq \gamma \int_a^b |M(t,s)[h(t,y(s)) - h(t,x(s))]|ds \\
&\leq \gamma \int_a^b M(t,s)\upsilon(x,y)|y(s) - x(s)|ds \\
&\leq \sup_{s\in[a,b]} |y(s) - x(s)|\gamma \int_a^b M(t,s)\mu(x,y)ds \\
&\leq \gamma\omega\|y - x\| \\
&\leq \|y - x\|.
\end{aligned}$$

Thus, we have that, for $\lambda\|x - Tx\| \leq \|x - y\|$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

Clearly, T satisfies condition (C_λ) and by Proposition 3.4, T is a generalized (α, β)-nonexpansive mapping and all the conditions in Theorem 3.16 are satisfied, as such T has a fixed point, that is the integral equation (4.1) has a solution. □

5. Numerical examples

Example 5.1. Define a mapping $T : [0, 1] \to [0, 1]$ as

$$Tx = \begin{cases} 1 - x \text{ if } x \in [0, \frac{1}{8}), \\ \frac{x+7}{8} \text{ if } x \in [\frac{1}{8}, 1]. \end{cases} \tag{5.1}$$

Then, it is easy to see that T satisfy condition (C), thus it is a generalized (α, β)-nonexpansive mapping.

In what follows, we numerically compare our new iteration process with some existing iterative processes.

Case I: Taking, $\alpha_n = \frac{1}{\sqrt{n^3+4}}$, $\gamma_n = \frac{3}{(n^3+200)}$, $\beta_n = \frac{2}{\sqrt{n^3+5}}$ and $x_0 = 0.5$.

Case II: Taking, $\alpha_n = \frac{1}{202}$, $\gamma_n = \frac{1}{1000}$, $\beta_n = \frac{1}{300}$ and $x_0 = 0.8$.

Case III: Taking, $\alpha_n = \frac{1}{\sqrt{n^{30}+40}}$, $\gamma_n = \frac{3}{300n^3}$, $\beta_n = \frac{1}{\sqrt{n^{10}+50}}$ and $x_0 = 0.3$.

Case IV: Taking, $\alpha_n = \frac{5}{300n^{30}}$, $\gamma_n = \frac{8}{1000n^{34}}$, $\beta_n = \frac{7}{200n^{20}}$ and $x_0 = 0.6$.

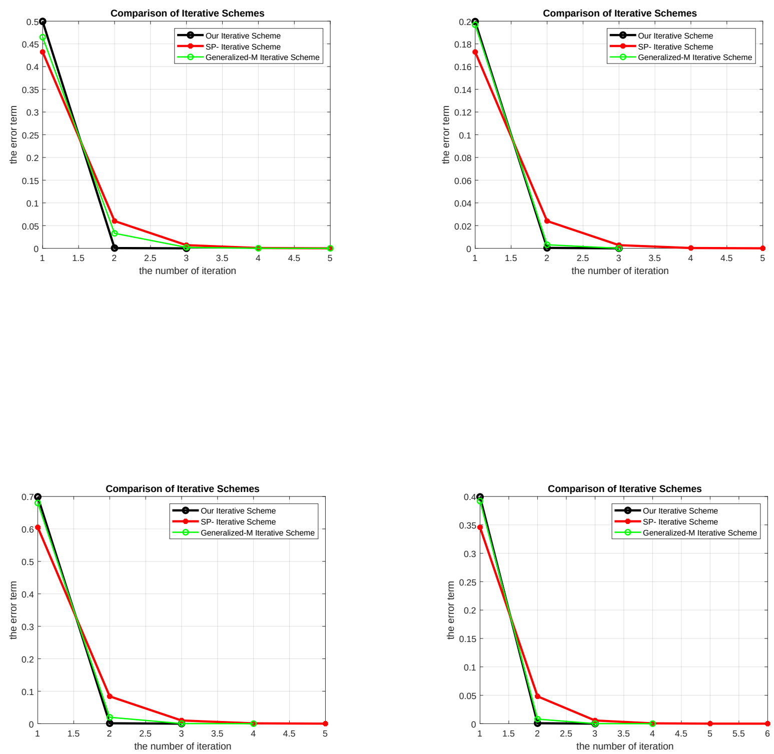

FIGURE 1. Example 5.1, Top Left: Case I; Top Right: Case II; Bottom Left: case III; Bottom Right: Case IV

Remark 5.2. The comparison shows that our iterative processes (3.9) converges faster than the iterative processes (1.1), (1.2) and consequently converges faster than some existing iterative schemes in the literature.

6. Conclusion and open problem

In this work, we present some fixed point results for a general class of nonexpansive mappings and also proposed a new iterative scheme for approximating the fixed point of this class of mappings in the frame work of uniformly convex Banach spaces. Our numerical experiment shows that our iterative method is better compare to some existing iterative methods in the literature. In addition, we gave the definition of a generalized (α, β)-nonexpansive type 1 mapping, and generalized (α, β)-nonexpansive type 2 mapping, we established some results for the type 1 mapping but not for type 2. In the light of this, we leave the type 2 mapping as an open problem for interested researchers in this area to explore.

References

[1] M. Abass and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems* , Matematicki Vesnik, **66** (2014), 223–234.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux S. équations intégrales*, Fund. Math., **3** (1922), 133–181.

[3] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space,* Proc. Nat. Acad. Sci. USA., **54** (1965), 1041–1044.

[4] P. Chuadchawna, A. Farajzadeh and A. Kaewcharoen, *Fixed-point approximation of generalized nonexpansive mappings via generalized M-iteration in hyperbolic spaces,* Inter.J. Math. Sci., (2020), ID 6435043, 1–8.

[5] J.G. Falset, E.L. Fuster and T. Suzuki *Fixed point theory for a class of generalizec nonexpansive mappings,* J. Math. Anal. Appl., **375** (2011), 185–195.

[6] K. Goebel and W.A. Kirk, *Topics in metric fixed point theory,* Cambridge University Press (1990).

[7] S. Ishikawa, *Fixed points by new iteration method,* Proc. Amer. Math. Soc. **149** (1974), 147–150.

[8] W.R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc. **4** (1953), 506–510.

[9] A.A. Mebawondu, C. Izuchukwu, H.A. Abass and O.T. Mewomo, *Some results on generalized mean nonexpansive mapping in complete metric space,* Bol. Soc. Paran. Mat., (2019), DOI:10.5269/bspm.44174.

[10] A.A. Mebawondu and O.T. Mewomo, *Suzuki-type fixed point results in Gb-metric spaces,* (2020), DOI:10.1142/S1793557121500704

[11] M.A. Noor, *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl., **251** (2000), 217–229.

[12] Z. Opial, *Weak convergence of the sequence of the successive approximation for nonexapansive mapping,* Bull Amer. Math. Soc., **73** (1967), 591–597.

[13] W. Phuengrattana and S. Suantai, *On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval,* J. Comput. Appl. Math., **235** (2011), 3006–3014.

[14] J. Schu, *Weak and strong convergence to the fixed points of asymptotically nonexpansive mappings,* Bull. Austral. Math. Soc., **43**(1) (1991), 153–159.

[15] H.F. Senter, and W.G. Dotson *Approximating fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **44** (1974), 375–380.

[16] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings,* J. Math. Anal. Appl., **340** (2008), 1088–1095.

[17] K. Ullah and M. Arshad, *Numerical reckoning fixed points for Suzuki generalized nonexpansive mappings via new iteration process,* Filomat, **32** (2018), 187–196.

[18] S. Zhang, *About fixed point theory for mean nonexpansive mapping in Banach spaces,* J. Sichuan University, **2** (1975), 67–68.

[19] Z. Zuo, *Fixed-point theorems for mean nonexpansive mappings in Banach spaces,* Abstr. Appl. Anal., (2014), 1–6.