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QUASI STRONGLY E-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this article, we introduce the quasi strongly E-convex function and pseudo strongly E-convex function on strongly E-convex set which generalizes strongly E-convex function defined by Youness [10]. Some non trivial examples have been constructed that show the existence of these functions. Several interesting properties of these functions have been discussed. An important characterization and relationship of these functions have been established. Furthermore, a nonlinear programming problem for quasi strongly E-convex function has been discussed.

1. Introduction

Convexity is an important branch of mathematical sciences which plays a significant role in solving practical problems. The generalization of convexity as E-convexity was introduced by [10] and later studied by many authors working in the field of pure and applied mathematical analysis, see [3, 4, 7]. Later, Youness [10, 11], presented the different class of E-convexity named as strongly E-convexity and discussed its various properties. Motivated by Youness [10], Chen [2], introduced semi E-convex function and proved some

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interesting properties related to these functions. The generalization of quasi convex and pseudoconvex function was given by Bazarra et al. [1]. Some interesting properties of E-quasi convex and E-pseudoconvex functions have been derived by Soleimani [6]. Many properties and results of nonlinear optimization theory have been developed for E-convex sets and E-convex functions.

Motivated and inspired by above research works, we introduce the concept of quasi strongly E-convex function, pseudo strongly E-convex function and establish several interesting properties of these functions. By the non-trivial examples, we show the existence of quasi strongly E-convex functions. An important characterization of quasi strongly E-convex function and its relation with strongly E-convex function have been discussed. We also define pseudo strongly E-convex function and study its relationship with strongly E-convex function in section (3). Furthermore, an application of quasi strongly E-convex function to non-linear programming problem has been presented in section (4).

2. Preliminaries

Let $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $E : \mathbb{R}^n \to \mathbb{R}^n$ be a map. Youness [8] introduced the concept of E-convexity as follows:

Definition 2.1. ([8]) A nonempty subset $S \subseteq \mathbb{R}^n$ is said to be an E-convex set with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$ if $\beta E(x) + (1-\beta)E(y) \in S$, for every $x, y \in S$ and $\beta \in [0, 1]$.

Definition 2.2. ([12]) A real valued function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be E-quasi convex on S with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$, if S is an E-convex set and for each $x, y \in S, \beta \in [0, 1]$,

$$f(\beta E(x) + (1-\beta)E(y)) \leq \max\{f(E(x)), f(E(y))\}.$$

The function f is said to be a strictly quasi E-convex if, for each $x, y \in S$ with $f(E(x)) \neq f(E(y))$ and for each $\beta \in (0, 1)$,

$$f(\beta E(x) + (1-\beta)E(y)) < \max\{f(E(x)), f(E(y))\}.$$

Definition 2.3. A real valued function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be E-pseudo convex with respect to a map E on S if there exists a strictly positive function $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $f(E(x)) < f(E(y))$ implies

$$f(\beta E(x) + (1-\beta)E(y)) \leq f(E(y)) + \beta(\beta - 1)b(E(x), E(y))$$

for all $x, y \in S$ and $\beta \in [0, 1]$.

Youness [10] generalized the concept of E-convexity and introduced strongly E-convexity as follows:

Definition 2.4. ([10]) A nonempty set $S \subseteq \mathbb{R}^n$ is said to be a strongly E-convex set with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$ if for any $x, y \in S$, $\alpha \in [0,1], \beta \in [0,1]$,

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in S.$$

Definition 2.5. ([10]) Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set. A real valued function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be a strongly E-convex with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$ on S, if

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \beta f(E(x)) + (1-\beta) f(E(y)),$$

for each $x, y \in S, \alpha \in [0,1]$ and $\beta \in [0,1]$.

If the above inequality is strict for all $x, y \in S$, $\alpha x + E(x) \neq \alpha y + E(y), \alpha \in [0,1]$ and $\beta \in (0,1)$, then f is called a strictly strongly E-convex function.

Definition 2.6. ([11]) Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set. A real valued function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be a semi strongly E-convex with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$ on S, if

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \leq \beta f(x) + (1-\beta) f(y),$$

for each $x, y \in S, \alpha \in [0,1]$ and $\beta \in [0,1]$.

If the above inequality is strict for all $x, y \in S$, $\alpha x + E(x) \neq \alpha y + E(y), \alpha \in [0,1]$ and $\beta \in (0,1)$, then f is called a strictly semi strongly E-convex function.

3. Quasi strongly E-convex functions

In this section, we introduce the generalized class of strongly E-convex functions named as quasi strongly E-convex functions on strongly E-convex set $S \subseteq \mathbb{R}^n$ and discuss some of their properties.

Definition 3.1. Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set. A real valued function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be quasi strongly E-convex function with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$ on S, if for each $x, y \in S, \alpha \in [0,1]$ and $\beta \in [0,1]$,

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \max\{f(E(x)), f(E(y))\}.$$

If the above inequality is strict for all $x, y \in S$, $f(E(x)) \neq f(E(y)), \alpha \in [0,1]$ and $\beta \in (0,1)$, then f is called a strictly quasi strongly E-convex function. Definition 3.1 reduces to quasi semi strongly E-convex function if $f(E(x))$ and $f(E(y))$ in the above inequality replaced by $f(x)$ and $f(y)$, see [11].

Remark 3.2. Every quasi strongly E-convex function with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$ is an E-quasi convex function if $\alpha = 0$.

Example 3.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1, & if\ x > 0, \\ -x, & if\ x \leq 0, \end{cases} \tag{3.1}$$

and $E : \mathbb{R} \to \mathbb{R}$ be a map defined as

$$E(x) = |x| = \begin{cases} x, & if\ x > 0, \\ -x, & if\ x \leq 0. \end{cases} \tag{3.2}$$

Then the function $f(x)$ is quasi strongly E-convex as well as E-quasi convex but it is not strongly E-convex function. In particular, at $x = 0, y = 1, \alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$, we have

$$f(\beta(\alpha 0 + E(0)) + (1-\beta)(\alpha 1 + E(1))) = f(\frac{1}{2}(\frac{1}{2} + 1)) = f(\frac{3}{4}) = 1.$$

However,

$$\beta f(E(0)) + (1-\beta) f(E(1)) = \frac{1}{2} f(E(0)) + \frac{1}{2} f(E(1)) = \frac{1}{2},$$

implies that

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \nleq \beta f(E(x)) + (1-\beta) f(E(y)).$$

Hence, it is not strongly E-convex function.

An E-quasi convex function with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$ need not be quasi strongly E-convex function as shown in the following example:

Example 3.4. The above function $f(x)$ is E-quasi convex with respect to a map $E : \mathbb{R} \to \mathbb{R}$ defined as $E(x) = -x^2$ but it is not a quasi strongly E-convex function. At $x = 0, y = -1, \alpha = 1$ and $\beta = 0$, we have

$$\begin{aligned} f(\beta(\alpha 0 + E(0)) + (1-\beta)(\alpha(-1) + E(-1))) &= f(-1-1) = 2 \\ &> \max\{f(E(0)), f(E(-1))\} = \max\{0, 1\} = 1. \end{aligned}$$

Hence, it is not a quasi strongly E-convex function.

Theorem 3.5. *If $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is a quasi strongly E-convex function on a strongly E-convex set S, then $f(\alpha y + E(y)) \leq f(E(y))$, for any $y \in S$, $\alpha \in [0,1]$.*

Proof. Since, f is a quasi strongly E-convex function on a strongly E-convex set S, for each $x, y \in S, \alpha \in [0,1]$ and $\beta \in [0,1]$, we have

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in S$$

and

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \max\{f(E(x)), f(E(y))\}.$$

Thus for $x = y$, we have $f(\alpha y + E(y)) \leq f(E(y))$, for any $y \in S$ and $\alpha \in [0, 1]$. $\square$

In the following result, we discuss an important characterization of quasi strongly E-convex function.

Theorem 3.6. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a quasi strongly E-convex function on a strongly E-convex set $S \subseteq \mathbb{R}^n$. Then, f is quasi semi strongly E-convex on set S if and only if $f(E(x)) \leq f(x)$ for each $x \in S$.*

Proof. If f is quasi semi strongly E-convex, then

$$f(\beta(\alpha x + E(x) + (1 - \beta)(\alpha y + E(y))) \leq \max\{f(x), f(y)\},$$

for each $x, y \in S, \alpha \in [0, 1]$ and $\beta \in [0, 1]$. Setting $x = y$ and $\alpha = 0$ in the above inequality, we have $f(E(x)) \leq f(x)$ for each $x \in S$.

Conversely, let $f(E(x)) \leq f(x)$ for each $x \in S$. Then from quasi strongly E-convexity of f, we have

$$\begin{aligned} f(\beta(\alpha x + E(x)) + (1 - \beta)(\alpha y + E(y))) &\leq \max\{f(E(x)), f(E(y))\} \\ &\leq \max\{f(x), f(y)\} \end{aligned}$$

for each $x, y \in S, \alpha \in [0, 1]$ and $\beta \in [0, 1]$. This completes the proof. $\square$

An important relation between strongly E-convex function and quasi strongly E-convex function is as follows:

Theorem 3.7. *A strongly E-convex function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ on a strongly E-convex set S, is quasi strongly E-convex if $f(E(x)) \leq f(E(y))$, for each $x, y \in S$.*

Proof. Suppose that the function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is a strongly E-convex function and $f(E(x)) \leq f(E(y))$, for all $x, y \in S$. For each $x, y \in S, \alpha \in [0, 1]$ and $\beta \in [0, 1]$, we have

$$\beta(\alpha x + E(x)) + (1 - \beta)(\alpha y + E(y)) \in S$$

and

$$f(\beta(\alpha x + E(x)) + (1 - \beta)(\alpha y + E(y))) \leq \beta f(E(x)) + (1 - \beta)f(E(y))$$

or

$$f(\beta(\alpha x + E(x)) + (1 - \beta)(\alpha y + E(y))) \leq f(E(y)) = \max\{f(E(x)), f(E(y))\},$$

that is,

$$f(\beta(\alpha x + E(x)) + (1 - \beta)(\alpha y + E(y))) \leq \max\{f(E(x)), f(E(y))\}.$$

Hence, f is a quasi strongly E-convex function on S. $\square$

Theorem 3.8. *Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set and $f : S \to \mathbb{R}$ be a quasi strongly E-convex function on S. Let $E(S) \subseteq S$ be a strongly E-convex set. Then, the restriction of f to $E(S)$, denoted by f_0, is a quasi strongly E-convex function on $E(S)$.*

Proof. Let $x, y \in E(S)$ and $\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in S$, for any $\alpha \in [0,1]$ and $\beta \in [0,1]$ and $f(x) = f_0(x)$ for every $x \in E(S)$. Since $E(S)$ is a strongly E-convex set, we have

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in E(S),$$

for any $\alpha \in [0,1]$ and $\beta \in [0,1]$. Therefore, we have

$$\begin{aligned} f_0(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) &= f(\beta(\alpha x + E(x)) \\ &\quad +(1-\beta)(\alpha y + E(y))) \\ &\leq \max\{f(E(x)), f(E(y))\} \\ &= \max\{f_0(E(x)), f_0(E(y))\}. \end{aligned}$$

Hence,

$$f_0(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \max\{f_0(E(x)), f_0(E(y))\}.$$

This completes the proof. □

Theorem 3.9. *Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set. If the functions $f_i : S \subseteq \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2..., k$ are nonnegative quasi strongly E-convex with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$ on S, then the linear combination of quasi strongly E-convex functions is also a quasi strongly E-convex function, that is, for $a_i \geq 0, i = 1, 2, ..., k$, the function*

$$h(x) = \sum_{i=1}^{k} a_i f_i(x)$$

is a quasi strongly E-convex function on S.

Proof. Since $f_i(x), i = 1, 2, ..., k$, are quasi strongly E-convex functions on a strongly E-convex set S. For any $x, y \in S, \alpha \in [0,1]$ and $\beta \in [0,1]$, we have

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in S$$

and

$$\begin{aligned} & h(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \\ &= \sum_{i=1}^{k} a_i f_i(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \end{aligned}$$

$$\leq \max\Big\{\sum_{i=1}^{k} a_i f_i(E(x)), \sum_{i=1}^{k} a_i f_i(E(y))\Big\}$$
$$= \max\{h(E(x)), h(E(y))\}.$$

Thus, $h(x)$ is a quasi strongly E-convex function on S. □

Theorem 3.10. *Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set and $\{f_j\}_{j\in I}$ be a family of real valued functions defined on S such that $\sup_{j\in I} f_j(x)$ exists in R, for all $x \in S$. Let $f : S \to \mathbb{R}$ be a real valued function defined by $f(x) = \sup_{j\in I} f_j(x)$, for all $x \in S$. If $f_j : S \to \mathbb{R}$ for any $j \in I$, are quasi strongly E-convex functions on S, then the function f is quasi strongly E-convex on S.*

Proof. Suppose that $f_j : S \to \mathbb{R}$, for all $j \in I$, are quasi strongly E-convex functions on S. Then, for every $x, y \in S, \alpha \in [0,1]$ and $\beta \in [0,1]$,

$$f_j(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \max\{f_j(E(x)), f_j(E(y))\}$$

and

$$\begin{aligned}
&\sup_{j\in I} f_j(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \\
&\quad \leq \sup_{j\in I} \max\{f_j(E(x)), f_j(E(y))\} \\
&\quad = \max\{sup_{j\in I} f_j(E(x)), sup_{j\in I} f_j(E(y))\} \\
&\quad = \max\{f(E(x)), f(E(y))\}.
\end{aligned}$$

Hence, we have

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \max\{f(E(x)), f(E(y))\}.$$

Therefore, f is a quasi strongly E-convex function on S. □

Theorem 3.11. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a quasi strongly E-convex function on a strongly E-convex set $S \subseteq \mathbb{R}^n$ and $\phi : \mathbb{R} \to \mathbb{R}$ be a positive homogeneous non-decreasing function. Then the function $\phi \circ f$ is quasi strongly E-convex on S.*

Proof. As S is a strongly E-convex set, then for any $x, y \in S, \alpha \in [0,1]$ and $\beta \in [0,1]$, we have

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in S.$$

Also, f is a quasi strongly E-convex function on S, we have

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \max\{f(E(x)), f(E(y))\}.$$

Since, ϕ is a positively homogeneous non-decreasing function, we have

$$\begin{aligned}(\phi \circ f)(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)))\\ &\leq \phi \circ \{\max\{f(E(x)), f(E(y)\}\}\\ &\leq \max\{(\phi \circ f)(E(x)), (\phi \circ f)(E(y))\},\end{aligned}$$

this implies that $\phi \circ f$ is a quasi strongly E-convex function on S. □

Theorem 3.12. *Let $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, ..., k$ be quasi strongly E-convex functions on $\mathbb{R}^n$ with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$. If $E(S) \subseteq S$, then the set*

$$S = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, 2, ..., k\}$$

is a strongly E-convex set.

Proof. Since $g_i(x), i = 1, 2, ..., k$, are quasi strongly E-convex functions, for each $x, y \in S \subseteq \mathbb{R}^n, \alpha \in [0, 1]$ and $\beta \in [0, 1]$, we have

$$\begin{aligned}g_i(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) &\leq \max\{(g_i \circ E)(x), (g_i \circ E)(y)\}\\ &= \max\{g_i(E(x)), g_i(E(y))\}\\ &\leq 0,\end{aligned}$$

where we used the assumption $E(S) \subseteq S$ to obtain the right most of the inequality above. Hence,

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in S.$$

Therefore, S is strongly E-convex set. □

Definition 3.13. Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set and let $f : S \to \mathbb{R}$ be a real valued function. The lower level set of f is defined as

$$K_\gamma = \{x \in S : f(x) \leq \gamma\}.$$

In the following theorem, we discuss the relationship between lower level set and quasi strongly E-convex function on strongly E-convex set.

Theorem 3.14. *Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set. If the lower level set K_γ is strongly E-convex for each $\gamma \in \mathbb{R}$, then the function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is quasi strongly E-convex on S.*

Proof. Assume that $S \subseteq \mathbb{R}^n$ be a strongly E-convex set and the set K_γ be a strongly E-convex set for each $\gamma \in \mathbb{R}$. For each $x, y \in S, \alpha \in [0, 1]$ and $\beta \in [0, 1]$, we have

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in S.$$

By using [Theorem 1 in [10]], we get $E(y) \in S$, for all $y \in S$. Let

$$\gamma = \max\{f(E(x)), f(E(y))\}$$

and $x, y \in K_\gamma$. Since K_γ is a strongly E-convex set, we have

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in K_\gamma,$$

which implies

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \gamma = \max\{f(E(x)), f(E(y))\}.$$

Hence, the function f is quasi strongly E-convex on S. $\square$

Theorem 3.15. *Let $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2, ..., k$ be quasi strongly E-convex functions on $\mathbb{R}^n$ with respect to a map $E : \mathbb{R}^n \to \mathbb{R}^n$. Then the set*

$$S = \bigcap_i^k \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, 2, ..., k\}$$

is strongly E-convex set.

Proof. By the Theorem 3.12, it is obvious that the set

$$S_i = \{x \in \mathbb{R}^n : g_i(x) \leq 0\}$$

is strongly E-convex, i=1,2,...,k, which implies that the set $S = \bigcap_i^k S_i$ is a strongly E-convex set. $\square$

Theorem 3.16. *Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set. If $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable quasi strongly E-convex function on a strongly E-convex set $S \subseteq \mathbb{R}^n$ with $f(E(x)) \leq f(E(y))$, then*

$$(E(x) - E(y))\nabla f(E(y)) \leq 0, \forall\ x, y \in S.$$

Proof. Since, f is a quasi strongly E-convex function on a strongly E-convex set S. Then for any $x, y \in S, \alpha \in [0, 1]$ and $\beta \in [0, 1]$, we have

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \max\{f(E(x)), f(E(y))\}.$$

As $f(E(x)) \leq f(E(y))$,

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq f(E(y)),$$

or

$$f(\alpha y + E(y)) + \beta(\alpha x + E(x) - \alpha y - E(y))) \leq f(E(y)).$$

Since, f is differentiable, we have

$$f(\alpha y + E(y)) + \beta[\alpha x + E(x) - \alpha y - E(y)]\nabla f(\alpha y + E(y)) + 0(\beta^2) \leq f(E(y)).$$

Now taking limit as $\alpha \to 0$, we get

$$f(E(y)) + \beta[E(x) - E(y)]\nabla f(E(y)) + 0(\beta^2) \leq f(E(y)).$$

Dividing the above inequality by $\beta > 0$ and taking $\beta \to 0$, we get

$$(E(x) - E(y))\nabla f(E(y)) \leq 0, \ \ \forall x, y \in S.$$

Hence, we have the desired result. □

An immediate consequence of Theorem 3.16 is as follows:

Corollary 3.17. *Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set. If the function $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable quasi strongly E-convex at $y \in S$ and y is a fixed point of the map E, then*

$$(E(x) - y)\nabla f(y) \leq 0, \ \ \forall\ x \in S.$$

Now, we define pseudo strongly E-convex function on strongly E-convex set as follows:

Definition 3.18. A real valued function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be pseudo strongly E-convex with respect to a map E on a strongly E-convex set $S \subseteq \mathbb{R}^n$ if there exists a strictly positive function $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $f(E(x)) < f(E(y))$, then

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq f(E(y)) + \beta(\beta - 1)b(E(x), E(y)),$$

for all $x, y \in S, \alpha \in [0, 1]$ and $\beta \in [0, 1]$.

For $\alpha = 0$, f reduces to E-pseudo convex function. For $\alpha = 0$ and if the map E is an identity map, then f reduces to pseudo convex function.

In the following theorem, we discuss a relationship between strongly E-convex function and pseudo strongly E-convex function.

Theorem 3.19. *A strongly E-convex function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ on a strongly E-convex set S, is a pseudo strongly E-convex function on S.*

Proof. Let $f(E(x)) < f(E(y))$. Since f is a strongly E-convex function on a strongly E-convex set $S \subseteq \mathbb{R}^n$, for all $x, y \in S, \alpha \in [0, 1]$ and $\beta \in [0, 1]$, we have

$$\begin{aligned} & f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \\ & \quad \leq \beta f(E(x)) + (1-\beta) f(E(y)) \\ & \quad = f(E(y)) + \beta(f(E(x)) - f(E(y)) \\ & \quad \leq f(E(y)) + \beta(1-\beta)(f(E(x)) - f(E(y)) \\ & \quad = f(E(y)) + \beta(\beta-1)(f(E(y)) - f(E(x)) \\ & \quad = f(E(y)) + \beta(\beta-1)(b(E(x), (E(y)), \end{aligned}$$

where $b(E(x), E(y)) = f(E(y)) - f(E(x)) > 0$. Therefore, f is a pseudo strongly E-convex function on strongly E-convex $S \subseteq \mathbb{R}^n$. □

4. NONLINEAR PROGRAMMING PROBLEM FOR QUASI STRONGLY E-CONVEX FUNCTIONS

In this section, we consider the quasi strongly E-convex programming problem which generalizes the results obtained by Youness [9] and Kaul et al. [5].

Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping and $f : \mathbb{R}^n \to \mathbb{R}$, $f_j : \mathbb{R}^n \to \mathbb{R}, j = 1, 2, ..., m$, be quasi strongly E-convex functions on $\mathbb{R}^n$. A quasi strongly E-convex programming problem is formulated as follows:

$$(P) \ \min f(x)$$

subject to

$$x \in S = \{x \in E(\mathbb{R}^n) : f_j(x) \leq 0, j = 1, 2, ..., m\}.$$

Theorem 4.1. *Let $S \subseteq \mathbb{R}^n$ be a strongly E-convex set, and $f : \mathbb{R}^n \to \mathbb{R}$ be a quasi strongly E-convex function on S. If $E(x^*) \in E(S)$, is a local minimum of problem (P), then $E(x^*)$ is a global minimum of problem (P) on S.*

Proof. Let $E(x^*) \in E(S)$ be a non global minimum of the problem (P) on S. Then there is $E(y) \in S$ such that $f(E(y)) \leq f(E(x^*))$. Since the function $f : \mathbb{R}^n \to \mathbb{R}$ is quasi strongly E-convex, it implies that

$$\begin{aligned} f(\beta(\alpha y + E(y)) + (1-\beta)(\alpha x^* + E(x^*))) &\leq \max\{f(E(y)), f(E(x^*))\} \\ &\leq f(E(x^*)). \end{aligned}$$

By putting $\alpha = 0$, we get

$$f(\beta E(y) + (1-\beta)E(x^*)) \leq f(E(x^*)),$$

for any small $\beta \in (0,1)$, which contradicts the local optimality of $f(E(x^*))$ for problem (P). Hence, $E(x^*)$ is a global minimum of problem (P) on S. □

Theorem 4.2. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly quasi strongly E-convex function on a strongly E-convex set $S \subseteq \mathbb{R}^n$, then the global optimal solution of problem (P) is unique.*

Proof. Let $E(x), E(y) \in S$ be two distinct global optimal solutions of the problem (P). Then, $f(E(x)) = f(E(y))$. Since S is a strongly E-convex set and $f : \mathbb{R}^n \to \mathbb{R}$ is a strictly quasi strongly E-convex function,

$$\begin{aligned} f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) &< \max\{f(E(x)), f(E(y))\} \\ &= f(E(x)), \end{aligned}$$

for each $\beta \in (0,1)$, which contradicts the optimality of $E(x)$ for problem (P). Hence, the global optimal solution of problem (P) is unique. □

Theorem 4.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a quasi strongly E-convex function on a strongly E-convex set $S \subseteq \mathbb{R}^n$ and let $\beta = \min_{x \in S} f(E(x))$. Then, the set $X = \{E(x) \in S : f(E(x)) = \beta\}$ of optimal solutions of the problem (P) is strongly E-convex. If f is a strictly quasi strongly E-convex function on a strongly E-convex set $S \subseteq \mathbb{R}^n$, then, the set X is a singleton.*

Proof. Let $E(x), E(y) \in S$ be two distinct global optimal solutions of the problem (P). Then, $f(E(x)) = \beta$, $f(E(y)) = \beta$. Since $f : \mathbb{R}^n \to \mathbb{R}$ is a quasi strongly E-convex function on a strongly E-convex set $S \subseteq \mathbb{R}^n$,

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) \leq \max\{f(E(y)), f(E(y))\} = \beta,$$

which implies that

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in X.$$

Thus, it follows that X is strongly E-convex.

For the other part, assume on contrary that $E(x), E(y) \in X$, such that $E(x) \neq E(y)$. For all $\alpha \in [0,1]$ and $\beta \in (0,1)$, we have

$$\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y)) \in S.$$

Further, since f is a strictly quasi strongly E-convex function on S, we have

$$f(\beta(\alpha x + E(x)) + (1-\beta)(\alpha y + E(y))) < \max\{f(E(y)), f(E(y))\} = \beta.$$

Which contradicts that $\beta = \min_{x \in S} f(E(x))$. Hence, we have the desired result. $\square$

5. CONCLUSION

The strong concept of quasi strongly E-convexity has been introduced, and few examples have been constructed to support our definitions. A characterization and relationships of these functions have been discussed. A nonlinear programming problem has been taken for quasi strongly E-convex functions and the existence and uniqueness of the global optimal solution has been discussed. Our findings extend previously established results and can be explored to Riemannian manifolds in the future.

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