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A GENERALIZED CLASS OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH AL-OBOUDI OPERATOR INVOLVING CONVOLUTION

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Abstract. In this paper, we have introduced a generalized class $S_H^i(m, n, \gamma, \phi, \psi; \alpha)$, $i \in \{0, 1\}$ of harmonic univalent functions in unit disc $\mathbb{U}$, a sufficient coefficient condition for the normalized harmonic function in this class is obtained. It is also shown that this coefficient condition is necessary for its subclass $\mathcal{T}S_H^i(m, n, \gamma, \phi, \psi; \alpha)$. We further obtained extreme points, bounds and a covering result for the class $\mathcal{T}S_H^i(m, n, \gamma, \phi, \psi; \alpha)$. Also, show that this class is closed under convolution and convex combination. While proving our results, certain conditions related to the coefficients of ϕ and ψ are considered, which lead to various well-known results.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real harmonic in $\mathbb{D}$. In any simply connected domain $\mathbb{D}$, we can write $f = h + \bar{g}$. We call h

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the analytic part and g the co-analytic part of function f. A necessary and sufficient condition for function f to be locally univalent and sense-preserving in $\mathbb{D}$ is that $\left|h'(z)\right| > \left|g'(z)\right|$, $z \in \mathbb{D}$ (see [4]).

Denote by S_H the class of functions $f = h + \bar{g}$ which are harmonic, univalent and sense-preserving in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad z \in U. \tag{1.1}$$

Therefore

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad |b_1| < 1.$$

Note that S_H reduces normalized analytic univalent functions to the class if the co-analytic part of function f is identically zero, i.e. $g \equiv 0$. For this class, $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

For more basic results on harmonic functions one may refer to the following book by Duren [8] (see also [1],[13],[14],[15]). For $f = h + \bar{g}$ with h and g are of the form (1.1), [2] defined the Al-Oboudi operator D_γ^n *for* $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by

$$D_\gamma^n f(z) = D_\gamma^n h(z) + (-1)^n \overline{D_\gamma^n g(z)}, \tag{1.3}$$

where

$$D_\gamma^n h(z) = z + \sum_{k=2}^{\infty} [1 + (k-1)\gamma]^n a_k z^k, \quad D_\gamma^n g(z) = \sum_{k=1}^{\infty} [1 + (k-1)\gamma]^n b_k z^k.$$

Several authors such as [5],[6],[7],[9],[12] and [16] introduced and studied various new subclasses of analytic univalent as well as harmonic univalent functions with the help of convolution.

We motivated by the earlier work of Jahangiri et al. [11] and Sharma et al. [17] for subclasses of S_H, in this paper, we define a generalized class $S_H^i(m, n, \gamma, \phi, \psi; \alpha)$ of functions $f = h + \bar{g} \in S_H$ satisfying for $i \in \{0, 1\}$, the condition

$$\Re\left\{ \frac{D_\gamma^m h(z) * \phi(z) + (-1)^{m+i} \overline{D_\gamma^m g(z) * \psi(z)}}{D_\gamma^n h(z) + (-1)^n \overline{D_\gamma^n g(z)}} \right\} > \alpha, \tag{1.4}$$

where, $m, n \in \mathbb{N}_0$, $m \geq n$, $0 \leq \alpha < 1$, $\gamma \geq 1$, $\phi(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k$ and $\psi(z) = z + \sum_{k=2}^{\infty} \mu_k z^k$ are analytic in open unit disk $\mathbb{U}$ with the conditions $\lambda_k \geq 1$, $\mu_k \geq 1$. The operator "$*$" stands for the Hadamard product or convolution of two power series.

We further denote by $\mathcal{T}S_H^i(m, n, \gamma, \phi, \psi; \alpha)$, a subclass of $S_H^i(m, n, \gamma, \phi, \psi; \alpha)$ consisting of functions $f = h + \overline{g} \in S_H$ such that h and g are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1. \tag{1.5}$$

Interestingly, we obtain the following known subclasses of S_H studied earlier by various researchers by specializing the parameters.

(i) Yalcin [20] has studied the subclasses
$S_H^0\left(m, n, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha\right) \equiv S_H(m, n; \alpha)$
and
$\mathcal{T}S_H^0\left(m, n, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha\right) \equiv \mathcal{T}S_H(m, n; \alpha)$;

(ii) Jahangiri et al [11] has studied the subclasses
$S_H^0\left(n+1, n, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha\right) \equiv S_H(n; \alpha)$
and
$\mathcal{T}S_H^0\left(n+1, n, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha\right) \equiv \mathcal{T}S_H(n; \alpha)$;

(iii) Jahangiri [10] has studied the subclasses
$S_H^0\left(1, 0, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha\right) \equiv S_H^*(\alpha)$
and
$\mathcal{T}S_H^0\left(1, 0, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha\right) \equiv \mathcal{T}S_H^*(\alpha)$;

(iv) Jahangiri [10] has studied the subclasses
$S_H^0\left(2, 1, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha\right) \equiv \mathcal{K}_H(\alpha)$
and
$\mathcal{T}S_H^0\left(2, 1, 1, \frac{z}{1-z}, \frac{z}{1-z}; \alpha\right) \equiv \mathcal{K}_H(\alpha)$;

(v) Frasin [9] has studied the subclasses
$S_H^1(0, 0, 1, \phi, \psi; \alpha) \equiv S_H(\phi, \psi; \alpha)$
and
$\mathcal{T}S_H^1(0, 0, 1, \phi, \psi; \alpha) \equiv \mathcal{T}S_H(\phi, \psi; \alpha)$;

(vi) Silverman [18], Silverman and Silvia [19] (also see [3]) has studied the subclasses $S_H^0\left(2,1,1,\frac{z}{1-z},\frac{z}{1-z};0\right)\equiv\mathcal{K}_H$,
$\mathcal{T}S_H^0\left(2,1,1,\frac{z}{1-z},\frac{z}{1-z};0\right)\equiv\mathcal{T}\mathcal{K}_H$,
$S_H^0\left(1,0,1,\frac{z}{1-z},\frac{z}{1-z};0\right)\equiv S_H^*$
and
$\mathcal{T}S_H^0\left(1,0,1,\frac{z}{1-z},\frac{z}{1-z};0\right)\equiv\mathcal{T}S_H^*$;

(vii) Sharma et al [17] has studied the subclasses
$S_H^i(m,n,1,\phi,\psi;\alpha)\equiv S_H^i(m,n,\phi,\psi;\alpha)$
and
$\mathcal{T}S_H^i(m,n,1,\phi,\psi;\alpha)\equiv\mathcal{T}S_H^i(m,n,\phi,\psi;\alpha)$.

In the present paper, we prove some sharp results including, coefficient inequality, bounds, extreme points, convolution and convex combination for functions in $\mathcal{T}S_H^i(m,n,\phi,\psi,\gamma;\alpha)$ under certain conditions on the coefficients of ϕ and ψ.

2. Main results

We begin with a sufficient coefficient condition for functions to be in class $S_H^i(m,n,\gamma,\phi,\psi;\alpha)$.

Theorem 2.1. *Let the function $f=h+\bar{g}$, where h and g are of the form* (1.1)*, satisfies*

$$\sum_{k=2}^{\infty}\frac{\lambda_k[1+(k-1)\gamma]^m-\alpha[1+(k-1)\gamma]^n}{1-\alpha}|a_k| \\ +\sum_{k=1}^{\infty}\frac{\mu_k[1+(k-1)\gamma]^m-(-1)^{m+i-n}\alpha[1+(k-1)\gamma]^n}{1-\alpha}|b_k|\le 1, \tag{2.1}$$

where, $i\in\{0,1\}$, $m\in\mathbb{N}_0$, $n\in\mathbb{N}_0$, $m\ge n$, $0\le\alpha<1$, $\gamma\ge 1$, $\lambda_k\ge 1$, $\mu_k\ge 1$, $k\ge 1$. In case $m=n=0$, $\lambda_k\ge k$, $\mu_k\ge k$, $k\ge 1$. Then f is sense-preserving, harmonic univalent in $\mathbb{U}$ and $f\in S_H^i(m,n,\gamma,\phi,\psi;\alpha)$.

Proof. Under the given hypothesis, we note that for

$$k\ge 1,\quad k\le\frac{\lambda_k[1+(k-1)\gamma]^m-\alpha[1+(k-1)\gamma]^n}{1-\alpha}, \\ k\le\frac{\mu_k[1+(k-1)\gamma]^m-(-1)^{m+i-n}\alpha[1+(k-1)\gamma]^n}{1-\alpha}. \tag{2.2}$$

Hence, for $f = h + \overline{g}$, where h and g are of the form (1.1), we get that

$$\begin{aligned}
\left|h^{'}(z)\right| &\geq 1 - \sum_{k=2}^{\infty} k|a_k|r^{k-1} > 1 - \sum_{k=2}^{\infty} k|a_k| \\
&> 1 - \sum_{k=2}^{\infty} \frac{\lambda_k \left[1+(k-1)\gamma\right]^m - \alpha \left[1+(k-1)\gamma\right]^n}{1-\alpha} |a_k| \\
&\geq \sum_{k=1}^{\infty} \frac{\mu_k \left[1+(k-1)\gamma\right]^m - (-1)^{m+i-n}\alpha \left[1+(k-1)\gamma\right]^n}{1-\alpha} |b_k| \\
&\geq \sum_{k=1}^{\infty} k|b_k| > \sum_{k=1}^{\infty} k|b_k|r^{k-1} \geq \left|g^{'}(z)\right|,
\end{aligned}$$

which proves that f is sense preserving in $\mathbb{U}$. To show that f is univalent in $\mathbb{U}$, suppose $z_1, z_2 \in \mathbb{U}$ such that $z_1 \neq z_2$, then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &= 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_2 - z_1) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)}\right| \\
&> 1 - \left|\frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_{k|}}\right| \\
&\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{\mu_k \left[1+(k-1)\gamma\right]^m - (-1)^{m+i-n}\alpha \left[1+(k-1)\gamma\right]^n}{1-\alpha} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{\lambda_k \left[1+(k-1)\gamma\right]^m - \alpha \left[1+(k-1)\gamma\right]^n}{1-\alpha} |a_k|} \\
&\geq 0.
\end{aligned}$$

Now, to show that $f \in S_H^i(m, n, \gamma, \phi, \psi; \alpha)$, we use the fact that $\mathfrak{Re}\{\omega\} \geq \alpha$, if and only if $|1 - \alpha + \omega| \geq |1 + \alpha - \omega|$.

Hence, it suffices to show that

$$Q(z) = |A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0, \tag{2.3}$$

where, $A(z) = D_\gamma^m h(z) * \phi(z) + (-1)^{m+i}\overline{D_\gamma^m g(z) * \psi(z)}$ and
$B(z) = D_\gamma^n h(z) + (-1)^n \overline{D_\gamma^n g(z)}$.

Substituting the corresponding series expansions in the expressions of $A(z)$ and $B(z)$, we obtain from (2.3), that

$$\begin{aligned}
Q(z) = &\left| \begin{array}{l} (2-\alpha)z + \displaystyle\sum_{k=2}^{\infty} \left\{ \begin{array}{l} \lambda_k \left[1+(k-1)\gamma\right]^m \\ +(1-\alpha)\left[1+(k-1)\gamma\right]^n \end{array} \right\} a_k z^k \\ +(-1)^{m+i} \displaystyle\sum_{k=1}^{\infty} \left\{ \begin{array}{l} \mu_k \left[1+(k-1)\gamma\right]^m \\ +(-1)^{m+i-n}(1-\alpha)\left[1+(k-1)\gamma\right]^n \end{array} \right\} \overline{b_k z^k} \end{array} \right| \\
&- \left| \begin{array}{l} -\alpha z + \displaystyle\sum_{k=2}^{\infty} \left\{ \begin{array}{l} \lambda_k \left[1+(k-1)\gamma\right]^m \\ -(1+\alpha)\left[1+(k-1)\gamma\right]^n \end{array} \right\} a_k z^k \\ +(-1)^{m+i} \displaystyle\sum_{k=1}^{\infty} \left\{ \begin{array}{l} \mu_k \left[1+(k-1)\gamma\right]^m \\ -(-1)^{m+i-n}(1+\alpha)\left[1+(k-1)\gamma\right]^n \end{array} \right\} \overline{b_k z^k} \end{array} \right| \\
&> 2 \left\{ \begin{array}{l} (1-\alpha) - \displaystyle\sum_{k=2}^{\infty} \lambda_k \left[\begin{array}{l} (1+(k-1)\gamma)^m \\ -\alpha(1+(k-1)\gamma)^n \end{array} \right] |a_k| \\ - \displaystyle\sum_{k=1}^{\infty} \mu_k \left[\begin{array}{l} (1+(k-1)\gamma)^m \\ -(-1)^{m+i-n}\alpha(1+(k-1)\gamma)^n \end{array} \right] |b_k| \end{array} \right\} \\
&\geq 0.
\end{aligned}$$

Hence inequality (2.3) satisfied. This proves the Theorem 2.1.

Sharpness of the coefficient inequality (2.1) can be seen by the function

$$\begin{aligned}
f(z) = &z + \sum_{k=2}^{\infty} \frac{1-\alpha}{\lambda_k(1+(k-1)\gamma)^m - \alpha(1+(k-1)\gamma)^n} x_k z^k \\
&+ \sum_{k=1}^{\infty} \frac{1-\alpha}{\mu_k(1+(k-1)\gamma)^m - (-1)^{m+i-n}\alpha(1+(k-1)\gamma)^n} \overline{y_k z^k},
\end{aligned}$$

where, $i \in \{0,1\}, m \in \mathbb{N}, n \in \mathbb{N}_0, m \geq n, 0 \leq \alpha < 1, \lambda_k \geq 1, \mu_k \geq 1, k \geq 1$.
In case $m = n = 0, \lambda_k \geq k, \mu_k \geq k, k \geq 1$, $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$. □

Next, we have show that the above sufficient coefficient condition is also necessary for functions in the class $\mathcal{T}S_H^i(m, n, \gamma, \phi, \psi; \alpha)$.

Theorem 2.2. *Let the function $f = h + \bar{g}$, be such that h and g are given by (1.5). Then, $f \in \mathcal{T}S^i_H(m, n, \gamma, \phi, \psi; \alpha)$ if and only if*

$$\sum_{k=1}^{\infty} \frac{\lambda_k [1+(k-1)\gamma]^m - \alpha [1+(k-1)\gamma]^n}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{\mu_k [1+(k-1)\gamma]^m - (-1)^{m+i-n}\alpha [1+(k-1)\gamma]^n}{1-\alpha}|b_k| \leq 2, \tag{2.4}$$

where, $a_1 = 1, m \in \mathbb{N}, n \in \mathbb{N}_0, m \geq n, 0 \leq \alpha < 1, \gamma \geq 1, \lambda_k \geq 1, \mu_k \geq 1, k \geq 1$. In case $m = n = 0, \lambda_k \geq k, \mu_k \geq k, k \geq 1$.

Proof. The if part, follows from Theorem 2.1. To prove the "only if" part, let $f \in \mathcal{T}S^i_H(m, n, \gamma, \phi, \psi; \alpha)$, then from (1.4), we have

$$\mathfrak{Re}\left\{ \frac{D^m_\gamma h(z) * \phi(z) + (-1)^{m+i}\overline{D^m_\gamma g(z) * \psi(z)}}{D^n_\gamma h(z) + (-1)^n \overline{D^n_\gamma g(z)}} - \alpha \right\} > 0, \quad z \in \mathbb{U},$$

which is equivalent to

$$\mathfrak{Re}\left\{ \frac{(1-\alpha)z - \sum_{k=2}^{\infty} \{\lambda_k [1+(k-1)\gamma]^m - \alpha [1+(k-1)\gamma]^n\} |a_k| z^k + (-1)^{2m+2i-1} \sum_{k=1}^{\infty} \left\{ \begin{array}{l} \mu_k [1+(k-1)\gamma]^m \\ -(-1)^{m+i-n}\alpha [1+(k-1)\gamma]^n \end{array} \right\} |b_k| \overline{z}^k}{z - \sum_{k=2}^{\infty} [1+(k-1)\gamma]^n |a_k| z^k + (-1)^{m+i-1} [1+(k-1)\gamma]^n |b_k| z^k} \right\} > 0.$$

If we choose z to be real and $z \to 1$, we get

$$\left\{ \begin{array}{l} \left(\dfrac{(1-\alpha) - \sum_{k=2}^{\infty} \{\lambda_k [1+(k-1)\gamma]^m - \alpha [1+(k-1)\gamma]^n\} |a_k|}{1 - \sum_{k=2}^{\infty} [1+(k-1)\gamma]^n |a_k| + (-1)^{m+i-1} [1+(k-1)\gamma]^n |b_k|} \right) \\ - \left(\dfrac{\sum_{k=1}^{\infty} \{\mu_k [1+(k-1)\gamma]^m - (-1)^{m+i-n}\alpha [1+(k-1)\gamma]^n\} |b_k|}{1 - \sum_{k=2}^{\infty} [1+(k-1)\gamma]^n |a_k| + (-1)^{m+i-1} [1+(k-1)\gamma]^n |b_k|} \right) \end{array} \right\} \geq 0$$

or, equivalently,

$$\left\{\begin{array}{l}\sum_{k=2}^{\infty}\{\lambda_k\left[1+(k-1)\gamma\right]^m-\alpha\left[1+(k-1)\gamma\right]^n\}\left|a_k\right| \\ +\sum_{k=1}^{\infty}\left\{\mu_k\left[1+(k-1)\gamma\right]^m-(-1)^{m+i-n}\alpha\left[1+(k-1)\gamma\right]^n\right\}\left|b_k\right|\end{array}\right\}\leq 1-\alpha,$$

which is the required condition (2.4). This completes the proof. □

For the classes $\mathcal{T}S_H(m,n,\gamma;\alpha)$ and $\mathcal{T}S_H(\phi,\psi;\alpha)$ mentioned in the main results, Theorem 2.2 yields the following results, which include the results for other known classes discussed in main results.

Corollary 2.3. ([16]) *Let the function $f=h+\bar{g}$, be such that h and g are given by* (1.5)*. Then, $f\in\mathcal{T}S_H\left(m,n,\gamma;\alpha\right)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{\left[1+(k-1)\gamma\right]^m-\left[1+(k-1)\gamma\right]^n}{1-\alpha}\left|a_k\right| + \sum_{k=1}^{\infty}\frac{\left[1+(k-1)\gamma\right]^m-(-1)^{m-n}\alpha\left[1+(k-1)\gamma\right]^n}{1-\alpha}\left|b_k\right|\leq 2,$$

where $a_1=1$, $m\in\mathbb{N}$, $n\in\mathbb{N}_0$, $m\geq n$, $0\leq\alpha<1$.

Corollary 2.4. ([17]) *Let the function $f=h+\bar{g}$, be such that h and g are given by* (1.5)*. Then, $f\in\mathcal{T}S^i_H\left(m,n,\phi,\psi;\alpha\right)$ if and only if*

$$\sum_{k=1}^{\infty}\frac{\lambda_k k^m-\alpha k^n}{1-\alpha}\left|a_k\right|+\frac{\mu_k k^m-(-1)^{m+i-n}\alpha k^n}{1-\alpha}\left|b_k\right|\leq 2,$$

where, $a_1=1, m\in\mathbb{N}, n\in\mathbb{N}_0, m\geq n, 0\leq\alpha<1, \lambda_k\geq 1, \mu_k\geq 1, k\geq 1$. In case $m=n=0, \lambda_k\geq k, \mu_k\geq k, k\geq 1$.

Corollary 2.5. *Let the function $f=h+\bar{g}$, be such that h and g are given by* (1.5)*. Then, $f\in\mathcal{T}S_H\left(\phi,\psi;\alpha\right)$ if and only if*

$$\sum_{k=1}^{\infty}\left\{\frac{\lambda_k-\alpha}{1-\alpha}\left|a_k\right|+\frac{\mu_k+\alpha}{1-\alpha}\left|b_k\right|\right\}\leq 2,$$

where $a_1=1, \lambda_k\geq k, \mu_k\geq k, k\geq 1, 0\leq\alpha<1$.

3. Bounds

Our next theorem provides the bounds for the functions in $\mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$ which is followed by a covering result for this class.

Theorem 3.1. *Let $f = h + \bar{g}$, with h and g are of the form (1.5) belongs to the class $\mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$ for functions ϕ and ψ with non-decreasing sequences $\{\lambda_k\}$ and $\{\mu_k\}$ satisfying $\lambda_2 \geq \alpha$, $\mu_1 \geq (2-\alpha)$, $\mu_k \geq \lambda_2$, $k \geq 2$, $\gamma \geq 1$, then*

$$|f(z)| \leq (1+|b_1|)r + \left[1 - \frac{1-(-1)^{m+i-n}\alpha}{1-\alpha}|b_1|\right] \frac{(1-\alpha)r^2}{2^m\lambda_2 - \alpha 2^n}, |z| = r < 1 \quad (3.1)$$

and

$$|f(z)| \geq (1-|b_1|)r - \left[1 - \frac{1-(-1)^{m+i-n}\alpha}{1-\alpha}|b_1|\right] \frac{(1-\alpha)r^2}{2^m\lambda_2 - \alpha 2^n}, |z| = r < 1. \quad (3.2)$$

Proof. We only prove the result for upper bound. The result for the lower bound can similarly be obtained.

Let $\mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$, then on taking the absolute value of function f, we get for $|z| = r < 1$,

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty}\left[|a_k| + |b_k|\right] r^k \leq (1+|b_1|)r + r^2\sum_{k=2}^{\infty}\left[|a_k| + |b_k|\right] \\
&\leq (1+|b_1|)r + \left[\frac{(1-\alpha)r^2}{2^m\lambda_2 - \alpha 2^n}\right] \\
&\quad \times \left\{ \begin{aligned} &\sum_{k=2}^{\infty} \frac{\left(\lambda_k\left[1+(k-1)\gamma\right]^m - \alpha\left[1+(k-1)\gamma\right]^n\right)}{1-\alpha}|a_k| \\ &+ \sum_{k=2}^{\infty} \frac{\left(\mu_k\left[1+(k-1)\gamma\right]^m - (-1)^{m+i-n}\alpha\left[1+(k-1)\gamma\right]^n\right)}{1-\alpha}|b_k| \end{aligned} \right\} \\
&\leq (1+|b_1|)r + \left\{1 - \frac{1-(-1)^{m+i-n}\alpha}{1-\alpha}|b_1|\right\} \frac{(1-\alpha)r^2}{2^m\lambda_2 - \alpha 2^n}, \quad by \quad (2.4).
\end{aligned}$$

The bounds (3.1) and (3.2) are sharp for the function given by

$$f(z) = z + |b_1|\bar{z} + \left\{1 - \frac{1-(-1)^{m+i-n}\alpha}{1-\alpha}|b_1|\right\} \frac{(1-\alpha)\bar{z}^2}{2^m\lambda_2 - \alpha 2^n},$$

for $|b_1| < \frac{(1-\alpha)}{(1-(-1)^{m+i-n}\alpha)}$. A covering result follows from (3.2). The proof of the theorem is complete. □

Corollary 3.2. *Let $f \in S_H^i(m,n,\gamma,\phi,\psi;\alpha)$ and for functions ϕ and ψ with non-decreasing sequences $\{\lambda_k\}$ and $\{\mu_k\}$ satisfying $\lambda_2 \geq \alpha, \mu_1 \geq (2-\alpha), \mu_k \geq \lambda_2,\ k \geq 2, \gamma \geq 1$. Then*

$$\left\{\omega : |\omega| < \left[1 - \frac{(1-\alpha)}{2^m\lambda_2 - \alpha 2^n}\right] + \left[\frac{1-(-1)^{m+i-n}\alpha}{2^m\lambda_2 - \alpha 2^n} - 1\right]|b_1|\right\} \subset f(\mathbb{U}).$$

Further, for the classes $\mathcal{T}S_H(m,n,\gamma;\alpha)$ and $\mathcal{T}S_H\phi,\psi;\alpha)$, Theorem 3.1 yields following results which include the results for other known classes discussed in main results.

Corollary 3.3. ([16]) *Let $f = h+\overline{g}$ with h and g are of the form (1.5) belongs to the class $\mathcal{T}S_H(m,n,\gamma;\alpha)$, $\gamma \geq 1, 0 \leq \alpha < 1$ with non-decreasing sequences $\{\lambda_k\}$ and $\{\mu_k\}$ satisfying $\lambda_2 \geq \alpha, \mu_1 \geq (2-\alpha), \mu_k \geq \lambda_2, k \geq 2, \gamma \geq 1$. Then*

$$|f(z)| \leq (1+|b_1|)r + \left\{1 - \frac{1-(-1)^{m-n}\alpha}{1-\alpha}|b_1|\right\}\frac{(1-\alpha)r^2}{2^m - \alpha 2^n}, \quad |z| = r < 1$$

and

$$|f(z)| \geq (1-|b_1|)r - \left\{1 - \frac{1-(-1)^{m-n}\alpha}{1-\alpha}|b_1|\right\}\frac{(1-\alpha)r^2}{2^m - \alpha 2^n}, \quad |z| = r < 1.$$

Further,

$$\left\{\omega : |\omega| < \left[1 - \frac{(1-\alpha)}{2^m - \alpha 2^n}\right] + \left[\frac{1-(-1)^{m+i-n}\alpha}{2^m - \alpha 2^n} - 1\right]|b_1|\right\} \subset f(\mathbb{U}).$$

Corollary 3.4. *Let $f = h+\overline{g}$ with h and g are of the form (1.5) belongs to the class $\mathcal{T}S_H(\phi,\psi;\alpha)$, for function ϕ and ψ with non-decreasing sequences $\{\lambda_k\}$ and $\{\mu_k\}$ satisfying $\lambda_2 \geq \alpha,\ \mu_1 \geq (2-\alpha),\ \mu_k \geq \lambda_2,\ k \geq 2,\ \gamma \geq 1$. Then*

$$|f(z)| \leq (1+|b_1|)r + \left\{1 - \frac{1+\alpha}{1-\alpha}|b_1|\right\}\frac{(1-\alpha)r^2}{\lambda_2 - \alpha}, \quad |z| = r < 1 \tag{3.3}$$

and

$$|f(z)| \geq (1-|b_1|)r - \left\{1 - \frac{1+\alpha}{1-\alpha}|b_1|\right\}\frac{(1-\alpha)r^2}{\lambda_2 - \alpha}, \quad |z| = r < 1. \tag{3.4}$$

Further,

$$\left\{\omega : |\omega| < \frac{1}{\lambda_2 - \alpha}\left[\lambda_2 - 1 + (1 - \lambda_2 + 2\alpha)\right]|b_1|\right\} \subset f(\mathbb{U}).$$

4. Extreme points

In this section we determine the extreme points of $\mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$.

Theorem 4.1. *Let*

$$h_1(z)=z,$$

$$h_k(z)=z-\frac{1-\alpha}{\lambda_k\left[1+(k-1)\gamma\right]^m-\alpha\left[1+(k-1)\gamma\right]^n}z^k\ \ (k\geq 2)$$

and

$$g_k(z)=z+\frac{(-1)^{m+i-1}(1-\alpha)}{\mu_k\left[1+(k-1)\gamma\right]^m-(-1)^{m+i-n}\alpha\left[1+(k-1)\gamma\right]^n}\overline{z}^k\ \ (k\geq 1).$$

Then $f\in\mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$ *if and only if it can be expressed as*

$$f(z)=\sum_{k=1}^{\infty}\left[x_kh_k(z)+y_kg_k(z)\right], \tag{4.1}$$

where, $x_k\geq 0,\ y_k\geq 0$ *and* $\sum_{k=1}^{\infty}(x_k+y_k)=1.$

In particular, the extreme points of $\mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. Suppose that

$$f(z)=\sum_{k=1}^{\infty}\left[x_kh_k(z)+y_kg_k(z)\right].$$

Then,

$$\begin{aligned}f(z)=&\sum_{k=1}^{\infty}\left[x_k+y_k\right]z-\sum_{k=2}^{\infty}\frac{1-\alpha}{\lambda_k\left[1+(k-1)\gamma\right]^m-\alpha\left[1+(k-1)\gamma\right]^n}x_kz^k\\&+(-1)^{m+i-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{\mu_k\left[1+(k-1)\gamma\right]^m-(-1)^{m+i-n}\alpha\left[1+(k-1)\gamma\right]^n}y_k\overline{z}^k\\=&z-\sum_{k=2}^{\infty}\frac{1-\alpha}{\lambda_k\left[1+(k-1)\gamma\right]^m-\alpha\left[1+(k-1)\gamma\right]^n}x_kz^k\\&+(-1)^{m+i-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{\mu_k\left[1+(k-1)\gamma)\right]^m-(-1)^{m+i-n}\alpha\left[1+(k-1)\gamma)\right]^n}y_k\overline{z}^k.\end{aligned}$$

We show that the function $f(z) \in \mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$.
Since,

$$\sum_{k=2}^{\infty}\left\{\begin{array}{l}\left(\dfrac{\lambda_k\left[1+(k-1)\gamma\right]^m-\alpha\left[1+(k-1)\gamma\right]^n}{1-\alpha}\right)\\ \times\left(\dfrac{1-\alpha}{\lambda_k(1+(k-1)\gamma)^m-\alpha(1+(k-1)\gamma)^n}\right)\end{array}\right\}x_k$$
$$+\sum_{k=1}^{\infty}\left\{\begin{array}{l}\left(\dfrac{\mu_k\left[1+(k-1)\gamma\right]^m-(-1)^{m+i-n}\alpha\left[1+(k-1)\gamma\right]^n}{1-\alpha}\right)\\ \times\left(\dfrac{1-\alpha}{\mu_k(1+(k-1)\gamma)^m-(-1)^{m+i-n}\alpha(1+(k-1)\gamma)^n}\right)\end{array}\right\}y_k$$
$$=\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k$$
$$=1-x_1$$
$$\leq 1.$$

Thus $f(z) \in \mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$.

Conversely, If $f \in \mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$, then

$$|a_k| \leq \frac{1-\alpha}{\lambda_k(1+(k-1)\gamma)^m-\alpha(1+(k-1)\gamma)^n}, \quad k \geq 2$$

and

$$|b_k| \leq \frac{1-\alpha}{\mu_k(1+(k-1)\gamma)^m-(-1)^{m+i-n}\alpha(1+(k-1)\gamma)^n}, \quad k \geq 1.$$

Setting

$$x_k = \frac{\lambda_k\left[1+(k-1)\gamma\right]^m-\alpha\left[1+(k-1)\gamma\right]^n}{1-\alpha}|a_k|, \quad k \geq 2$$

and

$$y_k = \frac{1-\alpha}{\mu_k(1+(k-1)\gamma)^m-(-1)^{m+i-n}\alpha(1+(k-1)\gamma)^n}|b_k|, \quad k \geq 1.$$

Then, by Theorem 2.2,

$$\sum_{k=2}^{\infty}x_k+\sum_{k=1}^{\infty}y_k \leq 1.$$

We define

$$x_1 = 1-\sum_{k=2}^{\infty}x_k-\sum_{k=1}^{\infty}y_k \geq 0.$$

Consequently, we can see that $f(z)$ can be expressed in the form (4.1). This completes the proof. □

5. Convolution and convex combination

In this section, we show that the class $\mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$ is invariant under convolution and convex combination of harmonic functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |B_k| \overline{z}^k.$$

We define the convolution

$$(f*F)(z) = f(z)*F(z) = z - \sum_{k=2}^{\infty} |a_k A_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k B_k| \overline{z}^k.$$

Theorem 5.1. *If $f \in \mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$ and $F \in \mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$ then $f*F \in \mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$, where $a_1 = A_1 = 1, m \in \mathbb{N}, n \in \mathbb{N}_0, m \geq n, 0 \leq \alpha < 1, \gamma \geq 1, \lambda_k \geq 1, \mu_k \geq 1, k \geq 1$. In case $m = n = 0, \lambda_k \geq k, \mu_k \geq k, k \geq 1$.*

Proof. Let

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

and

$$F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |B_k| \overline{z}^k$$

be in $\mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$. Then by Theorem 2.2, we have

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{\lambda_k [1+(k-1)\gamma]^m - \alpha [1+(k-1)\gamma]^n}{1-\alpha} |a_k| \\ &\quad + \sum_{k=1}^{\infty} \frac{\mu_k [1+(k-1)\gamma]^m - (-1)^{m+i-n} \alpha [1+(k-1)\gamma]^n}{1-\alpha} |b_k| \leq 1 \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{\lambda_k [1+(k-1)\gamma]^m - \alpha [1+(k-1)\gamma]^n}{1-\alpha} |A_k| \\ &\quad + \sum_{k=1}^{\infty} \frac{\mu_k [1+(k-1)\gamma]^m - (-1)^{m+i-n} \alpha [1+(k-1)\gamma]^n}{1-\alpha} |B_k| \leq 1. \end{aligned} \tag{5.2}$$

From (5.2), we conclude that $|A_k| \le 1, K = 2, 3, ...$ and $|B_k| \le 1, K = 1, 2, ...$ So, for $f * F$, we may write

$$\sum_{k=2}^{\infty} \frac{\lambda_k [1+(k-1)\gamma]^m - \alpha [1+(k-1)\gamma]^n}{1-\alpha} |a_k A_k|$$
$$+ \sum_{k=1}^{\infty} \frac{\mu_k [1+(k-1)\gamma]^m - (-1)^{m+i-n}\alpha [1+(k-1)\gamma]^n}{1-\alpha} |b_k B_k|$$
$$\le \sum_{k=2}^{\infty} \frac{\lambda_k [1+(k-1)\gamma]^m - \alpha [1+(k-1)\gamma]^n}{1-\alpha} |a_k|$$
$$+ \sum_{k=1}^{\infty} \frac{\mu_k [1+(k-1)\gamma]^m - (-1)^{m+i-n}\alpha [1+(k-1)\gamma]^n}{1-\alpha} |b_k| \le 1.$$

Thus $f * F \in \mathcal{T}S_H^i(m, n, \gamma, \phi, \psi; \alpha)$. This completes the proof. □

In the following theorem, we prove that $\mathcal{T}S_H^i(m, n, \gamma, \phi, \psi; \alpha)$ is closed under convex combination.

Theorem 5.2. *The class $\mathcal{T}S_H^i(m, n, \gamma, \phi, \psi; \alpha)$ is closed under convex combination, where $m \in \mathbb{N}, n \in \mathbb{N}_0, m \ge n, 0 \le \alpha < 1, \gamma \ge 1, \lambda_k \ge 1, \mu_k \ge 1, k \ge 1$. In case $m = n = 0, \lambda_k \ge k, \mu_k \ge k, k \ge 1$.*

Proof. For $j = 1, 2, ...$, suppose that $f_j \in \mathcal{T}S_H^i(m, n, \gamma, \phi, \psi; \alpha)$ where $f_j(z)$ is given by

$$f_j(z) = z - \sum_{k=2}^{\infty} |a_{j,k}| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_{j,k}| \bar{z}^k.$$

Then, by Theorem 2.2, we have

$$\sum_{k=1}^{\infty} \frac{\lambda_k [1+(k-1)\gamma]^m - \alpha [1+(k-1)\gamma]^n}{1-\alpha} |a_{j,k}|$$
$$+ \sum_{k=1}^{\infty} \frac{\mu_k [1+(k-1)\gamma]^m - (-1)^{m+i-n}\alpha [1+(k-1)\gamma]^n}{1-\alpha} |b_{j,k}| \le 2. \tag{5.3}$$

For $\sum_{j=1}^{\infty} t_j = 1, 0 \le t_j \le 1$, the convex combination of $f_j(z)$ may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z - \sum_{k=2}^{\infty} \sum_{j=1}^{\infty} t_j |a_{j,k}| z^k + (-1)^{m+i-1} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} t_j |b_{j,k}| \bar{z}^k.$$

Now

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\lambda_k \left[1+(k-1)\gamma\right]^m - \alpha\left[1+(k-1)\gamma\right]^n}{1-\alpha} \sum_{1=1}^{\infty} t_j \left|a_{j,k}\right| \\
&\quad + \sum_{k=1}^{\infty} \frac{\mu_k \left[1+(k-1)\gamma\right]^m - (-1)^{m+i-n}\alpha\left[1+(k-1)\gamma\right]^n}{1-\alpha} \sum_{j=1}^{\infty} t_j \left|b_{j,k}\right| \\
&= \sum_{j=1}^{\infty} t_i \sum_{k=1}^{\infty} \frac{\lambda_k \left[1+(k-1)\gamma\right]^m - \alpha\left[1+(k-1)\gamma\right]^n}{1-\alpha} \left|a_{j,k}\right| \\
&\quad + \sum_{j=1}^{\infty} t_i \sum_{k=1}^{\infty} \frac{\mu_k \left[1+(k-1)\gamma\right]^m - (-1)^{m+i-n}\alpha\left[1+(k-1)\gamma\right]^n}{1-\alpha} \left|b_{j,k}\right| \\
&\leq 2\sum_{j=1}^{\infty} t_i \\
&= 2.
\end{aligned}$$

and so, by Theorem 2.2, we have $\sum_{j=1}^{\infty} t_i f_i(z) \in \mathcal{T}S_H^i(m,n,\gamma,\phi,\psi;\alpha)$. This completes the proof. □

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