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FIXED POINT THEOREMS FOR (ξ, β)-EXPANSIVE MAPPING IN $\mathcal{G}$-METRIC SPACE USING CONTROL FUNCTION

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Abstract. In this paper, some fixed point theorems for new type of (ξ, β)-expansive mappings of type (S) and type (T) using control function and β-admissible function in $\mathcal{G}$-metric spaces are proved. Further, we prove certain fixed point results by relaxing the continuity condition.

1. Introduction

In 2011, Imdad et al. [6] generalized some common fixed point results for expansive mappings in symmetric spaces. Afterwards, some researchers established fixed point results for expansive mappings in complete metric spaces,

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cone metric spaces and 2-metric spaces (see [5], [12], [15]). In 2013, Shabani and Razani [14] investigated the solutions of minimization problem for noncyclic functions in the context of $\mathcal{G}$-metric spaces. In 2014, Karapinar [8] proved some interesting results for (ξ, α)-contractive mappings in generalized metric space. In 2010, Mustafa et al. [10] proved some fixed point results for expansive mappings in $\mathcal{G}$-metric spaces.

Afterwards, many researchers proved some fixed point results for another sort of contraction known as F-Suzuki contraction and α-type F-contraction in metric spaces and $\mathcal{G}$-metric spaces (see [2], [4], [9], [11]). In 2018, Jyoti et al. [7] introduced the notion of (β, ξ, ϕ)-expansive mappings in digital metric space. After then, some researchers established fixed point results in Hausdorff rectangular metric spaces and b-metric spaces with the help of C-functions (see [1], [3]).

Lemma 1.1. *Let $\{x_n\}$ be a Cauchy sequence in $(\mathcal{H}, \mathcal{G})$ with $\lim_{n\to\infty} \mathcal{G}(x_n, u, u) = 0$. Then $\mathcal{G}(x_n, t, t) = \mathcal{G}(u, t, t)$ for every $t \in \mathcal{H}$.*

Definition 1.2. ([13]) Let Ψ be the family of functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the followings:

(i) ψ is upper semi-continuous and strictly increasing;
(ii) $\{\psi^n(\kappa)\}$ tend to 0 as $n \to \infty$ for all $\kappa > 0$;
(iii) $\psi(\kappa) < \kappa$ for all $\kappa > 0$.

These functions are known as comparison functions.

Definition 1.3. ([13]) Let $h : \mathcal{H} \to \mathcal{H}$ be a given self-map in a metric space $(\mathcal{H}, \varpi)$. Then, h is said to be an (α, ψ)-contraction if there exist two maps $\psi \in \Psi$ and $\alpha : \mathcal{H} \times \mathcal{H} \to [0, +\infty)$ such that

$$\alpha(x, z)\varpi(hx, hz) \leq \psi(\varpi(x, z)),$$

for all $x, z \in \mathcal{H}$.

In 2012, Samet et al. introduced the notion of β-admissible functions as follows:

Definition 1.4. ([13]) Let $H : \mathcal{H} \to \mathcal{H}$ and $\beta : \mathcal{H} \times \mathcal{H} \times \mathcal{H} \to [0, +\infty)$. Then, H is said to be a β-admissible if $\beta(e, k, k) \geq 1$, then $\beta(He, Hk, Hk) \geq 1$, for all $e, k \in \mathcal{H}$.

2. Main results

In this section, we introduce (ξ, β)-expansive mappings of type (S) and type (T) and prove some fixed point theorems in a $\mathcal{G}$-metric space with the help of a β-admissible function.

Definition 2.1. Let $\mathcal{Q}:\mathcal{H}\to\mathcal{H}$ be a function in $(\mathcal{H},\mathcal{G})$. Then, $\mathcal{Q}$ is said to be a (ξ,β)-expansive mapping of type (S) if there are two mappings $\xi\in\Phi$ and $\beta:\mathcal{H}\times\mathcal{H}\times\mathcal{H}\to[0,\infty]$ such that

$$\begin{aligned}\xi(\mathcal{G}(\mathcal{Q}x,\mathcal{Q}y,\mathcal{Q}z))\geq\beta(x,y,z)\min\{&\mathcal{G}(x,y,z),\mathcal{G}(x,\mathcal{Q}x,\mathcal{Q}x),\mathcal{G}(y,\mathcal{Q}y,\mathcal{Q}y),\\ &\mathcal{G}(z,\mathcal{Q}z,\mathcal{Q}z),\mathcal{G}(x,\mathcal{Q}y,\mathcal{Q}y),\mathcal{G}(y,\mathcal{Q}z,\mathcal{Q}z)\},\end{aligned}\tag{2.1}$$

where Φ denote the class of all the mappings $\xi:[0,\infty)\to[0,\infty)$ satisfying the followings:

(i) ξ is upper semi-continuous;
(ii) $\xi(\kappa)<\kappa$ for any $\kappa>0$;
(iii) $\{\xi^n(\kappa)\}$ converges to zero when $n\to\infty$ for every $\kappa>0$.

Definition 2.2. Let $\mathcal{Q}:\mathcal{H}\to\mathcal{H}$ be a function in $(\mathcal{H},\mathcal{G})$. Then, $\mathcal{Q}$ is known as (ξ,β)-expansive function of type (T) if there exist two mappings $\xi\in\Phi$ and $\beta:\mathcal{H}\times\mathcal{H}\times\mathcal{H}\to[0,\infty]$ such that

$$\xi(\mathcal{G}(\mathcal{Q}x,\mathcal{Q}y,\mathcal{Q}z))\geq\beta(x,y,z)\min\left\{\mathcal{G}(x,y,z),\frac{\mathcal{G}(x,\mathcal{Q}z,\mathcal{Q}z)+\mathcal{G}(z,\mathcal{Q}y,\mathcal{Q}y)}{2}\right\}.\tag{2.2}$$

Theorem 2.3. *Let $\mathcal{Q}:\mathcal{H}\to\mathcal{H}$ be (ξ,β)-expansive mapping of type (S) in $(\mathcal{H},\mathcal{G})$ which is complete, symmetrical, one to one and onto. Also, $\mathcal{Q}$ satisfies the following conditions:*

(i) *$\mathcal{Q}$ is continuous;*
(ii) *$\mathcal{Q}^{-1}$ is β-admissible and there exist $x_0\in\mathcal{H}$ such that $\beta(x_0,\mathcal{Q}^{-1}x_0,\mathcal{Q}^{-1}x_0)\geq1$, $\beta(x_0,\mathcal{Q}^{-2}x_0,\mathcal{Q}^{-2}x_0)\geq1$.*

Then, $\mathcal{Q}$ has a fixed point in $\mathcal{H}$.

Proof. Let $\{x_n\}$ be the sequence such that $\mathcal{Q}x_{n+1}=x_n$, for every $n\in\mathbf{Z}_+$. If there exists a positive integer n such that $x_n=x_{n+1}$, then $\mathcal{Q}x_n=x_n$. So, x_n is a fixed point of $\mathcal{Q}$.

Let us assume that $x_{n+1}\neq x_n$, for every $n\in\mathbf{Z}_+$. Then,

$$\mathcal{G}(x_{n+1},x_n,x_n)>0,\ \forall\, n\in\mathbf{Z}_+.$$

From the assumption of the theorem, we have

$$\beta(x_0,\mathcal{Q}^{-1}x_0,\mathcal{Q}^{-1}x_0)=\beta(x_0,x_1,x_1)\geq1.$$

Since $\mathcal{Q}^{-1}$ is β-admissible, we have

$$\beta(\mathcal{Q}^{-1}x_0,\mathcal{Q}^{-1}x_1,\mathcal{Q}^{-1}x_1)=\beta(x_1,x_2,x_2)\geq1.$$

By induction on n,, we have

$$\beta(x_n,x_{n+1},x_{n+1})\geq1.\tag{2.3}$$

Proceeding in the same way, we obtain

$$\beta(x_0, \mathcal{Q}^{-2}x_0, \mathcal{Q}^{-2}x_0) = \beta(x_0, x_2, x_2) \geq 1$$

and

$$\beta(\mathcal{Q}^{-1}x_0, \mathcal{Q}^{-2}x_2, \mathcal{Q}^{-2}x_2) = \beta(x_1, x_3, x_3) \geq 1.$$

By repeating the same process, we obtain

$$\beta(x_n, x_{n+2}, x_{n+2}) \geq 1.$$

Now, we claim that $\lim_{n\to\infty} \mathcal{G}(x_n, x_{n+1}, x_{n+1}) = 0$.

Putting $x = x_n$ and $y = z = x_{n+1}$ in (2.1), we get

$$\begin{aligned}
&\xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1})) \\
&\quad \geq \beta(x_n, x_{n+1}, x_{n+1}) \min\{\mathcal{G}(x_n, x_{n+1}, x_{n+1}), \mathcal{G}(x_n, \mathcal{Q}x_n, \mathcal{Q}x_n), \\
&\qquad \mathcal{G}(x_{n+1}, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1}), \mathcal{G}(x_{n+1}, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1}), \\
&\qquad \mathcal{G}(x_n, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1}), \mathcal{G}(x_{n+1}, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1})\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1})) \\
&\geq \beta(x_n, x_{n+1}, x_{n+1}) \min\{\mathcal{G}(x_n, x_{n+1}, x_{n+1}), \mathcal{G}(x_n, x_{n-1}, x_{n-1}), \\
&\quad \mathcal{G}(x_{n+1}, x_n, x_n), \mathcal{G}(x_{n+1}, x_n, x_n)\mathcal{G}(x_n, x_n, x_n), \mathcal{G}(x_{n+1}, x_n, x_n)\}.
\end{aligned}$$

By using definition of ξ, we get

$$\mathcal{G}(x_{n-1}, x_n, x_n) > \xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1})).$$

Therefore, we get

$$\begin{aligned}
&\mathcal{G}(x_{n-1}, x_n, x_n) \\
&\quad > \beta(x_n, x_{n+1}, x_{n+1}) \min\{\mathcal{G}(x_n, x_{n+1}, x_{n+1}), \mathcal{G}(x_n, x_{n-1}, x_{n-1}), \\
&\qquad \mathcal{G}(x_{n+1}, x_n, x_n), \mathcal{G}(x_{n+1}, x_n, x_n), \mathcal{G}(x_n, x_n, x_n), \mathcal{G}(x_{n+1}, x_n, x_n)\}.
\end{aligned} \tag{2.4}$$

Since $(\mathcal{H}, \mathcal{G})$ is symmetrical, we have

$$\mathcal{G}(x_n, x_{n+1}, x_{n+1}) = \mathcal{G}(x_{n+1}, x_n, x_n).$$

By using (2.4), we obtain

$$\mathcal{G}(x_{n-1}, x_n, x_n) > \beta(x_n, x_{n+1}, x_{n+1}) min\{\mathcal{G}(x_{n+1}, x_n, x_{n+1}), \mathcal{G}(x_{n-1}, x_n, x_{n-1})\}.$$

If there exist $n \in \mathbf{Z}_+$ such that

$$\min\{\mathcal{G}(x_{n+1}, x_n, x_{n+1}), \mathcal{G}(x_{n-1}, x_n, x_{n-1}) = \mathcal{G}(x_{n-1}, x_n, x_{n-1}),$$

then making use of (2.3), the above inequality is equivalent to

$$\mathcal{G}(x_{n-1}, x_n, x_n) > \mathcal{G}(x_{n-1}, x_{n-1}, x_n),$$

a contradiction.

Consequently, we have

$$\min\{\mathcal{G}(x_{n+1}, x_n, x_{n+1}), \mathcal{G}(x_{n-1}, x_n, x_{n-1}) = \mathcal{G}(x_{n+1}, x_n, x_{n+1}).$$

Therefore, we have

$$\mathcal{G}(x_{n-1}, x_n, x_n) > \xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1})) \geq \mathcal{G}(x_n, x_{n+1}, x_{n+1}),$$

which gives that

$$\mathcal{G}(x_n, x_{n+1}, x_{n+1}) < \mathcal{G}(x_{n-1}, x_n, x_n). \tag{2.5}$$

Using mathematical induction, we obtain

$$\mathcal{G}(x_n, x_{n+1}, x_{n+1}) \leq \xi^n \mathcal{G}(x_0, x_1, x_1).$$

It follows from the definition of ξ that

$$\lim_{n\to\infty} \mathcal{G}(x_n, x_{n+1}, x_{n+1}) = 0.$$

Next, we assert that

$$\lim_{n\to\infty} \mathcal{G}(x_n, x_{n+2}, x_{n+2}) = 0.$$

Putting $x = x_n$ and $y = z = x_{n+2}$ in (2.1), we get

$$\begin{aligned}
&\xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+2}, \mathcal{Q}x_{n+2})) \\
&\quad \geq \beta(x_n, x_{n+2}, x_{n+2}) \min\{\mathcal{G}(x_n, x_{n+2}, x_{n+2}), \mathcal{G}(x_n, \mathcal{Q}x_n, \mathcal{Q}x_n), \\
&\qquad \mathcal{G}(x_{n+2}, \mathcal{Q}x_{n+2}, \mathcal{Q}x_{n+2}), \mathcal{G}(x_{n+2}, \mathcal{Q}x_{n+2}, \mathcal{Q}x_{n+2}), \\
&\qquad \mathcal{G}(x_n, \mathcal{Q}x_{n+2}, \mathcal{Q}x_{n+2}), \mathcal{G}(x_{n+2}, \mathcal{Q}x_{n+2}, \mathcal{Q}x_{n+2})\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+2}, \mathcal{Q}x_{n+2})) \\
&\quad \geq \beta(x_n, x_{n+2}, x_{n+2} \min\{\mathcal{G}(x_n, x_{n+2}, x_{n+2}), \mathcal{G}(x_n, x_{n-1}, x_{n-1}), \\
&\qquad \mathcal{G}(x_{n+2}, x_{n+1}, x_{n+1}), \mathcal{G}(x_{n+2}, x_{n+1}, x_{n+1}), \\
&\qquad \mathcal{G}(x_n, x_{n+1}, x_{n+1}), \mathcal{G}(x_{n+2}, x_{n+1}, x_{n+1})\}.
\end{aligned}$$

By making use of definition of ξ, we obtain

$$\mathcal{G}(x_{n-1}, x_{n+1}, x_{n+1}) > \xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+2}, \mathcal{Q}x_{n+2})).$$

Therefore, we have

$$\begin{aligned}
&\mathcal{G}(x_{n-1}, x_{n+1}, x_{n+1}) \\
&> \beta(x_n, x_{n+2}, x_{n+2}) \min\{\mathcal{G}(x_n, x_{n+2}, x_{n+2}), \mathcal{G}(x_n, x_{n-1}, x_{n-1}), \\
&\quad \mathcal{G}(x_{n+2}, x_{n+1}, x_{n+1}), \mathcal{G}(x_{n+2}, x_{n+1}, x_{n+1}), \\
&\quad \mathcal{G}(x_n, x_{n+1}, x_{n+1}), \mathcal{G}(x_{n+2}, x_{n+1}, x_{n+1})\}.
\end{aligned} \tag{2.6}$$

Since $(\mathcal{H}, \mathcal{G})$ is symmetrical and utilizing (2.3), (2.5), we have

$$\mathcal{G}(x_{n-1}, x_{n+1}, x_{n+1}) > \min\{\mathcal{G}(x_n, x_{n+1}, x_{n+1}), \mathcal{G}(x_{n-1}, x_n, x_n)\}. \tag{2.7}$$

Let $p_n = \mathcal{G}(x_{n+1}, x_{n+3}, x_{n+3})$ and $q_n = \mathcal{G}(x_{n+2}, x_{n+3}, x_{n+3})$. Then, from (2.7), we conclude that

$$\begin{aligned} p_{n-2} &= \mathcal{G}(x_{n-1}, x_{n+1}, x_{n+1}) \\ &> \xi(\mathcal{G}(x_{n-1}, x_{n+1}, x_{n+1})) \\ &= \xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+2}, \mathcal{Q}x_{n+2})) \\ &\geq \min\{\mathcal{G}(x_n, x_{n+1}, x_{n+1}), \mathcal{G}(x_{n-1}, x_n, x_n)\} \\ &= \min\{p_{n-1}, q_{n-1}\}. \end{aligned}$$

From (2.5), we have

$$q_{n-2} \geq q_{n-1} \geq \min\{p_{n-1}, q_{n-1}\}.$$

Therefore, we conclude that

$$\min\{p_{n-2}, q_{n-2}\} \geq \min\{p_{n-1}, q_{n-1}\}.$$

Hence, the sequence $\{\min\{p_n, q_n\}\}$ is monotonically decreasing sequence. Therefore, the sequence converges to $\ell \geq 0$.

Let us assume that $\ell > 0$. Then, we have

$$\lim_{n\to\infty} \sup(p_n) = \lim_{n\to\infty} \sup(\min\{p_n, q_n\}) = \lim_{n\to\infty} \min\{p_n, q_n\} = \ell.$$

Using (2.7), we get

$$\begin{aligned} \ell &= \lim_{n\to\infty} \sup(p_{n-2}) \\ &> \lim_{n\to\infty} \sup(\xi(\mathcal{G}(x_{n-1}, x_{n+1}, x_{n+1}))) \\ &\geq \lim_{n\to\infty} \sup(\min\{p_{n-1}, q_{n-1}\} = \ell, \end{aligned}$$

which is a contradiction. Therefore, we get

$$\mathcal{G}(x_n, x_{n+2}, x_{n+2}) = 0.$$

Now, we assert that $x_a \neq x_b$, for each $a \neq b$. Suppose, on the contrary that $x_a = x_b$ for some $a, b \in \mathbf{Z}_+$ where $a \neq b$. Let us suppose that $a > b$. Then

$$\begin{aligned} \xi(\mathcal{G}(x_b, x_{b-1}, x_{b-1})) &= \xi(\mathcal{G}(x_b, \mathcal{Q}x_b, \mathcal{Q}x_b)) \\ &= \xi(\mathcal{G}(x_a, \mathcal{Q}x_a, \mathcal{Q}x_a)) \\ &= \xi(\mathcal{G}(\mathcal{Q}x_{a+1}, \mathcal{Q}x_a, \mathcal{Q}x_a)) \\ &\geq \beta(x_{a+1}, x_a, x_a) H(x_{n+1}, x_n, x_n) \\ &\geq H(x_{n+1}, x_n, x_n), \end{aligned}$$

where

$$
\begin{aligned}
&H(x_{n+1}, x_n, x_n) \\
&\quad = \min\{\mathcal{G}(x_{a+1}, x_a, x_a), \mathcal{G}(x_{a+1}, \mathcal{Q}x_{a+1}, \mathcal{Q}x_{a+1}), \mathcal{G}(x_a, \mathcal{Q}x_a, \mathcal{Q}x_a), \\
&\qquad \mathcal{G}(x_a, \mathcal{Q}x_a, \mathcal{Q}x_a), \mathcal{G}(x_{a+1}, \mathcal{Q}x_a, \mathcal{Q}x_a), \mathcal{G}(x_a, \mathcal{Q}x_a, \mathcal{Q}x_a)\} \\
&\quad = \min\{\mathcal{G}(x_{a+1}, x_a, x_a), \mathcal{G}(x_{a+1}, x_a, x_a), \mathcal{G}(x_a, x_{a-1}, x_{a-1}), \\
&\qquad \mathcal{G}(x_a, x_{a-1}, x_{a-1}), \mathcal{G}(x_{a+1}, x_{a-1}, x_{a-1}), \mathcal{G}(x_a, x_{a-1}, x_{a-1})\} \\
&\quad = \min\{\mathcal{G}(x_{a+1}, x_a, x_a), \mathcal{G}(x_a, x_{a-1}, x_{a-1})\}.
\end{aligned}
$$

If $\min\{\mathcal{G}(x_{a+1}, x_a, x_a), \mathcal{G}(x_a, x_{a-1}, x_{a-1})\} = \mathcal{G}(x_{a+1}, x_a, x_a)$, then we have

$$\xi(\mathcal{G}(x_b, x_{b-1}, x_{b-1})) \geq \mathcal{G}(x_{a+1}, x_a, x_a),$$

implies that

$$
\begin{aligned}
\mathcal{G}(x_{a+1}, x_a, x_a) &\leq \xi(\mathcal{G}(x_b, x_{b-1}, x_{b-1})) \\
&\leq \xi^{b-a}\mathcal{G}(x_{a+1}, x_a, x_a).
\end{aligned} \tag{2.8}
$$

If $\min\{\mathcal{G}(x_{a+1}, x_a, x_a), \mathcal{G}(x_a, x_{a-1}, x_{a-1})\} = \mathcal{G}(x_a, x_{a-1}, x_{a-1})$, then we have

$$\xi(\mathcal{G}(x_b, x_{b-1}, x_{b-1})) \geq \mathcal{G}(x_a, x_{a-1}, x_{a-1}),$$

that is,

$$
\begin{aligned}
\mathcal{G}(x_a, x_{a-1}, x_{a-1}) &\leq \xi(\mathcal{G}(x_b, x_{b-1}, x_{b-1})) \\
&\leq \xi^{b-a+1}\mathcal{G}(x_a, x_{a-1}, x_{a-1}).
\end{aligned} \tag{2.9}
$$

Using (2.8) and (2.9), we have

$$\mathcal{G}(x_{a+1}, x_a, x_a) \leq \xi^{b-a}\mathcal{G}(x_{a+1}, x_a, x_a)$$

and

$$\mathcal{G}(x_a, x_{a-1}, x_{a-1}) \leq \xi^{b-a+1}\mathcal{G}(x_a, x_{a-1}, x_{a-1}).$$

In both cases, this is a contradiction. So, $x_a \neq x_b$, for each $a \neq b$.

Next, we assert that $\{x_n\}$ is a Cauchy sequence, that is,

$$\lim_{n\to\infty} \mathcal{G}(x_n, x_{n+m}, x_{n+m}) = 0. \tag{2.10}$$

We have proved (2.10) for cases $m = 1$ and $m = 2$, respectively.

Let us take $m \geq 3$. Now, two cases arise.

Case 1 : For $m = 2r$ where $r \geq 2$.

Using (2.8) and definition of $(\mathcal{H}, \mathcal{G})$, we obtain

$$\begin{aligned}\mathcal{G}(x_n, x_{n+m}, x_{n+m}) &= \mathcal{G}(x_n, x_{n+2r}, x_{n+2r}) \\ &\leq \mathcal{G}(x_n, x_{n+2}, x_{n+2}) + \mathcal{G}(x_{n+2}, x_{n+3}, x_{n+3}) \\ &\quad + \cdots + \mathcal{G}(x_{n+2r-1}, x_{n+2r}, x_{n+2r}) \\ &\leq \mathcal{G}(x_n, x_{n+2}, x_{n+2}) + \sum_{d=n+2}^{n+2r-1} \xi^d(\mathcal{G}(x_0, x_1, x_1)) \\ &\leq \mathcal{G}(x_n, x_{n+2}, x_{n+2}) + \sum_{d=n}^{\infty} \xi^d(\mathcal{G}(x_0, x_1, x_1)) \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Case 2 : For $m = 2r + 1$ where $r \geq 1$.

Using (2.8) and definition of $(\mathcal{H}, \mathcal{G})$, we obtain

$$\begin{aligned}\mathcal{G}(x_n, x_{n+m}, x_{n+m}) &= \mathcal{G}(x_n, x_{n+2r+1}, x_{n+2r+1}) \\ &\leq \mathcal{G}(x_n, x_{n+1}, x_{n+1}) + \mathcal{G}(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + \cdots + \mathcal{G}(x_{n+2r}, x_{n+2r+1}, x_{n+2r+1}) \\ &\leq \sum_{d=n}^{n+2r} \xi^d(\mathcal{G}(x_0, x_1, x_1)) \\ &\leq \sum_{d=n}^{\infty} \xi^d(\mathcal{G}(x_0, x_1, x_1)) \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

In both cases $\lim_{n\to\infty} \mathcal{G}(\mathrm{x}_n, \mathrm{x}_{n+m}, \mathrm{x}_{n+m}) = 0$, which yields that $\{\mathrm{x}_n\}$ is Cauchy. Since $(\mathcal{H}, \mathcal{G})$ is complete, there exist $u \in \mathcal{H}$ such that

$$\lim_{n\to\infty} \mathcal{G}(\mathrm{x}_n, u, u) = 0.$$

Using the first assumption of the Theorem 2.3, we get

$$\lim_{n\to\infty} \mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}u, \mathcal{Q}u) = \lim_{n\to\infty} \mathcal{G}(x_{n+1}, \mathcal{Q}u, \mathcal{Q}u) = 0.$$

Therefore, we have $\mathcal{Q}u = \lim_{n\to\infty} x_{n+1} = u$. So, $\mathcal{Q}$ has a fixed point $u \in \mathcal{H}$. $\square$

Theorem 2.4. *Let $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ be a (ξ, β)-expansive mapping of type (T) in $(\mathcal{H}, \mathcal{G})$, which is complete, symmetrical, one to one and onto. Also, $\mathcal{Q}$ satisfies the conditions of Theorem* 2.3. *Then, $\mathcal{Q}$ has a fixed point in $\mathcal{H}$.*

Proof. Let $\{x_n\}$ be a sequence such that $\mathcal{Q}x_{n+1} = x_n$, for each $n \in \mathbf{Z}_+$. Then, by using Theorem 2.3, we get

$$\beta(x_n, x_{n+2}, x_{n+2}) \geq 1.$$

Next, we assert that $\lim_{n\to\infty} \mathcal{G}(x_{n+1}, x_n, x_{n+1}) = 0$.

Putting $x = x_n$ and $y = z = x_{n+1}$ in (2.1), we get

$$\begin{aligned}
&\xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1}))\\
&\quad = \xi(\mathcal{G}(\mathcal{Q}x_n, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1}))\\
&\quad \geq \beta(x_n, x_{n+1}, x_{n+1}) \min\Big\{\mathcal{G}(x_n, x_{n+1}, x_{n+1}),\\
&\quad \frac{\mathcal{G}(x_{n+1}, \mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1}) + \mathcal{G}(x_{n+1}\mathcal{Q}x_{n+1}, \mathcal{Q}x_{n+1})}{2}\Big\}\\
&\quad = \beta(x_n, x_{n+1}, x_{n+1}) \min\{\mathcal{G}(x_n, x_n, x_n), \mathcal{G}(x_{n+1}, x_n, x_n)\}.
\end{aligned}$$

By using identical steps as in proof of Theorem 2.3, we can show that $\mathcal{Q}$ has a fixed point in $\mathcal{H}$. □

Theorem 2.5. *Let $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ be a (ξ, β)-expansive mapping of type (S) in $(\mathcal{H}, \mathcal{G})$, which is complete, symmetrical, one to one and onto. Also, $\mathcal{Q}$ satisfies the following conditions:*

(i) *If $\{\mathrm{x}_n\}$ is a sequence in $\mathcal{H}$ such that $\beta(\mathrm{x}_n, \mathrm{x}_{n+1}, \mathrm{x}_{n+1}) \geq 1$ and $\{x_n\}$ tends to x when $n \to \infty$, then there exist a subsequence $\{x_{n_t}\}$ of $\{x_n\}$ in order that $\beta(x_{n_t}, x, x) \geq 1$;*

(ii) *$\mathcal{Q}^{-1}$ is β-admissible and there exists $x_0 \in \mathcal{H}$ such that $\beta(x_0, \mathcal{Q}^{-1}x_0, \mathcal{Q}^{-1}x_0) \geq 1$, $\beta(x_0, \mathcal{Q}^{-2}x_0, \mathcal{Q}^{-2}x_0) \geq 1$.*

Then, $\mathcal{Q}$ has a fixed point in $\mathcal{H}$.

Proof. Let $\{x_n\}$ be the sequence in $\mathcal{H}$ such that $x_n = \mathcal{Q}x_{n+1}$. By using identical steps as in proof of Theorem 2.3, we can prove that $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}$, which converges to $w \in \mathcal{H}$.

Using Lemma 1.1, we have

$$\lim_{n\to\infty} \mathcal{G}(x_{n_t+1}, \mathcal{Q}w, \mathcal{Q}w) = \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w). \tag{2.11}$$

Now, we assert that $\mathcal{Q}w = w$. Assume on the contrary that $\mathcal{Q}w \neq w$. Using the assumption (i) of the Theorem 2.5, there exist a subsequence $\{x_{n_t}\}$ of $\{x_n\}$ such that $\beta(x_{n_t}, w, w) \geq 1$. Letting $t \to \infty$ and using (2.1), (2.11), we obtain

$$\begin{aligned}
&\mathcal{G}(x_{n_t-1}, w, w)\\
&\quad > \xi(\mathcal{G}(\mathcal{Q}\mathrm{x}_{n_t}, \mathcal{Q}w, \mathcal{Q}w)\\
&\quad \geq \beta(x_{n_t}, w, w) \min\{\mathcal{G}(x_{n_t}, w, w), \mathcal{G}(x_{n_t}, \mathcal{Q}x_{n_t}, \mathcal{Q}x_{n_t}),\\
&\qquad \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w), \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w), \mathcal{G}(x_{n_t}, \mathcal{Q}w, \mathcal{Q}w), \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w)\}\\
&\quad \geq \min\{\mathcal{G}(x_{n_t}, w, w), \mathcal{G}(x_{n_t}, x_{n_t-1}, x_{n_t-1}), \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w),\\
&\qquad \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w), \mathcal{G}(x_{n_t}, \mathcal{Q}w, \mathcal{Q}w), \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w)\}\\
&\quad \geq \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w).
\end{aligned} \tag{2.12}$$

By definition of ξ, we obtain

$$\xi(\mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w)) < \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w). \tag{2.13}$$

By combining (3.12) and (3.13), we have

$$\mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w) \leq \xi(\mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w)) < \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w),$$

which is a contradiction. So, $\mathcal{Q}w = w$. Hence, w is a fixed point of $\mathcal{Q}$. □

Theorem 2.6. *Let $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ be a (ξ, β)-expansive mapping of type(T) in $(\mathcal{H}, \mathcal{G})$ which is complete, symmetrical, one to one and onto. Also, $\mathcal{Q}$ satisfies the conditions of Theorem* 2.5*. Then, $\mathcal{Q}$ has a fixed point in $\mathcal{H}$.*

Proof. Let $\{x_n\}$ a sequence in $\mathcal{H}$ such that $x_n = \mathcal{Q}x_{n+1}$. By using identical steps as in proof of Theorem 2.4, we can prove that $\{x_n\}$ is a cauchy sequence in $\mathcal{H}$, which converges to $w \in \mathcal{H}$.

Using Lemma 1.1, we have

$$\lim_{n\to\infty} \mathcal{G}(x_{n_t+1}, \mathcal{Q}w, \mathcal{Q}w) = \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w). \tag{2.14}$$

Now, we claim that $\mathcal{Q}w = w$. Suppose on the contrary that $\mathcal{Q}w \neq w$. Letting $t \to \infty$, using (2.1) and (2.14), we obtain

$$\begin{aligned}
&\mathcal{G}(x_{n_t-1}, w, w)\\
&\quad > \xi(\mathcal{G}(\mathcal{Q}x_{n_t}, \mathcal{Q}w, \mathcal{Q}w)\\
&\quad \geq \beta(x_{n_t}, w, w)\min\left\{\mathcal{G}(x_{n_t}, w, w), \frac{\mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w)+G(w, \mathcal{Q}w, \mathcal{Q}w)}{2}\right\}\\
&\quad \geq \min\{\mathcal{G}(x_{n_t}, w, w), \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w)\}.
\end{aligned} \tag{2.15}$$

Letting $t \to \infty$ in (2.15), we have

$$\xi(\mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w) \geq \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w). \tag{2.16}$$

By definition of ξ, we obtain

$$\xi(\mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w)) < \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w). \tag{2.17}$$

By combining (2.16) and (2.17), we have

$$\mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w) \leq \xi(\mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w)) < \mathcal{G}(w, \mathcal{Q}w, \mathcal{Q}w),$$

which is a contradiction. So $\mathcal{Q}w = w$. Hence, w is a fixed point of $\mathcal{Q}$. □

3. Conclusion

In this manuscript, some common fixed point theorems are proved for (ξ, β)-expansive mappings of type (S) and type (T) using control function and β-admissible function in $\mathcal{G}$-metric space.

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