Nonlinear Functional Analysis and Applications
Vol. 26, No. 3 (2021), pp. 581-600
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.03.09
http://nfaa.kyungnam.ac.kr/journal-nfaa

FIXED POINT THEOREMS IN ORDERED b-METRIC SPACES WITH ALTERNATING DISTANCE FUNCTIONS

Radia Bouhafs[1], Abdalla Ahmed Tallafha[2] and Wasfi Shatanawi[3]

[1]Department of Mathematics, Faculty of Science
Jordan, Amman, Jordan
e-mail: rad9170450@ju.edu.jo; radia.93.bou@gmail.com

[2]Department of Mathematics, Faculty of Science
Jordan, Amman, Jordan
e-mail: a.tallafha@ju.edu.jo

[3]Department of Mathematics and General Sciences, Prince Sultan University
Riyadh, Saudi Arabia
Department of Mathematics, Hashemite University, Zarqa, Jordan
e-mail: wshatanawi@psu.edu.sa

Abstract. In this paper we obtain a unique common fixed point theorem for four self-maps which are involved in (ϕ, ψ)-weak contraction of a partially ordered b-metric space. The necessary condition has been given to a space for the existence of an unique common fixed of the maps. And our work changed conditions and nonlinear contraction, and search for the unique common fixed point of the maps.

1. Introduction

The Banach contraction principle is one of the basic results in fixed point theory which asserts that every contraction function in a complete metric space has a unique fixed point. Many authors extended this crucial theorem to many directions, see ([1],[2],[6],[8],[9],[10],[11]).

Bakhtin in [7] extended the notion of metric space to the notion of b-metric space. For some work in b-metric spaces, see ([3],[4],[13],[14],[17],[20],[21],[22], [30],[31],[45]).

[0]Received December 17, 2020. Revised March 2, 2021. Accepted March 7, 2021.
[0]2010 Mathematics Subject Classification: 47H09, 47H10, 37C25.
[0]Keywords: Metric space, b-metric space, complete partially ordered b-metric space.
[0]Corresponding author: R. Bouhafs(rad9170450@ju.edu.jo).

Many authors studied many fixed point theorems, for example see ([9], [15], [16], [18], [19], [22]-[29], [32]-[44]). The benefit of fixed point theorems is to prove the existence and uniqueness of such equations in partial differential equations, integral equations, and ordinary differential equations.

Definition 1.1. A metric space is a pair (X, d), where X is a nonempty set and $d : X \times X \longrightarrow \mathbb{R}$ is a function such that for all $x, y, z \in X$, the following conditions hold:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

Definition 1.2. ([5]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

(X, d, s) is said to be b-metric space, with coefficient $s \geq 1$.

Remark 1.3. Every b-metric space is a metric space with coefficient $s = 1$, but the converse is not true.

Example 1.4. Let $X = \mathbb{R}$ and define $d : \mathbb{R}\times\mathbb{R} \longrightarrow [0, \infty)$ by $d(x, y) = (x-y)^2$. Then d is a b-metric space which is not a metric space.

Definition 1.5. A metric d on X together with a partially ordered relation $\leq$ is called a partially ordered metric space. It is denoted by $(X, d, \leq)$.

Definition 1.6. If the b-metric d is complete, then $(X, d, \leq)$ is called a complete partially ordered b-metric space.

Proposition 1.7. ([12]) *In a b-metric space (X, d), the following assertions hold:*

(1) *A b-convergent sequence has a unique limit.*
(2) *Each b-convergent sequence is b-Cauchy.*
(3) *In general, a b-metric is continuous.*

2. Previous results

In the sequel, we have to recall previous notations and results. Let f and g be self-mappings on a set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called the point of coincidence of f and g.

Two self mappings f and g are said to be weakly compatible if they commute at their coincidence point, that is, if $fx = gx$, then $fgx = gfx$. Now, consider $(X, \leq)$ to be partially ordered set. Two self-mappings f and g are said to be compatible, if for any sequence $\{x_n\}$ with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = \mu$, then $\lim_{n\to\infty} d(gfx_n, fgx_n) = 0$. Also g is called monotone f nondecreasing, if $fx \leq fy$ then $gx \leq gy$ for any $x, y \in X$.

Definition 2.1. Let f, g, S and T be self-maps on a partial b-metric space $(X, p, \leq)$ with $(s > 1)$. Then f and g are said to satisfy almost generalized (S, T)-contractive condition if there $\delta \in [0, 1)$ such that

$$s^2 p(fx, gy) \leq \delta \max\left\{p(Sx, Ty), p(fx, Sx), p(gy, Ty), \frac{p(Sx, gy) + p(fx, Ty)}{2s}\right\}, \tag{2.1}$$

for all $x, y \in X$.

3. Main result

Let Ψ is the family of all functions $\psi : [0, \infty) \longrightarrow [0, \infty)$ such that

(1) ψ is continuous and nondecreasing,
(2) $\psi(t) = 0$ if and only if $t = 0$.

Also, let Φ denote all functions $\phi : [0, \infty) \times [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ such that

(1) ϕ is continuous,
(2) $\phi(t, s, u) = 0$ if and only if $u = s = t = 0$.

If $\psi \in \Psi$, then Ψ is called an altering distance function (see [18]).

Now, we introduce our definition.

Definition 3.1. Let f, g, S, and T be self-mappings on a b-metric space (X, d). Then f and g are said to satisfy the almost nonlinear (S, T, ψ, ϕ)-contractive condition if there exist $\psi \in \Psi$, $\phi \in \Phi$ such that

$$\begin{aligned} &\psi(s^2 d(fx, gy)) \\ &\leq \psi\left(\max\left\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(Sx, gy) + d(fx, Ty)}{2s}\right\}\right) \\ &\quad - \phi(d(Sx, Ty), d(Sx, gy), d(fx, Ty)), \end{aligned} \tag{3.1}$$

for all $x, y \in X$, where $\phi(x, y, z) = \psi(x, x, x, x)$, for all $x \in [0, +\infty)$.

Theorem 3.2. *Let $(X,d,\leq)$ be a complete ordered b-metric space. Suppose $f,\ g,\ T,\ S : X \longrightarrow X$ are continuous mappings such that f and g satisfy the almost nonlinear (S,T,ψ,ϕ)-contractive condition for any two comparable element $x,y \in X$. Suppose that f, g, S and T satisfy the following conditions:*

(1) $fX \subseteq TX$,
(2) $gX \subseteq SX$,
(3) *one of four mappings f, g, S, T is continuous,*
(4) *$\{f,S\}$ and $\{g,T\}$ are compatible.*

Then f,g,S, and T have a common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary. From $fX \subseteq TX$, $gX \subseteq SX$, construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $fx_{2n} = Tx_{2n+1} = y_{2n}$, $gx_{2n+1} = Sx_{2n+2} = y_{2n+1}$. Putting $y = x_{2n+1}$, $x = x_{2n+2}$.

Suppose that $y_{2n} = y_{2n-1}$, we have

$$\begin{aligned}
&\psi(s^2 d(y_{2n}, y_{2n+1}))\\
&= \psi(s^2 d(fx_{2n}, gx_{2n+1}))\\
&\leq \psi\Bigg(\max\Bigg\{ d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}),\\
&\qquad \frac{d(Sx_{2n}, gx_{2n+1}) + d(fx_{2n}, Tx_{2n+1})}{2s} \Bigg\}\Bigg)\\
&\quad - \phi(d(Sx_{2n}, Tx_{2n+1}), d(gx_{2n+1}, Sx_{2n}), d(fx_{2n}, Tx_{2n+1})).\\
&= \psi\Bigg(\max\Bigg\{ d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \frac{d(y_{2n-1}, y_{2n+1})}{2s} \Bigg\}\Bigg)\\
&\quad - \phi(d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n-1}), d(y_{2n}, y_{2n}))\\
&= \psi\Bigg(\max\Bigg\{ 0, 0, d(y_{2n+1}, y_{2n}), \frac{d(y_{2n+1}, y_{2n})}{2} \Bigg\}\Bigg)\\
&\quad - \phi(0, d(y_{2n+1}, y_{2n-1}), 0))\\
&\leq \psi(d(y_{2n+1}, y_{2n})).
\end{aligned}$$

Therefore $\phi(0, d(y_{2n+1}, y_{2n-1}), 0)) = 0$ and hence $y_{2n-1} = y_{2n+1} = y_{2n}$. Similarly, we may show that $y_{2n+2} = y_{2n+1}$. Thus $\{y_n\}$ is a constant sequence in X, hence it is a Cauchy sequence in (X,d).

Suppose $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. If n is even, then $n = 2t$ for some $t \in \mathbb{N}$. Since x_{2t} and x_{2t+1} are comparable, we have

$$\begin{aligned}
&\psi(s^2 d(y_n, y_{n+1}))\\
&= \psi(s^2 d(fx_{2t}, gx_{2t+1}))
\end{aligned}$$

$$
\begin{aligned}
&\le \psi\Bigg(\max\Bigg\{ d(Sx_{2t}, Tx_{2t+1}), d(fx_{2t}, Sx_{2t}), d(gx_{2t+1}, Tx_{2t+1}),\\
&\quad \frac{d(Sx_{2t}, gx_{2t+1}) + d(fx_{2t}, Tx_{2t+1})}{2s} \Bigg\}\Bigg)\\
&\quad - \phi(d(Sx_{2t}, Tx_{2t+1}), d(gx_{2t+1}, Sx_{2t}), d(fx_{2t}, Tx_{2t+1})).\\
&= \psi\Bigg(\max\Bigg\{ d(y_{2t-1}, y_{2t}), d(y_{2t}, y_{2t-1}), d(y_{2t+1}, y_{2t}), \frac{d(y_{2t-1}, y_{2t+1})}{2s} \Bigg\}\Bigg)\\
&\quad - \phi(d(y_{2t-1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), d(y_{2t}, y_{2t}))\\
&= \psi\Bigg(\max\Bigg\{ d(y_{2t-1}, y_{2t}), d(y_{2t}, y_{2t-1}), d(y_{2t+1}, y_{2t}),\\
&\quad \frac{d(y_{2t-1}, y_{2t}) + d(y_{2t}, y_{2t+1})}{2} \Bigg\}\Bigg) - \phi(d(y_{2t-1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), 0)\\
&\le \psi(\max\{d(y_{2t-1}, y_{2t}), d(y_{2t+1}, y_{2t})\}).
\end{aligned}
$$

If $\max\{d(y_{2t-1}, y_{2t}), d(y_{2t+1}, y_{2t})\} = d(y_{2t+1}, y_{2t})$, then

$$\phi(d(y_{2t-1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), 0) = 0$$

and hence

$$d(y_{2t-1}, y_{2t}) = d(y_{2t-1}, y_{2t+1}) = 0.$$

Thus $y_{2t} = y_{2t-1}$. That is, $y_n = y_{n-1}$ which is a contradiction. Thus,

$$\max\{d(y_{2t-1}, y_{2t}), d(y_{2t+1}, y_{2t})\} = d(y_{2t-1}, y_{2t}). \tag{3.2}$$

Therefore,

$$\psi(s^2 d(y_{2t}, y_{2t+1})) \le \psi(d(y_{2t-1}, y_{2t})) - \phi(d(y_{2t+1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), 0). \tag{3.3}$$

If n is odd, then $n = 2t + 1$ for some $t \in \mathbb{N}$. Since x_{2t+2} and x_{2t+1} are comparable, we have

$$
\begin{aligned}
&\psi(s^2 d(y_n, y_{n+1}))\\
&= \psi(s^2 d(y_{2t+2}, y_{2t+1}))\\
&= \psi(s^2 d(fx_{2t+2}, gx_{2t+1}))\\
&\le \psi\Bigg(\max\Bigg\{ d(Sx_{2t+2}, Tx_{2t+1}), d(fx_{2t+2}, Sx_{2t+2}), d(gx_{2t+1}, Tx_{2t+1}),\\
&\quad \frac{d(Sx_{2t+2}, gx_{2t+1}) + d(fx_{2t+2}, Tx_{2t+1})}{2s} \Bigg\}\Bigg)\\
&\quad - \phi(d(Sx_{2t+2}, Tx_{2t+1}), d(gx_{2t+1}, Sx_{2t+2}), d(fx_{2t+2}, Tx_{2t+1})).
\end{aligned}
$$

$$
\begin{aligned}
&= \psi\left(\max\left\{ d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1}), \frac{d(y_{2t+2}, y_{2t})}{2s} \right\}\right) \\
&\quad - \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t})). \\
&\leq \psi\left(\max\left\{ d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1}), \frac{d(y_{2t+2}, y_{2t+1}) + d(y_{2t+1}, y_{2t})}{2} \right\}\right) \\
&\quad - \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t})). \\
&= \psi(\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\}) \\
&\quad - \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t})). \\
&\leq \psi(\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\}).
\end{aligned}
$$

If $\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\} = d(y_{2t+2}, y_{2t+1})$, then

$$\phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t})) = 0,$$

and hence $d(y_{2t+1}, y_{2t}) = d(y_{2t+2}, y_{2t})$. Thus $y_{2t+1} = y_{2t}$ which is a contradiction. So we have

$$\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\} = d(y_{2t+1}, y_{2t}). \tag{3.4}$$

Therefore,

$$\psi(s^2 d(y_{2t+2}, y_{2t+1})) \leq \psi(d(y_{2t+1}, y_{2t}) - \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t+2}, y_{2t})).$$

From (3.2) and (3.4), we have

$$d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n).$$

Therefore $\{d(y_{n+1}, y_n) : n \in \mathbb{N}\}$ is a nonincreasing sequence. Thus there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = r.$$

On taking limsup in (3.3) and (3.4), we have

$$\psi(s^2 r) \leq \psi(r) - \liminf_{t\to\infty} \phi(d(y_{2t-1}, y_{2t}), d(y_{2t-1}, y_{2t+1}), 0)$$

and

$$
\begin{aligned}
\psi(s^2 r) &\leq \psi(r) - \liminf_{t\to\infty} \phi(d(y_{2t+1}, y_{2t}), 0, d(y_{2t}, y_{2t+2})) \\
&= \psi(r).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\liminf_{t\to\infty} d(y_{2t-1}, y_{2t}) &= \liminf_{t\to\infty} d(y_{2t-1}, y_{2t+1}) \\
&= \liminf_{t\to\infty} d(y_{2t}, y_{2t+2}) \\
&= \liminf_{t\to\infty} d(y_{2t}, y_{2t+1}) \\
&= 0.
\end{aligned}
$$

Therefore, $r = 0$ and hence

$$\lim_{n \longrightarrow \infty} d(y_n, y_{n+1}) = 0. \tag{3.5}$$

We show that $\{y_n\}$ is a Cauchy sequence in the metric space (X, d). Suppose to the contrary, that is, $\{y_{2n}\}$ is not a Cauchy sequence in (X, d). Then there exists $\epsilon \geq 0$ and two subsequences $\{y_{2m(i)}\}$ and $\{y_{2n(i)}\}$ of $\{y_{2n}\}$ such that $n(i)$ is the smallest index for which, $n(i) > m(i) > i$,

$$d(y_{2m(i)}, y_{2n(i)}) \geq \epsilon$$

and

$$d(y_{2m(i)}, y_{2n(i)-1}) < \epsilon. \tag{3.6}$$

From (3.5), (3.6) and the triangular inequality, we get that

$$\begin{aligned}
\epsilon &\leq d(y_{2m(i)}, y_{2n(i)}) \\
&\leq sd(y_{2m(i)}, y_{2n(i)-1}) \\
&\quad + sd(y_{2n(i)-1}, y_{2n(i)}) \\
&\leq s\epsilon + sd(y_{2n(i)-1}, y_{2n(i)}).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
d(y_{2n(i)}, y_{2m(i)+1}) &\leq sd(y_{2n(i)}, y_{2n(i)-1}) \\
&\quad + sd(y_{2n(i)-1}, y_{2m(i)+1}) \\
&\leq sd(y_{2n(i)}, y_{2n(i)-1}) \\
&\quad + s^2 d(y_{2n(i)-1}, y_{2m(i)}) \\
&\quad + s^2 d(y_{2m(i)}, y_{2m(i)+1}) \\
&\leq sd(y_{2n(i)}, y_{2n(i)-1}) + s^2\epsilon \\
&\quad + s^2 d(y_{2m(i)}, y_{2m(i)+1}).
\end{aligned}$$

Also, from triangular inequality, we get

$$\begin{aligned}
\epsilon &\leq d(y_{2m(i)}, y_{2n(i)}) \\
&\leq sd(y_{2m(i)}, y_{2m(i)-1}) + sd(y_{2m(i)-1}, y_{2n(i)}) \\
&\leq sd(y_{2m(i)}, y_{2m(i)-1}) + s^2 d(y_{2m(i)-1}, y_{2m(i)+1}) + s^2 d(y_{2m(i)+1}, y_{2n(i)}) \\
&\leq sd(y_{2m(i)}, y_{2m(i)-1}) + s^3 d(y_{2m(i)-1}, y_{2m(i)}) + s^3 d(y_{2m(i)}, y_{2m(i)+1}) \\
&\quad + s^2 d(y_{2m(i)+1}, y_{2n(i)}).
\end{aligned}$$

Letting $i \to +\infty$ in above inequalities in (3.5) and (3.6), we get

$$\limsup_{i \to \infty} d(y_{2m(i)}, y_{2n(i)}) \leq s\epsilon,$$

and

$$\frac{\epsilon}{s^2} \leq \limsup_{i\to\infty} d(y_{2n(i)}, y_{2m(i)+1}) \leq s^2\epsilon.$$

Similarly, we get

$$\frac{\epsilon}{s^2} \leq \liminf_{i\to\infty} d(y_{2n(i)}, y_{2m(i)+1}) \leq s^2\epsilon.$$

Also,

$$\begin{aligned} d(y_{2n(i)-1}, y_{2m(i)+1}) &\leq sd(y_{2n(i)-1}, y_{2m(i)}) + sd(y_{2m(i)}, y_{2m(i)+1}) \\ &\leq s\epsilon + sd(y_{2m(i)}, y_{2m(i)+1}). \end{aligned}$$

Letting $i \longrightarrow +\infty$ in above inequalities in (3.5) and (3.6), we get

$$\limsup_{i\to\infty} d(y_{2n(i)-1}, y_{2m(i)+1}) \leq s\epsilon.$$

Since $x_{2n(i)}$ and $x_{2m(i)+1}$ are comparable, we have

$$\begin{aligned} &\psi(s^2 d(y_{2n(i)}, y_{2m(i)+1})) \\ &= \psi(s^2 d(fx_{2n(i)}, gx_{2m(i)+1})) \\ &\leq \psi\Bigg(\max\Bigg\{ d(Sx_{2n(i)}, Tx_{2m(i)+1}), d(fx_{2n(i)}, Sx_{2n(i)}), d(gx_{2m(i)+1}, Tx_{2m(i)+1}), \\ &\qquad \frac{d(Sx_{2n(i)}, gx_{2m(i)+1}) + d(fx_{2n(i)}, Tx_{2m(i)+1})}{2s} \Bigg\}\Bigg) \\ &\quad - \phi(d(Sx_{2n(i)}, Tx_{2m(i)+1}), d(Sx_{2n(i)}, gx_{2m(i)+1}), d(fx_{2n(i)}, Tx_{2m(i)+1})). \\ &= \psi\Bigg(\max\Bigg\{ d(y_{2n(i)-1}, y_{2m(i)}), d(y_{2n(i)-1}, y_{2m(i)}), d(y_{2m(i)+1}, y_{2m(i)}), \\ &\qquad \frac{d(y_{2n(i)-1}, y_{2m(i)+1}) + d(y_{2n(i)}, y_{2m(i)})}{2s} \Bigg\}\Bigg) \\ &\quad - \phi(d(y_{2n(i)-1}, y_{2m(i)}), d(y_{2n(i)-1}, y_{2m(i)+1}), d(y_{2n(i)}, y_{2m(i)})). \end{aligned}$$

Letting $i \to +\infty$. And using the continuity of ψ, we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon, s\epsilon, s\epsilon) < \psi(\epsilon).$$

So $\phi(\epsilon, s\epsilon, s\epsilon) = 0$, hence $\epsilon = 0$. Thus $\{y_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there is $z \in X$ such that $y_n \to z$ in the metric space (X, d). Thus $d(y_n, z) = 0$, for all $n \to +\infty$. Hence, by the compatibility of S and f, we obtain

$$\lim_{n\longrightarrow\infty} d(f(Sx_{2n}), S(fx_{2n})) = 0,$$

further by using triangular inequality,

$$d(Sz, fz) \leq sd(Sz, S(fx_{2n})) \\ + s^2 d(S(fx_{2n}), f(Sx_{2n})) \\ + s^2 d(fz, f(Sx_{2n})). \tag{3.7}$$

Therefore, we arrive at $d(Sz, fz) = 0$ as $n \to +\infty$ in (3.7). Hence, z is a coincidence point for S and f in X.

Putting $x = z$, $y = x_{2n+1}$, we get

$$\psi(s^2 d(Sz, z)) \leq \psi(d(Sz, z)),$$

hence $Sz = z$ and $fz = z$. Since $z = fz \in fX \subseteq TX$, there exists $u \in X$ such that $z = Tu$. Putting $x = x_{2n}$, $y = u$, and letting $n \to \infty$, we get $z = gu$, so $gu = Tu$ since (g, T) is compatible, we have $gz = Tz$. Putting $x = x_{2n}$, $y = z$, we get $gz = z$ so that $Tz = z$. This completes the proof. $\square$

Example 3.3. Let $X = \{0, 1, 2, 3, 4\}$.

$$\begin{cases} d(x, y) = 0, & if \quad x = y, \\ d(x, y) = (x + y)^2, & if \quad x \neq y. \end{cases}$$

Then (X, d) is a b-metric space with constant $s > 1$. Let $f, g, S, T : X \longrightarrow X$ be defined by

$$f(x) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 0 & 0 \end{pmatrix}, \; S(x) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \; g(x) = 1, T(x) = x.$$

Then, we know that $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$, and the pairs $\{f, S\}$ and $\{g, T\}$ are compatible. Let the control functions $\psi : [0, \infty) \longrightarrow [0, \infty)$, $\phi : [0, \infty) \times [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ be defined by $\psi(t) = t$ and $\phi(t_1, t_2, t_3) = t_1 + t_2 + t_3$. Then all the conditions of Theorem 3.2 are satisfied except the contractive condition of this theorem. Hence f, g, S, T have no common fixed point in X.

Remark 3.4. If inequality of Theorem 3.2 is replaced by

$$\psi(s^2 d(fx, gy)) \\ \leq k\psi \left(\max \left\{ d(Sx, Ty), d(fx, Sy), d(gy, Ty), \frac{d(Sx, gy) + d(fx, Ty)}{2s} \right\} \right) \tag{3.8}$$

for all $x, y \in X$, where $0 < k < 1$, then the result of the Theorem 3.2 holds.

Example 3.5. Let $X = \{1, 2, 3, 4\}$, define $d : X \times X \to \mathbb{R}$ by

$$\begin{cases} d(x,y) = 0, & if \quad x = y \\ d(x,y) = 1, & if \quad 2 \neq x \neq y \neq 3. \\ d(x,y) = \frac{1}{2}, & if \quad x = 2, y = 3. \end{cases}$$

Also define $f, g : X \to X$, $f1 = f4 = 2$, $f2 = f3 = 2$, $g2 = 2$, $g4 = 3$, $g1 = g3 = 4$. Moreover take $S, T = I$(Identity map) in Theorem 3.2. Then f, g, S, T have a common fixed point.

Theorem 3.6. *Let $(X, d, s, \leq)$ be a complete partially ordered b-metric space with parameter $s \geq 1$ and $f, g, S, T : X \to X$ be self-mappings such that*

(1)

$$\phi(sd(fx, gy)) \leq \phi(M(x, y)) - \psi(M(x, y)),$$

where

$$M(x,y) = \max\left\{ d(gy, Ty)\frac{[1 + d(fx, Sx)]}{1 + d(fx, gy)}, \frac{d(fx, Ty) + d(gy, Sx)}{2s}, \right. \\ \left. d(fx, Sx), d(gy, Ty), d(fx, gy) \right\}$$

for $x, y \in X$, and for some $\phi \in \Phi$ and $\psi \in \Psi$.

(2) *One of four mappings f, g, S, T is continous nondecreasing map with regerds to $\leq$ such that there exists x_0 with $x_0 \leq Sx_0$ (if S is continous and nondecreasing map).*
(3) $fX \subseteq TX$.
(4) $gX \subseteq SX$.
(5) *f is monotone T-nondecreasing map.*
(6) *g is monotone S-nondecreasing map.*
(7) *$\{f, s\}$ and $\{g, T\}$ are weakly compatible.*

Then f, g, S and T have a common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary. From $fX \subseteq TX$, $gX \subseteq SX$, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$fx_{2n} = Tx_{2n+1} = y_{2n}, \; gx_{2n+1} = Sx_{2n+2} = y_{2n+1}.$$

Putting $y = x_{2n+1}$, $x = x_{2n+2}$. Suppose $y_{2n} = y_{2n-1}$, we have

$$
\begin{aligned}
&\phi(sd(y_{2n}, y_{2n+1}))\\
&\quad = \phi(sd(fx_{2n}, gx_{2n+1}))\\
&\quad \leq \phi\Bigg(\max\Bigg\{ d(gx_{2n+1}, Tx_{2n+1})\frac{[1 + d(fx_{2n}, Sx_{2n})]}{1 + d(fx_{2n}, gx_{2n})},\\
&\qquad\quad \frac{d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n})}{2s},\\
&\qquad\quad d(fx_{2n}, Sx_{2n}), d(gx_{2n+1}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1})\Bigg\}\Bigg)\\
&\qquad - \psi\Bigg(\max\Bigg\{ d(gx_{2n+1}, Tx_{2n+1})\frac{[1 + d(fx_{2n}, Sx_{2n})]}{1 + d(fx_{2n}, gx_{2n})},\\
&\qquad\quad \frac{d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n})}{2s}, d(fx_{2n}, Sx_{2n}),\\
&\qquad\quad d(gx_{2n+1}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1})\Bigg\}\Bigg)\\
&\quad = \phi\Bigg(\max\Bigg\{ d(y_{2n+1}, y_{2n})\frac{[1 + d(y_{2n}, y_{2n-1})]}{1 + d(y_{2n}, y_{2n+1})}, \frac{d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})}{2s},\\
&\qquad\quad d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+1})\Bigg\}\Bigg)\\
&\qquad - \psi\Bigg(\max\Bigg\{ d(y_{2n+1}, y_{2n})\frac{[1 + d(y_{2n}, y_{2n-1})]}{1 + d(y_{2n}, y_{2n+1})}, \frac{d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})}{2s},\\
&\qquad\quad d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+1})\Bigg\}\Bigg)\\
&\quad \leq \phi\Bigg(\max\Bigg\{ \frac{d(y_{2n+1}, y_{2n})}{1 + d(y_{2n}, y_{2n+1})}, \frac{d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})}{2},\\
&\qquad\quad d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+1})\Bigg\}\Bigg)\\
&\qquad - \psi\Bigg(\max\Bigg\{ d(\frac{d(y_{2n+1}, y_{2n})}{1 + d(y_{2n}, y_{2n+1})}, \frac{d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})}{2},\\
&\qquad\quad d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n+1})\Bigg\}\Bigg)\\
&\quad = \phi(d(y_{2n+1}, y_{2n})) - \psi(d(y_{2n+1}, y_{2n}))\\
&\quad \leq \phi(d(y_{2n+1}, y_{2n}).
\end{aligned}
$$

Therefore $\psi(d(y_{2n+1}, y_{2n})) = 0$ and hence $y_{2n-1} = y_{2n+1} = y_{2n}$.

Similarly, we may show that $y_{2n+2} = y_{2n+1}$. Thus $\{y_n\}$ is a constant sequence in X, hence it is a Cauchy sequence in (X, d).

Suppose $y_n \neq y_{n+1}$ and $y_{n-1} = y_{n+1}$ for all $n \in \mathbb{N}$. If n is even, then $n = 2t$ for some $t \in \mathbb{N}$. Since x_{2t} and x_{2t+1} are comparable, we have

$$\begin{aligned}
&\phi(sd(y_n, y_{n+1})) \\
&= \phi(sd(fx_{2t}, gx_{2t+1})) \\
&\leq \phi\Bigg(\max\Bigg\{ d(gx_{2t+1}, Tx_{2t+1}) \frac{[1 + d(fx_{2t}, Sx_{2t})]}{1 + d(fx_{2t}, gx_{2t})}, \\
&\qquad \frac{d(fx_{2t}, Tx_{2t+1}) + d(gx_{2t+1}, Sx_{2t})}{2s}, \\
&\qquad d(fx_{2t}, Sx_{2t}), d(gx_{2t+1}, Tx_{2t+1}), d(fx_{2t}, gx_{2t+1}) \Bigg\} \Bigg) \\
&\quad - \psi\Bigg(\max\Bigg\{ d(gx_{2t+1}, Tx_{2t+1}) \frac{[1 + d(fx_{2t}, Sx_{2t})]}{1 + d(fx_{2t}, gx_{2t+1})}, \\
&\qquad \frac{d(fx_{2t}, Tx_{2t+1}) + d(gx_{2t+1}, Sx_{2t})}{2s}, d(fx_{2t}, Sx_{2t}), \\
&\qquad d(gx_{2t+1}, Tx_{2t+1}), d(fx_{2t}, gx_{2t+1}) \Bigg\} \Bigg) \\
&= \phi\Bigg(\max\Bigg\{ d(y_{2t+1}, y_{2t}) \frac{[1 + d(y_{2t}, y_{2t-1})]}{1 + d(y_{2t}, y_{2t+1})}, \\
&\qquad \frac{d(y_{2t}, y_{2t}) + d(y_{2t+1}, y_{2t-1})}{2s}, \\
&\qquad d(y_{2t}, y_{2t-1}), d(y_{2t+1}, y_{2t}), d(y_{2t}, y_{2t+1}) \Bigg\} \Bigg) \\
&\quad - \psi\Bigg(\max\Bigg\{ d(y_{2t+1}, y_{2t}) \frac{[1 + d(y_{2t}, y_{2t-1})]}{1 + d(y_{2t}, y_{2t+1})}, \\
&\qquad \frac{d(y_{2t}, y_{2t}) + d(y_{2t+1}, y_{2t-1})}{2s}, \\
&\qquad d(y_{2t}, y_{2t-1}), d(y_{2t+1}, y_{2t}), d(y_{2t}, y_{2t+1}) \Bigg\} \Bigg) \\
&= \phi(d(y_{2t-1}, y_{2t})) - \psi(d(y_{2t-1}, y_{2t})).
\end{aligned}$$

Therefore,

$$\phi(sd(y_{2t}, y_{2t+1})) \leq \phi(d(y_{2t-1}, y_{2t})) - \psi(d(y_{2t-1}, y_{2t})). \tag{3.9}$$

If n is odd, then $n = 2t + 1$ for some $t \in \mathbb{N}$. Since x_{2t+2} and x_{2t+1} are comparable, we have

$$
\begin{aligned}
&\phi(sd(y_n, y_{n+1}))\\
&= \phi(sd(y_{2n+2}, y_{2n+1})) = \phi(sd(fx_{2t+2}, gx_{2t+1}))\\
&\leq \phi\Bigg(\max\Bigg\{ d(gx_{2t+1}, Tx_{2t+1})\frac{[1+d(fx_{2t+2}, Sx_{2t+2})]}{1+d(fx_{2t+2}, gx_{2t+1})},\\
&\quad \frac{d(fx_{2t+2}, Tx_{2t+1}) + d(gx_{2t+1}, Sx_{2t+2})}{2s},\\
&\quad d(fx_{2t+2}, Sx_{2t+2}), d(gx_{2t+1}, Tx_{2t+1}), d(fx_{2t+2}, gx_{2t+1}) \Bigg\}\Bigg)\\
&\quad - \psi\Bigg(\max\Bigg\{ d(gx_{2t+1}, Tx_{2t+1})\frac{[1+d(fx_{2t+2}, Sx_{2t+2})]}{1+d(fx_{2t+2}, gx_{2t+1})},\\
&\quad \frac{d(fx_{2t+2}, Tx_{2t+1}) + d(gx_{2t+1}, Sx_{2t+2})}{2s},\\
&\quad d(fx_{2t+2}, Sx_{2t+2}), d(gx_{2t+1}, Tx_{2t+1}), d(fx_{2t+2}, gx_{2t+1}) \Bigg\}\Bigg)\\
&= \phi\Bigg(\max\Bigg\{ d(y_{2t+1}, y_{2t})\frac{[1+d(y_{2t+2}, y_{2t+1})]}{1+d(y_{2t+2}, y_{2t+1})},\\
&\quad \frac{d(y_{2t+2}, y_{2t}) + d(y_{2t+1}, y_{2t+1})}{2s}, d(y_{2t+2}, y_{2t+1}),\\
&\quad d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1}) \Bigg\}\Bigg)\\
&\quad - \psi\Bigg(\max\Bigg\{ d(y_{2t+1}, y_{2t})\frac{[1+d(y_{2t+2}, y_{2t+1})]}{1+d(y_{2t+2}, y_{2t+1})},\\
&\quad \frac{d(y_{2t+2}, y_{2t}) + d(y_{2t+1}, y_{2t+1})}{2s}, d(y_{2t+2}, y_{2t+1}),\\
&\quad d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1}) \Bigg\}\Bigg)\\
&= \phi(\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+1}, y_{2t+2})\})\\
&\quad - \psi(\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+1}, y_{2t+2})\}.
\end{aligned}
$$

If

$$\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\} = d(y_{2t+2}, y_{2t+1}),$$

then, $\psi(d(y_{2t+2}, y_{2t+1}) = 0$, thus $y_n = y_{n+1}$ which is a contraction. So,

$$\max\{d(y_{2t+1}, y_{2t}), d(y_{2t+2}, y_{2t+1})\} = d(y_{2t+1}, y_{2t}). \tag{3.10}$$

Therefore,

$$\phi(sd(y_{2n+2}, y_{2n+1})) \leq \phi(d(y_{2t+1}, y_{2t})) - \psi(d(y_{2t+1}, y_{2t})).$$

From (3.9) and (3.10), we have

$$d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n).$$

Therefore $\{d(y_{n+1}, y_n) : n \in \mathbb{N}\}$ is a nonincreasing sequence. Thus there exists $r \geq 0$ such that

$$\lim_{t\to\infty} d(y_n, y_{n+1}) = r.$$

On taking limsup in (3.9) and (3.10), we have

$$\begin{aligned}\phi(sr) &\leq \phi(r) - \psi(r)\\ &= \phi(r).\end{aligned}$$

Thus,

$$\begin{aligned}\liminf_{t\to\infty} d(y_{2t-1}, y_{2t}) &= \liminf_{t\to\infty} d(y_{2t-1}, y_{2t+1})\\ &= \liminf_{t\to\infty} d(y_{2t}, y_{2t+2})\\ &= \liminf_{t\to\infty} d(y_{2t}, y_{2t+1})\\ &= 0.\end{aligned}$$

Therefore, $r = 0$ and hence

$$\lim_{n\longrightarrow\infty} d(y_n, y_{n+1}) = 0. \tag{3.11}$$

To show that $\{y_{2n}\}$ is a Cauchy sequence. If $\{y_{2n}\}$ is not Cauchy, there exists an $\epsilon > 0$, and monotone increasing sequence of natural numbers $\{2m(k)\}$ and $\{2n(k)\}$ such that $n(k) > m(k)$,

$$d(y_{2m(k),2n(k)}) \geq \epsilon$$

and

$$d(y_{2m(k)}, y_{2n(k)-1}) < \epsilon. \tag{3.12}$$

From (3.11), (3.12) and the triangular inequality, we obtain that

$$\begin{aligned}\epsilon &\leq d(y_{2m(k)}, y_{2n(k)})\\ &\leq sd(y_{2m(k)}, y_{2n(k)-1}) + sd(y_{2n(k)-1}, y_{2n(k)})\\ &\leq s\epsilon + sd(y_{2n(k)-1}, y_{2n(k)}).\end{aligned}$$

Furthermore,

$$\begin{aligned}&d(y_{2n(k)}, y_{2m(k)+1})\\ &\leq sd(y_{2n(k)}, y_{2n(k)-1}) + sd(y_{2n(k)-1}, y_{2m(k)+1})\\ &\leq sd(y_{2n(k)}, y_{2n(k)-1}) + s^2 d(y_{2n(k)-1}, y_{2m(k)})\\ &\quad + s^2 d(y_{2m(k)}, y_{2m(k)+1})\\ &\leq s^2\epsilon + s^2 d(y_{2m(k)}, y_{2m(k)+1}).\end{aligned}$$

Also, from triangular inequality, we get

$$\begin{aligned}\epsilon \le d(y_{2m(k)}, y_{2n(k)}) &\le sd(y_{2m(k)}, y_{2m(k)-1}) + sd(y_{2m(k)-1}, y_{2n(k)})\\ &\le sd(y_{2m(k)}, y_{2m(k)-1}) + s^2 d(y_{2m(k)-1}, y_{2m(k)+1})\\ &\quad + s^2 d(y_{2m(k)+1}, y_{2n(k)})\\ &\le sd(y_{2m(k)}, y_{2m(k)-1}) + s^3 d(y_{2m(k)-1}, y_{2m(k)})\\ &\quad + s^3 d(y_{2m(k)}, y_{2m(k)+1})\\ &\quad + s^2 d(y_{2m(k)+1}, y_{2n(k)}).\end{aligned}$$

Letting $k \to +\infty$ in above inequalities and using (3.11) and (3.12), we get

$$\limsup_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) \le s\epsilon$$

and

$$\frac{\epsilon}{s^2} \le \limsup_{k\to\infty} d(y_{2n(k)}, y_{2m(k)+1}) \le s^2\epsilon.$$

Similarly, we get

$$\frac{\epsilon}{s^2} \le \liminf_{k\to\infty} d(y_{2n(k)}, y_{2m(k)+1}) \le s^2\epsilon.$$

Since $x_{2n(k)}$ and $x_{2m(k)+1}$ are comparable, we have

$$\begin{aligned}&\phi(sd(y_{2n(k)}, y_{2m(k)+1}))\\ &\le \phi\Bigg(\max\Bigg\{d(y_{2m(k)+1}, y_{2m(k)})\frac{[1+d(y_{2n(k)}, y_{2n(k)-1})]}{1+d(y_{2n(k)}, y_{2m(k)+1})},\\ &\qquad \frac{d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)+1}, y_{2m(k)-1})}{2s},\\ &\qquad d(y_{2n(k)}, y_{2n(k)-1}), d(y_{2m(k)+1}, y_{2m(k)}), d(y_{2n(k)}, y_{2m(k)+1})\Bigg\}\Bigg)\\ &\quad - \psi\Bigg(\max\Bigg\{d(y_{2m(k)+1}, y_{2m(k)})\frac{[1+d(y_{2n(k)}, y_{2n(k)-1})]}{1+d(y_{2n(k)}, y_{2m(k)+1})},\\ &\qquad \frac{d(y_{2n(k)}, y_{2m(k)}) + d(y_{2m(k)+1}, y_{2m(k)-1})}{2s},\\ &\qquad d(y_{2n(k)}, y_{2n(k)-1}), d(y_{2m(k)+1}, y_{2m(k)}), d(y_{2n(k)}, y_{2m(k)+1})\Bigg\}\Bigg).\end{aligned}$$

Letting $k \longrightarrow \infty$ and using (3.11) and (3.12), we have

$$\begin{aligned}\phi(s^2\epsilon) &\le \phi\left(\max\left(0, \frac{\epsilon}{2}, 0, 0, s^2\epsilon\right)\right) - \psi\left(\max\left(0, \frac{\epsilon}{2}, 0, 0, s^2\epsilon\right)\right)\\ &< \phi(s^2\epsilon).\end{aligned}$$

So $\psi(s^2\epsilon) = 0$, and hence $\epsilon = 0$. Thus $\{y_{2n}\}$ is a Cauchy sequence in (X, d). Since X is complete, there exists $z \in X$such that $y_n \to z$ as $n \to \infty$.

Now, we show that z is the fixed point of g and T. Assume that TX is closed, since $\{y_{2n} = Tx_{2n+1}\}$ is a sequence in TX converging to z, we have $z \in TX$. So, there exists $u \in X$ such that $z = Tu$. Therefore,

$$\begin{aligned}\lim_{n\to\infty} fx_{2n} &= \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2}\\ &= z = Tu.\end{aligned}$$

Now, we show that $gu = z$. Since $x_{2n} \leq fx_{2n}$ and $y_{2n} = fx_{2n} \to z$, we have $x_{2n} \leq z$. Since the mapping f is monotone T nondecreasing, we obtain $x_{2n} \leq z = Tu \leq fTu \leq u$ and

$$\begin{aligned}&\phi(sd(y_{2n}, gu))\\ &= \phi(sd(fx_{2n}, gu))\\ &\leq \phi\Bigg(\max\Bigg\{ d(gu, Tu)\frac{[1 + d(fx_{2n}, Sx_{2n})]}{1 + d(fx_{2n}, gu)}, \frac{d(fx_{2n}, Tu) + d(gu, S_{2n})}{2S},\\ &\qquad d(fx_{2n}, Sx_{2n}), d(gu, Tu), d(fx_{2n}, gu)\Bigg\}\Bigg)\\ &= \phi\Bigg(\max\Bigg\{ d(gu, y)\frac{[1 + d(y_{2n}, y_{2n-1})]}{1 + d(y_{2n}, gu)}, \frac{d(y_{2n}, y) + d(gu, y_{2n-1})}{2S},\\ &\qquad d(y_{2n}, y_{2n-1}), d(gu, y), d(y_{2n}, gu)\Bigg\}\Bigg).\end{aligned}$$

Letting $n \to +\infty$ in the above inequalities and using (3.11) we get

$$\phi(sd(y, gu)) \leq \phi(d(y, gu)) - \psi(d(y, gu)).$$

Therefore $\psi(d(y, gu)) = 0$, and hence $d(y, gu) = 0$. Thus $y = gu$. Since g and T are weakly compatible, $gu = gTu = Tgu = Ty$.

Again, since x_{2n} and y are comparable, we have

$$\begin{aligned}&\phi(sd(y_{2n}, gy)) = \phi(sd(fx_{2n}, gy))\\ &\leq \phi\Bigg(\max\Bigg\{ d(gy, Ty)\frac{[1 + d(fx_{2n}, Sx_{2n})]}{1 + d(fx_{2n}, gy)}, \frac{d(fx_{2n}, Ty) + d(gy, S_{2n})}{2S},\\ &\qquad d(fx_{2n}, Sx_{2n}), d(gy, Ty), d(fx_{2n}, gy)\Bigg\}\Bigg)\\ &= \phi\Bigg(\max\Bigg\{ d(gy, gy)\frac{[1 + d(y_{2n}, y_{2n-1})]}{1 + d(y_{2n}, gy)}, \frac{d(y_{2n}, y) + d(gy, y_{2n-1})}{2S},\\ &\qquad d(y_{2n}, y_{2n-1}), d(gy, gy), d(y_{2n}, gy)\Bigg\}\Bigg).\end{aligned}$$

Letting $n \to \infty$ in the above inequalities and using (3.11), we have

$$\phi(sd(y, gy)) \leq \phi(d(y, gy)) - \psi(d(y, gy)).$$

Therefore, $\psi(d(y, gy)) = 0$ and hence $d(y, gy) = 0$. Thus $y = gy$.

Finally, we have to show that y is also a fixed point of f and T. Since $gX \subseteq SX$, there exists $a \in X$ such that $y = gy = Sa$. Since g is monotone S-nondecreasing map, we have $y \leq gy = Sa \leq gSa \leq a$. Thus y and a are comparable. □

Corollary 3.7. *Let $(X, d, s, \leq)$ be a complete partially b-metric space parameter $s \geq 1$. Suppose $S : X \to X$ is a continuous, nondecreasing map with regards to $\leq$ such that there exists x_0 with $x_0 \leq Sx_0$. Suppose that*

$$\phi(sd(Sx, Sy)) \leq \phi(M(x, y)) - \psi(M(x, y)), \tag{3.13}$$

where $\phi \in \Phi, \psi \in \Psi$, for any $x, y \in X$ with $x \leq y$ and,

$$M(x, y) = \max\left\{\frac{d(y, Sy)}{1 + d(x, y)}, \frac{d(x, Sy) + d(y, Sx)}{2s}, d(x, Sx), d(x, y)\right\}. \tag{3.14}$$

Then S has a fixed point in X.

Example 3.8. Define a metric $d : X \longrightarrow X$ as below and $\leq$ is an usual order on X, where $X = \{1, 2, 3, 4\}$,

$$\begin{array}{ll} d(x, y) = 0 & if \;\; x, y = 1, 2, 3 \text{ and } x = y, \\ d(x, y) = 1 & if \;\; x, y = 1, 2, 3 \text{ and } x \neq y, \\ d(x, y) = 5 & if \;\; x, y = 1, 2 \text{ and } y = 4, \\ d(x, y) = 20 & if \;\; x = 3 \text{ and } y = 4. \end{array}$$

Define a map $f : X \to X$ by $f1 = f2 = f3 = 1$, $f4 = 2$ and let $\phi(t) = t$, $\psi(t) = \frac{t}{2}$ for $t \in [0, \infty)$. Then f has a fixed point in X. In fact, it is apparent that, $(X, d, s, \leq)$ is a complete partially ordered b-metric space for $s = 2$. Consider the possible cases for x, y in X:

Case 1. Suppose $x, y \in \{1, 2, 3\}$ and $x < y$. Then

$$\phi(2d(fx, fy)) = 0 \leq \phi(M(x, y)) - \psi(M(x, y)).$$

Case 2. Suppose $x \in \{1, 2, 3\}$ and $y = 4$. Then $d(fx, fy) = d(1, 2) = 1$, $M(x, 4) = 5$ if $x = \{1, 2\}$ and $M(3, 4) = 20$. Therefore, we have the following inequality,

$$\phi(2d(fx, fy)) \leq \phi(M(x, y)) - \psi(M(x, y)).$$

Thus, condition (3) of Corollary 3.7 holds. Then, f has a fixed point in X.

Corollary 3.9. *If the inequality* (1) *of Theorem* 3.6 *is replaced by*

$$\begin{aligned}
&\phi(sd(fx,gy)) \\
&\le \phi\left(\max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(Sx,gy)+d(fx,Ty)}{2s}\right\}\right) \\
&\quad - \phi(d(Sx,Ty), d(Sx,gy), d(fx,Ty)) \\
&\quad + L\min\{d(Sx,Ty), d(Sx,gy), d(fx,Ty)\},
\end{aligned}$$

for all $x, y \in X, L \in [0,\infty)$. *Then* f, g, S *and* T *have a common fixed point in* X.

References

[1] T. Abdeljawad and M. Keeler, *$\alpha-\epsilon$-contractive fixed and common fixed point theorems*, Fixed Point Theory Appl., **2013**:19 (2013).

[2] K. Abodayeh, N. Mlaiki, T. Bdeljawad and W. Shatanawi, *Relations between partial metric spaces and M-metric, Caristi Kirks theorem in M-metric type spaces*, J. Math. Anal., **7**(3) (2016), 1–12.

[3] H. Alsamir, M. Salmi, MD. Noorani, W. Shatanawi and F. Shaddad, *Generalized Berinde-type $(\eta,\xi,\vartheta,\theta)$ contractive mapping in b-metric spaces with an application*, J. Math. Anal., **7**(6) (2016), 1–12.

[4] E. Ameer, M. Arshad and W. Shatanawi, *Common fixed point results for generalized $\alpha_*-\psi-$contraction multivalued mappings in $b-$metric spaces*, J. Fixed Point Theory Appl., **19** (2017), 3069-3086.

[5] H. Aydi, M. Bota, E. Karapinar and S. Mitrovic, *A fixed point theorem for set valued quasi-contractions in b-metric spaces,* Fixed Point Theory Appl., **2012**:88 (2012) doi:10.1186/16871812201288.

[6] H. Aydi, W. Shatanawi, M, Postolache, Z. Mustafa and N. Tahat, *Theorems for Boyd-Wong-type contractions in ordered metric spaces*, Abst. Appl. Anal., **2012** (2012), Articl No.: 359054.

[7] A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., Gos. Ped. Inst. Unianowsk, **30** (1989), 26-37.

[8] S. Banach, *Sur les operations dans les ensembles et leur application aux equation sitegrales*, Fundam. Math., **3** (1922), 133–181.

[9] A. Bataihah, W. Shatanawi and A. Tallafha, *Fixed point results with simulation functions*, Nonlinear Funct Anal. Appl., **25**(1) (2020), 13–23.

[10] A. Bataihah, A. Tallafha and W. Shatanawi, *Fixed point results with $\Omega-$distance by utilizing simulation functions*, Italian J. Pure and Appl. Math., **43** (2020), 185–196.

[11] A. Bataihah, W. Shatanawi, T. Qawasmeh and R. Hatamleh, *On H-simulation functions and fixed point results in the setting of wt-distance mappings with application on matrix equations*, Mathematics, **8**(5) (2020), Article No. 837.

[12] M. Boriceanu, *Strict fixed point theorem for multivalued operators in b-metric space,* Int. J. Mod. Math., **4** (2009), 285-301.

[13] S. Czerwik, *Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav., **1** (1993), 5-11.

[14] Y. Hao and H. Guan, *Common fixed point theorems in b-metric spaces*, Turkish J. Anal. Number Theory, **7**(4) (2019), 117-123, DOI:10.12691/tjant-7-4-4.

[15] J. Harjani, B. Lopez and K. Sadarangani, *Fixed point theorems for weakly C-contractive mappings in ordered metric spaces*, Comput. Math. Appl., **61**(4) (2011), 790–796.

[16] E. Karapinar and W. Shatanawi, *On weakly (C, ψ, ϕ)-contractive mappings in ordered partial metric spaces*, Abst. Appl. Anal., **2012** (2012), Article No: 4495892, doi: 10.1155/2012/495892.

[17] H. Kerim, W. Shatanawi and A. Tallafha, *Common fixed point theorem for set-valued maps on modular b-gauge spaces*, Palestine J. Math., **9**(2) (2020), 1-10.

[18] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points,* Bull. Austr. Math. Soc., **30**(1) (2009), 1–9.

[19] K. Kukiċ, W. Shatanawi, M.G. Filipoviċ, Khan and Ċiriċ *Contraction principles in almost $b-$metric space*, U.P.B. Sci. Bull., Series A, **82**(1) (2020), 133-144.

[20] A. Mukheimer, *$\alpha - \psi - \phi$-contractive mappings in ordered partial b-metric spaces,* J. Nonlinear Sci. Appl., **7**(3) (2014), 168–179.

[21] J.R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei and W. Shatanawi, *Common fixed points of almost generalized $(\psi, \phi)_s$-contractive mappings in ordered b-metric spaces*, Fixed Point Theory Appl., **2013**:159 (2013), 1–23.

[22] A. Mukheimer, N. Mlaiki, K. Abdayeh and W. Shatanawi, *New theorems on extended b-metric spaces under new contractions*, Nonlinear Anal.: Model. Contral., **24**(6) (2019), 870–883.

[23] H. Qawaqneh, M.S. Noorani, H. Aydi and W. Shatanawi, *On common fixed point results for new contractions with applications to graph and integral equations*, Mathematics, **7**(11) (2019). Doi:10.3390/math7111082.

[24] H. Qawaqneh, M.S.M. Noorani and W. Shatanawi, *Fixed point theorems for (α, k, θ)-contractive multi-valued mapping in b-metric space and applications*, Int. J. Math. Comput. Sci., **14**(1) (2019), 263–283.

[25] K.P.R. Rao, W. Shatanawi, G.N.V. Kishore, K. Abodayeh and D.R. Prasad, *Existences and uniqueness of Suzuki type results in S_b-metric spaces with applications to integral equations*, Nonlinear Funct. Anal. Appl., **23**(2) (2018), 225-245.

[26] H. Qawaqneh, M.S.M. Noorani, S. Shatanawi, H. Aydi and H. Alsamir, *Fixed point results for multi-valued contractions in b-metric spaces and an application*, Mathematics, **7**(2) (2019), Article No. 132.

[27] T. Qawasmeh, A. Tallafha and W. Shatanawi, *Fixed and common fixed point theorems through modified ω-distances mappings*, Nonlinear Funct. Anal. Appl., **24**(2) (2019), 221-239.

[28] J.R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei and W. Shatanawi, *Common fixed points of almost generalized $(\psi, \phi)_s$-contractive mappings in ordered b-metric spaces*, Fixed Point Theory Appl., **2013**: 159(1) (2013), doi:10.1186/1687-1812-2013-159.

[29] S. Sedghi, N. Shobkolaei, J. Rezaei Roshan and W. Shatanawi, *Coupled fixed point theorems in G_b-metric spaces*, Matematicki Vesnik, **66**(2) (2014), 190-201.

[30] W. Shatanawi, *Fixed and common fixed point for mapping satisfying some nonlinear contraction in b-metric spaces*, J. Math. Anal.,**7**(4) (2016), 1–12.

[31] W. Shatanawi, A. Pitea and Lazovic, *Contraction conditions using comparison functions on b-metric spaces*, Fixed Point Theory Appl., **2014**:135 (2014), doi:10.1186/1687-1812-2014-135.

[32] W. Shatanawi, M. Postolache, A.H. Ansari and W. Kassab,*Common fixed points of dominating and weak annihilators in ordered metric spaces via C-class functions*, J. Math. Anal., **8**(3) (2017), 54–68.

[33] W. Shatanawi and M. Postolache, *Common fixed point theorems for dominating and weak annihilator mappings in ordered metric spaces*, Fixed Point Theory Appl., **2013**:271 (2013). doi.org/10.1186/1687-1812-2013-271.

[34] W. Shatanawi, *Fixed point theorems for nonlinear weakly C-contractive mappings in metric spaces*, Math. Comput. Model., **54**(11-12) (2011), 2816-2826.

[35] W. Shatanawi and A. Pitea, *Fixed and coupled fixed point theorems of omega-distance for nonlinear contraction*, Fixed Point Theory Appl., **2013**:275 (2013), doi: 10.1186/1687-1812-2013-275.

[36] W. Shatanawi, Z. Mustafa and N. Tahat, *Some coincidence point theorems for nonlinear contraction in ordered metric spaces*, Fixed Point Theory Appl.,**2011**:68 (2011), doi:10.1186/1687-1812-2011-68.

[37] W. Shatanawi, A. Bataihah and A. Tallafha, *Four-step iteration scheme to approximate fixed point for weak contractions*, Comput. Mate. Continua, **64**(3) (2020), 1491-1504.

[38] W. Shatanawi, *On w-compatible mappings and common coupled coincidence point in cone metric spaces*, Appl. Math. Letter, **25**(6) (2012), 925–931

[39] W. Shatanawi, *Some fixed point results for a generalized $\psi-$weak contraction mappings in orbitally metric spaces*, Chaos, Solitons and Fractals, **45**(4) (2012), 520–526.

[40] W. Shatanawi, E. Karapnar and H. Aydi, *Coupled coincidence points in partially ordered cone metric spaces with a c-distance*, Journal of Applied Mathematics, Volume **2012**, 2012, Article number 312078

[41] W. Shatanawi, V.C. Rajic, S.C. Radenovic and A. Al-Rawashdeh, *Mizoguchi-Takahashi-type theorems in tvs-cone metric spaces*, Fixed Point Theory Appl., **2012**:106 (2012). Article No. 106.

[42] W. Shatanawi and A. Al-Rawashdeh: *Common fixed point of almost generalized (ψ,ϕ)-contractive mappings in ordered metric space.* Fixed point theory Appl. **2012**:80 (2012), Article No. 80.

[43] W. Shatanawi and S. Manro, *Fixed point results for cyclic (ψ,ϕ,A,B)-contraction in partial metric space.* Fixed point Theory Appl., **2012**:165 (2012), Article No. 165.

[44] W. Shatanawi, Z.D. Mitrovic, N. Hussain and S. Radenovic, *On generalized Hardy Rogers type $\alpha-$admissible mappings in cone b-metric spaces over Banach algebras*, Symmetry, **12**(1), 81 (2020). doi:10.3390/sym12010081.

[45] W. Shatanawi, K. Abodayeh and A. Mukheimer, *Some fixed point theorems in extended $b-$metric spaces*, U.P.B. Sci. Bull. Series A, **80**(4) (2018), 71-78.