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CONVERGENCE THEOREM FOR A GENERALIZED φ-WEAKLY CONTRACTIVE NONSELF MAPPING IN METRICALLY CONVEX METRIC SPACES

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Abstract. A convergence theorem for a generalized φ-weakly contractive mapping is proved which satisfy a generalized contraction condition on a complete metrically convex metric space. The result in this paper generalizes the relevant results due to Rhoades [18], Alber and Guerre-Delabriere [1], Khan and Imdad [14], Xue [20] and others. An illustrative example is also furnished in support of our main result.

1. Introduction and preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X.$$

Theorem 1.1. ([4]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction mapping. Then T has a unique fixed point.*

There are so many extensions and generalizations of the theorem mentioned above (see [5]-[9]).

A mapping $T : X \to X$ is said to be a *φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi^{-1}(0) = \{0\}$

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and $\lim_{t\to\infty} \varphi(t) = \infty$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \forall\, x, y \in X. \tag{1.1}$$

If X is bounded, then the infinity condition can be omitted.

The concept of the φ-weak contraction was introduced by Alber and Guerre-Delabriere [1] in 1997, who proved the existence of fixed points in Hilbert spaces. Later Rhoades [18] in 2001 extended the results of [1] to metric spaces.

Theorem 1.2. ([18]) *Let (X, d) be a complete metric space and $T : X \to X$ be a φ-weakly contractive self-map on X. Then T has a unique fixed point p in X.*

Remark 1.3. Theorem 1.2 is one of generalizations of the Banach contraction principle because it takes $\varphi(t) = (1 - \alpha)t$ for $\alpha \in (0, 1)$, then φ-weak contraction contains contraction as special cases.

In 2016, Xue [20] introduced a new contraction type mapping as follows.

Definition 1.4. ([20]) A mapping $T : X \to X$ is said to be a *generalized φ-weak contraction* if there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \quad \forall\, x, y \in X \tag{1.2}$$

holds.

We notice immediately that if $T : X \to X$ is a φ-weak contraction, then T is a generalized φ-weak contraction. However, the converse is not true in general.

Example 1.5. ([15]) Let $X = (-\infty, +\infty)$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ and let $Tx = \frac{2}{5}x$ for each $x \in X$.

Define $\varphi(t) : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = \frac{4}{3}t$. Then T satisfies (1.2), but T does not satisfy inequality (1.1). Indeed,

$$\begin{aligned} d(Tx, Ty) &= \left| \frac{2}{5}x - \frac{2}{5}y \right| \\ &\leq |x - y| - \frac{4}{3} \cdot \frac{2}{5}|x - y| \\ &= d(x, y) - \varphi(d(Tx, Ty)) \end{aligned}$$

and

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{2}{5}x - \frac{2}{5}y\right| \\ &\geq |x-y| - \frac{4}{3}|x-y| \\ &= d(x,y) - \varphi(d(x,y)) \end{aligned}$$

for all $x, y \in X$.

Example 1.6. ([20]) Let $X = [0, +\infty)$ be endowed by $d(x,y) = |x-y|$ and let $Tx = \frac{x}{1+x}$ for each $x \in X$. Define $\varphi : [0,+\infty) \to [0,+\infty)$ by $\varphi(t) = \frac{t^2}{1+t}$. Then

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{x}{1+x} - \frac{y}{1+y}\right| = \frac{|x-y|}{(1+x)(1+y)} \\ &\leq \frac{|x-y|}{1+|x-y|} = |x-y| - \frac{|x-y|^2}{1+|x-y|} \\ &= d(x,y) - \varphi(d(x,y)) \end{aligned}$$

holds for all $x, y \in X$. So T is a φ-weak contraction. However, T is not a contraction.

Remark 1.7. The above examples show that the class of generalized φ-weak contractions properly includes the class of φ-weak contractions and the class of φ-weak contractions properly includes the class of contractions. In fact, let $T : X \to X$ be a contraction, there exists $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \leq \alpha \cdot d(x,y), \quad \forall\, x, y \in X.$$

Then

$$\begin{aligned} d(Tx,Ty) \leq \alpha \cdot d(x,y) &= d(x,y) - (1-\alpha)d(x,y) \\ &= d(x,y) - \varphi(d(x,y)), \end{aligned}$$

where, $\varphi(d(x,y)) = (1-\alpha)d(x,y)$. So, T is a φ-weak contraction. Moreover, let T be a φ-weak contraction, from property of φ, we have $d(Tx,Ty) \leq d(x,y)$ and

$$\varphi(d(Tx,Ty)) \leq \varphi(d(x,y)).$$

From (1.1),

$$\begin{aligned} d(Tx,Ty) &\leq d(x,y) - \varphi(d(x,y)) \\ &\leq d(x,y) - \varphi(d(Tx,Ty)), \quad \forall\, x, y \in X. \end{aligned}$$

Therefore, T is a generalized φ-weak contraction.

In many applications the mappings involved is not always a self map. So, it is interested to find sufficient conditions for such mappings will guarantee the existence of a fixed point. Assad and Kirk [3] initiated the study of fixed point of nonself mappings in metrically convex spaces. The technique due to Assad and Kirk [3] has been utilized by many researchers and there exists considerable literature on this topic. To mentioned a few, we cite [2], [10]-[17], [19], [21].

In this paper, we prove a convergence theorem for single valued nonself mapping by utilizing the idea of Rhoades [18] which either partially or completely generalize the results due to Rhoades [18], Alber and Guerre-Delabriere [1], Khan and Imdad [14] and others.

Definition 1.8. Let K be a nonempty subset of a metric space (X, d). A mapping $T : K \to X$ is said to be a generalized φ-weak contraction if $Tx \in K$, $Tx \cap K$ is nonempty and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(Tx, Ty)), \tag{1.3}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function with $\varphi(t) = 0$ iff $t = 0$.

Definition 1.9. ([3]) A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X$, $x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Remark 1.10. ([3]) If K is a closed subset of the complete and convex metric space X and if $x \in K$, $y \notin K$, then there exists a point z in the boundary of K such that

$$d(x, z) + d(z, y) = d(x, y).$$

We use the symbol ∂K to denote the boundary of K.

2. Main result

In this section, we will prove convergence theorem for a single valued generalized φ-weak contractive nonself mapping.

Theorem 2.1. *Let (X, d) be a complete metrically convex metric space and K be a nonempty closed subset of X. Let $T : K \to X$ be a generalized φ-weak contraction defined as (1.3) for each $x \in \partial K$ and $Tx \in K$. Then T has a unique fixed point in K.*

Proof. We proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x_0 \in K$. Define $y_1 = Tx_0$. If $y_1 \in K$, set $y_1 = x_1$. If $y_1 \notin K$, then choose $x_1 \in \partial K$ such that

$$d(x_0, x_1) + d(x_1, y_1) = d(x_0, y_1).$$

Define $y_2 = Tx_1$. If $y_2 \in K$, set $y_2 = x_2$. If $y_2 \notin K$, then choose $x_2 \in \partial K$ such that

$$d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2).$$

Continuing this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{n+1} = Tx_n$,

(ii) $\begin{cases} x_n = y_n, \quad \text{if } y_n \in K, \\ \text{choose } x_n \in \partial K \text{ such that} \\ \qquad d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n), \quad \text{if } y_n \notin K. \end{cases}$

Here, we have two types of sets we denote as follows:

$$\begin{aligned} P &= \{x_i \in \{x_n\} : x_i = y_i,\ i = 1, 2, \cdots\} \\ &= \{x_i \in \{x_n\} : y_i = Tx_{i-1} \in K,\ i = 1, 2, \cdots\} \end{aligned}$$

and

$$\begin{aligned} Q &= \{x_i \in \{x_n\} : x_i \neq y_i,\ i = 1, 2, \cdots\} \\ &= \{x_i \in \{x_n\} : y_i = Tx_{i-1} \notin K,\ i = 1, 2, \cdots\} \\ &= \{x_i \in \{x_n\} : d(x_{i-1}, x_i) + d(x_i, y_i) = d(x_{i-1}, y_i),\ x_i \in \partial K,\ i = 1, 2, \cdots\}. \end{aligned}$$

Observe that if $x_n \in Q$ for some n, then x_{n-1} and $x_{n+1} \in P$.

Now, we wish to estimate $d(x_n, x_{n+1})$. Now we distinguish the following cases.

Case 1. If x_n and $x_{n+1} \in P$, then

$$\begin{aligned} d(x_n, x_{n+1}) = d(y_n, y_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\le d(x_{n-1}, x_n) - \varphi(d(x_n, x_{n+1})) \\ &\le d(x_{n-1}, x_n), \end{aligned}$$

by monotonicity of φ.

Case 2. If $x_n \in P$ and $x_{n+1} \in Q$, then

$$d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1}).$$

Therefore

$$\begin{aligned} d(x_n, x_{n+1}) &\le d(x_n, y_{n+1}) = d(y_n, y_{n+1}) = d(Tx_{n-1}, Tx_n) \\ &\le d(x_{n-1}, x_n) - \varphi(d(y_n, y_{n+1})) \\ &\le d(x_{n-1}, x_n). \end{aligned}$$

Case 3. If $x_n \in Q$ and $x_{n+1} \in P$. Since $x_n \in Q$ and $x_{n-1} \in P$, we have $x_{n-1} = y_{n-1}$. It follows

$$\begin{aligned} d(x_n, x_{n+1}) &\le d(x_n, y_n) + d(y_n, x_{n+1}) = d(x_n, y_n) + d(y_n, y_{n+1}) \\ &= d(x_n, y_n) + d(Tx_{n-1}, Tx_n) \\ &\le d(x_n, y_n) + d(x_{n-1}, x_n) - \varphi(d(y_n, y_{n+1})) \\ &\le d(x_{n-1}, y_n) = d(y_{n-1}, y_n) = d(Tx_{n-2}, Tx_{n-1}) \\ &\le d(x_{n-2}, x_{n-1}) - \varphi(d(y_{n-1}, y_n)) \\ &\le d(x_{n-2}, x_{n-1}). \end{aligned}$$

Case 4. The only other possibility, $x_n \in Q$ and $x_{n+1} \in Q$, cannot occur.

Thus in all the cases, we have

$$\begin{aligned} &d(x_n, x_{n+1}) \\ &\le \max\{d(x_{n-1}, x_n) - \varphi(d(y_n, y_{n+1})), d(x_{n-2}, x_{n-1}) - \varphi(d(y_{n-1}, y_n))\} \\ &\le \max\{d(x_{n-1}, x_n), d(x_{n-2}, x_{n-1})\}. \end{aligned} \tag{2.1}$$

By (2.1), it follows that the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing. Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.2}$$

Next, we prove that the sequence $\{x_n\}$ is Cauchy. Suppose that the sequence $\{x_n\}$ is not Cauchy. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{n_l}\}$ such that

$$d(x_{n_k}, x_{n_l}) \ge \varepsilon.$$

Corresponding to each n_k, we can find n_l in such a way that the smallest positive integer $n_l > n_k$ satisfying

$$d(x_{n_k}, x_{n_l-1}) < \varepsilon.$$

Then we have

$$\varepsilon \le d(x_{n_k}, x_{n_l}) \le d(x_{n_k}, x_{n_l-1}) + d(x_{n_l-1}, x_{n_l}) < \varepsilon + d(x_{n_l-1}, x_{n_l}).$$

Letting $n \to \infty$, we have

$$\lim_{n\to\infty} d(x_{n_k}, x_{n_l}) = \varepsilon. \tag{2.3}$$

By (2.1), if $d(x_{n-1}, x_n) - \varphi(d(y_n, y_{n+1})) \geq d(x_{n-2}, x_{n-1}) - \varphi(d(y_{n-1}, y_n))$, then

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) - \varphi(d(y_n, y_{n+1})).$$

So we have

$$\begin{aligned} d(x_{n_k}, x_{n_l}) &\leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k+2}) + \cdots + d(x_{n_l-1}, x_{n_l}) \\ &\leq d(x_{n_k-1}, x_{n_k}) - \varphi(d(y_{n_k}, y_{n_k+1})) \\ &\quad + d(x_{n_k}, x_{n_k+1}) - \varphi(d(y_{n_k+1}, y_{n_k+2})) \\ &\quad \vdots \\ &\quad + d(x_{n_l-2}, x_{n_l-1}) - \varphi(d(y_{n_l-1}, y_{n_l})). \end{aligned} \tag{2.4}$$

If $d(x_{n-1}, x_n) - \varphi(d(y_n, y_{n+1})) \leq d(x_{n-2}, x_{n-1}) - \varphi(d(y_{n-1}, y_n))$, then

$$d(x_n, x_{n+1}) \leq d(x_{n-2}, x_{n-1}) - \varphi(d(y_{n-1}, y_n)).$$

So we get

$$\begin{aligned} d(x_{n_k}, x_{n_l}) &\leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k+2}) + \cdots + d(x_{n_l-1}, x_{n_l}) \\ &\leq d(x_{n_k-2}, x_{n_k-1}) - \varphi(d(y_{n_k-1}, y_{n_k})) \\ &\quad + d(x_{n_k-1}, x_{n_k}) - \varphi(d(y_{n_k}, y_{n_k+1})) \\ &\quad \vdots \\ &\quad + d(x_{n_l-3}, x_{n_l-2}) - \varphi(d(y_{n_l-2}, y_{n_l-1})). \end{aligned} \tag{2.5}$$

From (2.2), (2.3) and $\varepsilon > 0$, on letting $n \to \infty$ in (2.4) and (2.5), we have a contradiction. Thus the sequence $\{x_n\}$ is Cauchy and since X is complete and K is closed, $\{x_n\}$ converges to a point $p \in K$. Also observe that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ each of whose terms is in the set P, that is

$$x_{n_j} = y_{n_j}, \quad j = 1, 2, \cdots .$$

Thus by condition (i),

$$y_{n_j} = Tx_{n_j-1}, \quad j = 1, 2, \cdots .$$

Since $x_{n_j-1} \to p$ as $j \to \infty$, we have $Tx_{n_j-1} \to Tp$ as $j \to \infty$, that is,

$$\lim_{j \to \infty} y_{n_j} = Tp.$$

Now, we will show that p is a fixed point of T. Suppose that $p \neq Tp$. Then $d(Tp, p) > 0$. So we get

$$\begin{aligned} d(T_p, x_{n_j+1}) &= d(Tp, y_{n_j+1}) = d(Tp, Tx_{n_j}) \\ &\leq d(p, x_{n_j}) - \varphi(d(Tp, x_{n_j+1})), \end{aligned}$$

that is,

$$d(Tp, x_{n_j+1}) + \varphi(d(Tp, x_{n_j+1})) \leq d(p, x_{n_j}).$$

Taking the limit as $j \to \infty$ to above inequality,

$$d(Tp, p) + \varphi(d(Tp, p)) \leq d(p, p) = 0.$$

Since $d(Tp, p) > 0$, this is a contradiction. Therefore

$$Tp = p.$$

Finally, we will prove that the uniqueness of fixed point. Assume that p and q are two fixed points of T. Then

$$\begin{aligned} d(p, q) &= d(Tp, Tq) \\ &\leq d(p, q) - \varphi(d(Tp, Tq)) \\ &= d(p, q) - \varphi(d(p, q)), \end{aligned}$$

that is,

$$\varphi(d(p, q)) \leq 0.$$

By the condition of φ, we have

$$d(p, q) = 0.$$

Therefore

$$p = q.$$

This completes the proof. □

Remark 2.2. By setting $K = X$ in Theorem 2.1, one deduces a theorem due to Xue [20].

3. An illustrative example

In this section, we furnish an example to establish the utility of our result.

Example 3.1. Let $X = \mathbb{R}$ with Euclidean metric and $K = [0, 1]$. Define $T : K \to X$ and $\varphi : [0, \infty) \to [0, \infty)$ by

$$Tx = \frac{2}{5}(x - x^2), \quad \varphi(t) = \frac{4}{3}t.$$

Since boundary of K, $\partial K = \{0, 1\}$, we have

$$T0 = 0 \in K, \quad T1 = 0 \in K.$$

For the verification of generalized φ-weak contraction condition (1.3) the following cases arise:

$$\begin{aligned}
d(Tx,Ty) &= |Tx-Ty| \\
&= \frac{2}{5}|(x-y)-(x-y)(x+y)| \\
&= \frac{2}{5}|(x-y)|\cdot|1-x-y| \\
&\le |x-y| - |x-y|\cdot|1-x-y| + \frac{2}{5}|(x-y)|\cdot|1-x-y| \\
&= |x-y| - \frac{3}{5}|x-y|\cdot|1-x-y| \\
&< |x-y| - \frac{8}{15}|x-y|\cdot|1-x-y| \\
&= |x-y| - \frac{4}{3}\cdot\frac{2}{5}|x-y|\cdot|1-x-y| \\
&= d(x,y) - \varphi(d(Tx,Ty)) \quad \forall\, x,y \in K.
\end{aligned}$$

Thus the generalized φ-weak contraction condition (1.3) and all other conditions of the Theorem 2.1 are satisfied. Moreover, 0 is the unique fixed point of T.

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