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EXISTENCE OF SOLUTION FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS VIA TOPOLOGICAL DEGREE METHOD

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ABSTRACT. This paper is studied the existence of a solution for the impulsive Cauchy problem involving the Caputo fractional derivative in Banach space by using topological structures. We based on using topological degree method and fixed point theorem with some suitable conditions. Further, some topological properties for the set of solutions are considered. Finally, an example is presented to demonstrate our results.

1. INTRODUCTION

Fractional differential equations have proved to be effective modeling of many physical phenomena and various fields for more details, see Kilbas et al. [1], Miller and Ross [2], Podlubny [3], Deimling [4]. Topological degree method is one of the important tools that procedure needs weakly compact conditions instead of strongly compact conditions. In fact, topological methods become very closely to study the existence of solutions of fractional differential equations in the last decades, see Feckan [5], and Mawhin [6]. The fractional differential equations in Banach space have recently been receiving more attention by many researchers such as Agarwal et al. [7], Balachandran and Park [8], and Zhang [9]. In 2009, Benchohra and Seba [10], considered the existence of solutions for impulsive fractional differential equations in a Banach space by Monch's fixed point theorem and the technique of measures of non compactness. In 2010, Ahmad and Sivasundaram [11], studied the existence of solutions for impulsive integral boundary value problems with fractional order by applying the contraction mapping principle and Krasnoselskii's fixed point theorem. In 2012, Wang et.al [12, 13], studied existence, uniqueness and data dependence for the solutions for impulsive Cauchy problems with fractional order by degree method for condensing maps by a singular Gronwall inequality. In 2012, Feckan et. al [14], corrected a formula of solutions for impulsive fractional differential

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equations which cited in the previous paper and they established some sufficient conditions for existence of the solutions by using fixed point methods. Motivated from some cited results, our aim in this paper is to confirm some new results on the following impulsive Cauchy problem (ICP) for fractional differential equations involving the Caputo fractional derivative by topological degree method and fixed point theorem.

$$\begin{cases} {}^c\mathcal{D}^q x(t) & = \xi(t, x(t)) \quad t \in \mathcal{J}/\{t_1, t_2, ..., t_m\}, \mathcal{J} := [0, T] \\ x(0) = x_0, & \\ \Delta x(t_k) & = I_k(x(t_k)) \qquad\qquad k = 1, 2, ..., m, \end{cases} \tag{1.1}$$

where ${}^c\mathcal{D}^q$ is the Caputo fractional derivative of order $q \in (0, 1)$, x_0 is an element of $\mathcal{X}$, $\xi : \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ is a given jointly continuous linear map, and $\mathcal{PC}(\mathcal{J}, \mathcal{X})$ is a Banach space with the norm $\|x\|_{PC} = sup\{\|x(t)\| : t \in \mathcal{J}\}$, $I_k : \mathcal{X} \to \mathcal{X}$ is a continuous map and t_k satisfies, $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = T$.

2. Preliminaries

In this section, we introduce some necessary definitions and theorems which are needed throughout this paper.

We define a Banach space $\mathcal{PC}(\mathcal{J}, \mathcal{X}) = \{x : \mathcal{J} \to \mathcal{X} : x \in \mathcal{C}((t_k, t_{k-1}], \mathcal{X})\}$, for $k = 0, ..., m$ and there exist $x(t_k^+)$ and $x(t_k^-)$ such that $x(t_k^-) = x(t_k),\ x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$ and $x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$.

Definition 2.1. *([2]) For a given function ξ on the closed interval $[a, b]$, the qth fractional order integral of ξ is defined by;*

$$\mathcal{I}^q_{a+}\xi(t) = \frac{1}{\Gamma(q)} \int_a^t (t - s)^{q-1} \xi(s) ds,$$

wherever Γ is the gamma function.

Definition 2.2. *([2]) For a given function ξ on the closed interval $[a, b]$, the qth Riemann-Liouville fractional-order derivative of ξ, is defined by;*

$$(\mathcal{D}^q_{a+}\xi)(t) = \frac{1}{\Gamma(n - q)} (\frac{d}{dt})^n \int_a^t (t - s)^{n-q-1} \xi(s) ds.$$

Here $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Definition 2.3. *([2]) For a given function ξ on the closed interval $[a, b]$, the Caputo fractional order derivative of ξ, is defined by;*

$$({}^c\mathcal{D}^q_{a+}\xi)(t) = \frac{1}{\Gamma(n - q)} \int_a^t (t - s)^{n-q-1} \xi^{(n)}(s) ds,$$

where $n = [q] + 1$.

Theorem 2.1. *(Banach contraction mapping principle)([15])*
Let $\mathcal{X}$ be a Banach space, and $\psi : \mathcal{X} \to \mathcal{X}$ is a contraction mapping with contraction constant $\mathcal{K}$, then ψ has a unique fixed point.

Theorem 2.2. *(Schaefer's fixed point theorem)(*[15]*)*
Let $\mathcal{K}$ be a nonempty convex, closed and bounded subset of a Banach space $\mathcal{X}$. If $\psi : \mathcal{K} \to \mathcal{K}$ is a complete continuous operator such that $\psi(\mathcal{K}) \subset \mathcal{X}$, then ψ has at least one fixed point in $\mathcal{K}$.

Lemma 2.1. *(*[14]*) Let $q \in (0,1)$ and $\xi : \mathcal{X} \times \mathcal{J} \to \mathcal{X}$ be continuous. A function $x \in \mathcal{C}(\mathcal{J}, \mathcal{X})$ is said to be a solution of the fractional integral equation*

$$x(t) = x_0 - \frac{1}{\Gamma(q)} \int_0^a (a-s)^{q-1} \xi(s, x(s)) ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \xi(s, x(s)) ds,$$

if and only if x is a solution of the following fractional Cauchy problems

$$\begin{cases} {}^c\mathcal{D}^q x(t) & = \xi(t, x(t)), \ t \in \mathcal{J}, \\ x(a) & = x_0, \qquad a > 0 \end{cases}$$

3. Main Results

First of all, let us define the mean of a solution of the ICP(1.1).

Definition 3.1. *If a function $x \in \mathcal{PC}(\mathcal{J}, \mathcal{X})$ satisfies the equation ${}^c\mathcal{D}^q x(t) = \xi(t, x(t))$ almost everywhere on $\mathcal{J}$, and the condition $\Delta x(t_k) = I_k(x(t_k)), k = 1, 2, ..., m$ and $x(0) = x_0$ then, x is said to be a solution of the fractional ICP*(1.1).

In order to treat the problem of existence for a solution of ICP(1.1), we need the following assumptions:

H1: $\xi : \mathcal{J} \times \mathcal{X} \to \mathcal{X}$ is jointly continuous.
H2: For arbitrary $x, y \in \mathcal{X}$, there exists a constant $\delta_\xi > 0$, such that

$$\|\xi(t, x) - \xi(t, y)\| \leq \delta_\xi \|x - y\|$$

H3: For arbitrary $(t, x) \in \mathcal{J} \times \mathcal{X}$, there exist $\delta_1, \delta_2 > 0$, $q_1 \in [0, 1)$ such that

$$\|\xi(t, x)\| \leq \delta_1 \|x\|^{q_1} + \delta_2.$$

H4: $I_k : \mathcal{X} \to \mathcal{X}$ is continuous and there is a constant $\gamma_I \in [0, \frac{1}{m})$ such that

$$\|I_k(x) - I_k(y)\| \leq \gamma_I \|x - y\|, \quad for\ all\ x, y \in \mathcal{X},\ k = 1, 2, ..., m$$

H5: For arbitrary $x \in \mathcal{X}$, there exist $\gamma_1, \gamma_2 > 0,\ q_2 \in [0, 1)$ such that

$$\|I_k(x)\| \leq \gamma_1 \|x\|^{q_2} + \gamma_2, \ \ k = 1, 2, ..., m$$

Lemma 3.1. *The fractional integral*

$$\begin{aligned} x(t) = x_0 &+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \xi(s, x(s)) ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} \xi(s, x(s)) ds + \sum_{0<t_k<t} I_k(x(t_k)) \end{aligned} \tag{3.1}$$

has a solution $x \in \mathcal{PC}(\mathcal{J},\mathcal{X})$, for $t \in (t_k, t_{k+1}),\ k = 1,2,...,m$ if and only if x is a solution of the fractional ICP(1.1).

Proof. First, assume $x \in \mathcal{PC}(\mathcal{J},\mathcal{X})$ satisfies ICP(1.1), we have to show that the fractional integral Eq. (3.1) has at least one solution $x \in \mathcal{PC}(\mathcal{J},\mathcal{X})$. Consider the operator $\mathcal{F} : \mathcal{PC}(\mathcal{J},\mathcal{X}) \to \mathcal{PC}(\mathcal{J},\mathcal{X})$ defined by;

$$(\mathcal{F}x)(t) = x(t) = x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \xi(s, x(s)) ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \xi(s, x(s)) ds + \sum_{0<t_k<t} I_k(x(t_k)), \quad k = 1,2,...,m.$$

It obvious that $\mathcal{F}$ is well defined due to [H1] and [H4]. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $\mathcal{PC}(\mathcal{J},\mathcal{X})$. Then, for each $t \in \mathcal{J}$ we consider

$$\|(\mathcal{F}x_n)(t) - (\mathcal{F}x_m)(t)\| \leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x_m(s))\| ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x_m(s))\| ds + \sum_{0<t_k<t} \|I_k(x_n(t_k)) - I_k(x_m(t_k))\|$$

$$\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s)) + \xi(s, x(s)) - \xi(s, x_m(s))\| ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s)) + \xi(s, x(s)) - \xi(s, x_m(s))\| ds$$
$$+ \sum_{0<t_k<t} \|I_k(x_n(t_k)) - I_k(x(t_k)) + I_k(x(t_k)) - I_k(x_m(t_k))\|$$

$$\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds$$
$$+\frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|\xi(s, x_m(s)) - \xi(s, x(s))\| ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|\xi(s, x_n(s)) - \xi(s, x(s))\| ds$$
$$+\frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|\xi(s, x_m(s)) - \xi(s, x(s))\| ds$$
$$+ \sum_{0<t_k<t} \|I_k(x_n(t_k)) - I_k(x(t_k))\| + \sum_{0<t_k<t} \|I_k(x_m(t_k)) - I_k(x(t_k))\|$$

Since I_k is continuous and ξ is also jointly continuous, then we have
$\|\xi(s,x_n(s))-\xi(s,x(s))\| \to 0 \ as \ n \to \infty$, also, $\sum_{0<t_k<t}\|I_k(x_n(t_k))-I_k(x(t_k))\| \to 0 \ as \ n \to \infty$, therefore $\|(\mathcal{F}x_n)(t)-(\mathcal{F}x_m)(t)\| \to 0 \ as \ n,m \to \infty$. Consequently, by assumptions [H2] and [H4], it is not difficult to obtain that $\|(\mathcal{F}x_n)(t)-(\mathcal{F}x)(t)\| \to 0 \ as \ n \to \infty$, as follows;

$$\|(\mathcal{F}x_n)(t)-(\mathcal{F}x)(t)\| \leq \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\|\xi(s,x_n(s))-\xi(s,x(s))\|ds$$

$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\|\xi(s,x_n(s))-\xi(s,x(s))\|ds+\sum_{0<t_k<t}\|I_k(x_n(t_k))-I_k(x(t_k))\|$$

$$\leq \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\delta_\xi\|x_n-x\|ds$$

$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\delta_\xi\|x_n-x\|ds+\sum_{0<t_k<t}\gamma_I\|x_n-x\| \to 0 \ as \ n \to \infty$$

Thus, $\mathcal{F}$ is continuous and completely continuous. Consequently, by Schaefer's fixed point theorem, one can deduce that $\mathcal{F}$ has at least one fixed point on $\mathcal{PC}(\mathcal{J},\mathcal{X})$ which is a solution of the fractional ICP(1.1).
Conversely, assume that x satisfies the fractional integral Eq. (3.1). If $t \in (0,t_1]$ then $x(0)=x_0$ and by using the fact that ${}^cD_t^q$ is the left inverse of I_t^q and by Lemma (2.1), one can obtain ${}^cD_t^qx(t)=\xi(t,x(t))$. If $t \in (t_k,t_{k+1}], k=1,...,m$ also by Lemma (2.1) and using that fact the Caputo derivative of a constant is equal to zero. It can deduced that ${}^cD_t^qx(t)=\xi(t,x(t)) \ for \ t \in (t_k,t_{k+1}]$ and $x(t_k^+)=x(t_k^-)+I_k(x(t_k))$ which completes the proof. □

Lemma 3.2. *The operator $\mathcal{F}:\mathcal{PC}(\mathcal{J},\mathcal{X}) \to \mathcal{PC}(\mathcal{J},\mathcal{X})$ is bounded.*

Proof. It is sufficient to show that for any $\mu>0$, there exists a constant $\mathcal{K}>0$ such that for each $x \in \beta_\mu=\{\|x\|_{PC} \leq \mu : x \in \mathcal{PC}(\mathcal{J},\mathcal{X})\}$, then we have $\|\mathcal{F}x\|_{PC} \leq \mathcal{K}$. Now, let $\{x_n\}$ be a sequence on a bounded subset $\mathcal{M} \subset \beta_\mu$, for every $x_n \in \mathcal{M}$ by assumptions $[H3] \ and \ [H5]$, we have

$$\|(\mathcal{F}x_n)(t)\|_{pc} \leq \|x_0\|+\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\|\xi(s,x_n(s))\|ds$$

$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\|\xi(s,x_n(s))\|ds+\sum_{0<t_k<t}\|I_k(x_n(t_k))\|, \quad k=1,2,...,m$$

$$\leq \|x_0\|+\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}[\delta_1\|x_n\|^{q_1}+\delta_2]ds$$

$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}[\delta_1\|x_n\|^{q_1}+\delta_2]ds+\sum_{0<t_k<t}[\gamma_1\|x_n\|^{q_2}+\gamma_2],\quad k=1,2,...,m$$

Which implies that

$$\|(\mathcal{F}x_n)(t)\|_{PC}\leq\|x_0\|+\frac{(m+1)[\delta_1\mu^{q_1}+\delta_2]T^q}{\Gamma(q+1)}+m[\gamma_1\mu^{q_2}+\gamma_2]:=\mathcal{K}.$$

Therefore $(\mathcal{F}x_n)$ is uniformly bounded on $\mathcal{M}$, which implies $\mathcal{F}(\mathcal{M})$ is bounded in $\beta_\mu\subseteq\mathcal{PC}(\mathcal{J},\mathcal{X})$. □

Lemma 3.3. *The operator $\mathcal{F}:\mathcal{PC}(\mathcal{J},\mathcal{X})\to\mathcal{PC}(\mathcal{J},\mathcal{X})$ is equicontinuous.*

Proof. Let $\{x_n\}$ be a sequence on a bounded subset $\mathcal{M}\subset\beta_\mu$ as we defined in Lemma (3.2). For $t_1,t_2\in\mathcal{J},\ and\, t_1<t_2$, we consider

$$\|(\mathcal{F}x_n)(t_2)-(\mathcal{F}x_n)(t_1)\|=\|\frac{1}{\Gamma(q)}\sum_{0<t_k<t_2}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\xi(s,x_n(s))ds$$
$$-\frac{1}{\Gamma(q)}\sum_{0<t_k<t_1}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}\xi(s,x_n(s))ds+\frac{1}{\Gamma(q)}\int_{t_k}^{t_2}(t_2-s)^{q-1}\xi(s,x_n(s))ds$$
$$-\frac{1}{\Gamma(q)}\int_{t_k}^{t_1}(t_1-s)^{q-1}\xi(s,x_n(s))ds+\sum_{0<t_k<t_2-t_1}I_k(x_n(t_k))\|,\quad k=1,2,...,m$$

$$\|(\mathcal{F}x_n)(t_2)-(\mathcal{F}x_n)(t_1)\|=\|\frac{1}{\Gamma(q)}\int_0^{t_1}(t_2-s)^{q-1}\xi(s,x_n(s))ds$$
$$+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\xi(s,x_n(s))ds+\frac{1}{\Gamma(q)}\int_{t_2}^{t_k}(t_2-s)^{q-1}\xi(s,x_n(s))ds$$
$$-\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}\xi(s,x_n(s))ds-\frac{1}{\Gamma(q)}\int_{t_1}^{t_k}(t_1-s)^{q-1}\xi(s,x_n(s))ds$$
$$+\frac{1}{\Gamma(q)}\int_{t_k}^{t_2}(t_2-s)^{q-1}\xi(s,x_n(s))ds-\frac{1}{\Gamma(q)}\int_{t_k}^{t_1}(t_1-s)^{q-1}\xi(s,x_n(s))ds$$
$$+\sum_{0<t_k<t_2-t_1}I_k(x_n(t_k))\|$$
$$\leq\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}]\|\xi(s,x_n(s))\|ds$$
$$+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\|\xi(s,x_n(s))\|ds+\sum_{0<t_k<t_2-t_1}\|I_k(x_n(t_k))\|$$

As $t_2\to t_1$, then it is easy to deduce that the right hand side of the above inequality tends to zero. Therefore, $(\mathcal{F}x_n)$ is equicontinuous. □

Lemma 3.4. *The operator $\mathcal{F} : \mathcal{PC}(\mathcal{J}, \mathcal{X}) \to \mathcal{PC}(\mathcal{J}, \mathcal{X})$ is compact.*

Proof. Consider a closed subset $\mathcal{H} \subseteq \mathcal{PC}(\mathcal{J}, \mathcal{X})$. Since $\mathcal{F} : \mathcal{PC}(\mathcal{J}, \mathcal{X}) \to \mathcal{PC}(\mathcal{J}, \mathcal{X})$ is bounded and equicontinuous then by the Arzela Ascoli theorem, we get $\mathcal{F} : \mathcal{PC}(\mathcal{J}, \mathcal{X}) \to \mathcal{PC}(\mathcal{J}, \mathcal{X})$ is completely continuous which implies $\mathcal{F}(\mathcal{H})$ is a relatively compact subset of $\mathcal{PC}(\mathcal{J}, \mathcal{X})$. Therefore $\mathcal{F} : \mathcal{PC}(\mathcal{J}, \mathcal{X}) \to \mathcal{PC}(\mathcal{J}, \mathcal{X})$ is compact. □

Theorem 3.1. *Assume that $[H1] - [H5]$ hold, then the fractional ICP(1.1) has at least one solution.*

Proof. It is clear that the fixed points of the operator $\mathcal{F}$ are solutions of the ICP(1.1). Obviously the operator $\mathcal{F} : \mathcal{PC}(\mathcal{J}, \mathcal{X}) \to \mathcal{PC}(\mathcal{J}, \mathcal{X})$ is continuous and completely continuous, then we shall prove that $\mathcal{S}(\mathcal{F}) = \{x \in \mathcal{PC}(\mathcal{J}, \mathcal{X}) : x = \kappa \mathcal{F}x, \ for\ some\ \kappa \in [0, 1]\}$ is bounded. Let $x \in \mathcal{S}(\mathcal{F})$, then $x = \kappa \mathcal{F}x$ for some $\kappa \in [0, 1]$.

$$\|x(t)\|_{Pc} \leq \kappa \|(\mathcal{F}x)(t)\| \leq \|x_0\| + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|\xi(s, x(s))\| ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \|\xi(s, x(s))\| ds + \sum_{0<t_k<t} \|I_k(x(t_k))\|, \quad k = 1, 2, ..., m$$

$$\leq \|x_0\| + \frac{(m+1)[\delta_1 \mu^{q_1} + \delta_2] T^q}{\Gamma(q+1)} + m[\gamma_1 \mu^{q_2} + \gamma_2].$$

The above inequality at the same time with $q_1, q_2 \in [0, 1)$ and by result of Lemma (3.2) show that $\mathcal{S}$ is bounded in $\mathcal{PC}(\mathcal{J}, \mathcal{X})$. As a consequence of Schaefer's fixed point theorem, we can deduce that $\mathcal{F}$ has a fixed point which is a solution of the fractional ICP(1.1). □

Theorem 3.2. *Assume that $[H1] - [H5]$ hold, then the set of solutions for the fractional ICP(1.1) is convex.*

Proof. By Theorem (3.1), it is obvious that the fractional ICP(1.1) has a solution in $\mathcal{PC}(\mathcal{J}, \mathcal{X})$. Set $\kappa = 1$, then the set solutions will be defined as $\mathcal{S}(\mathcal{F}) = \{x \in \mathcal{PC}(\mathcal{J}, \mathcal{X}) : x = \mathcal{F}x, \}$. For each $x_1, x_2 \in \mathcal{S}(\mathcal{F}), \lambda \in [0, 1]$ and $t \in \mathcal{J}$, then by definition of $\mathcal{F}$, we have

$$\lambda x_1(t) + (1 - \lambda) x_2(t) = \lambda (\mathcal{F}x_1)(t) + (1 - \lambda)(\mathcal{F}x_2)(t)$$

$$= \lambda [x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \xi(s, x_1(s)) ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \xi(s, x_1(s)) ds$$

$$+ \sum_{0<t_k<t} I_k(x_1(t_k))] + (1 - \lambda)[x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \xi(s, x_2(s)) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t - s)^{q-1} \xi(s, x_2(s)) ds + \sum_{0<t_k<t} I_k(x_2(t_k))]$$

$$= x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} [\lambda\xi(s, x_1(s)) + (1-\lambda)\xi(s, x_2(s))] ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} [\lambda\xi(s, x_1(s)) + (1-\lambda)\xi(s, x_2(s))] ds$$

$$+ \sum_{0<t_k<t} [\lambda I_k(x_1(t_k)) + (1-\lambda) I_k(x_2(t_k))]$$

$$= x_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \xi(s, [\lambda x_1 + (1-\lambda)x_2](s)) ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \xi(s, [\lambda x_1 + (1-\lambda)x_2](s)) ds + \sum_{0<t_k<t} I_k([\lambda x_1 + (1-\lambda)x_2](t_k))$$

Thus,

$$[\lambda x_1 + (1-\lambda)x_2](t) = (\mathcal{F}[\lambda x_1 + (1-\lambda)x_2])(t)$$

Therefore, $\lambda x_1 + (1-\lambda)x_2 \in \mathcal{S}(\mathcal{F})$ which implies $\mathcal{S}(\mathcal{F})$ is convex. Hence, the set solutions of ICP(1.1) is convex. □

Theorem 3.3. *Assume that* $[H1]-[H5]$ *hold, then the fractional ICP*(1.1) *has a unique solution on* $\mathcal{PC}(\mathcal{J}, \mathcal{X})$.

Proof. It can be easily shown that $\mathcal{F}$ is a contraction mapping on $\mathcal{PC}(\mathcal{J}, \mathcal{X})$ by [H2] and [H4] as follows, for arbitrary $x, y \in \mathcal{PC}(\mathcal{J}, \mathcal{X})$, we have

$$\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|\xi(s, x(s)) - \xi(s, y(s))\| ds$$

$$+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|\xi(s, x(s)) - \xi(s, y(s))\| ds + \sum_{0<t_k<t} \|I_k(x(t_k)) - I_k(y(t_k))\|$$

$$\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \delta_\xi \|x - y\| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \delta_\xi \|x - y\| ds$$

$$+ \sum_{0<t_k<t} \gamma_I \|x - y\|, \quad k = 1, 2, ..., m$$

$$\|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\| \leq [\frac{(m+1)\delta_\xi T^q}{\Gamma(q+1)} + m\gamma_I] \|x - y\|$$

Thus, $\mathcal{F}$ is a contraction mapping on $\mathcal{PC}(\mathcal{J}, \mathcal{X})$ with a contraction constant $[\frac{(m+1)\delta_\xi T^q}{\Gamma(q+1)} + m\gamma_I]$. By applying the Banach's contraction mapping principle we deduce that the operator $\mathcal{F}$ has a unique fixed point on $\mathcal{PC}(\mathcal{J}, \mathcal{X})$. Therefore, the ICP(1.1) has a unique solution which completes the proof. □

Example 3.1. *Consider the following fractional ICP*

$$\begin{cases} {}^{c}\mathcal{D}^{\frac{2}{3}}x(t) = \frac{|x(t)|}{(1+e^{t})(1+|x(t)|)} \quad, t \in [0,1]\backslash\{\frac{1}{2}\}, \\ x(0) = 0, \\ \Delta x(\frac{1}{2}) = \frac{1}{9}|x(\frac{1}{2})|. \end{cases} \tag{3.2}$$

Set $q = \frac{2}{3}$, for $(t, x) \in [0, 1] \times [0, +\infty)$, we can define $\xi(t, x) = \frac{x}{(1+e^{t})(1+x)}$. Also, for $t \in [0, 1]$ we have $x(t) = \frac{1}{1+e^{t}}$, and $I_k(x(t_k)) = \frac{1}{9}x(\frac{1}{2}),\ K = 1$. By Theorem (3.1), we have

$$|\xi(t,x) - \xi(t,y)| = \frac{1}{(1+e^{t})}|\frac{x}{1+x} - \frac{y}{1+y}|, \quad t \in [0,1]$$

$$\leq \frac{1}{2}|\frac{x-y}{(1+x)(1+y)}|$$

$$\leq \frac{1}{2}|x-y| \Rightarrow \delta_{\xi} = \frac{1}{2}$$

And,

$$|\xi(t,x)| = |\frac{x}{(1+e^{t})(1+x)}|, \quad t \in [0,1]$$

$$\leq \frac{1}{2}|\frac{x}{1+x}| \leq \frac{1}{2}|x| \Rightarrow \delta_1 = \frac{1}{2}, q_1 = 1, \delta_2 = 0$$

Next,

$$|I(x) - I(y)| = \frac{1}{9}|x(\frac{1}{2}) - y(\frac{1}{2})| \Rightarrow \gamma_I = \frac{1}{9},$$

$$|I(x)| = \frac{1}{9}|x(\frac{1}{2})| \Rightarrow \gamma_1 = \frac{1}{9}, q_2 = 1, \gamma_2 = 0$$

Obviously, it is not difficult to see that all assumptions in Theorem (3.1) are satisfied. Therefore, our results can be used to solve the problem (3.2).

Conclusion

We established sufficient conditions for existence of a solution for the ICP(1.1) by using Schaefer's fixed point theorem, Banach contraction mapping principle besides to topological technique of approximate solutions. Moreover, we studied some of topological properties for the set of solutions. Finally, an example was presented to clarify our results.

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