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GREEN'S FUNCTION APPROACH TO THERMAL DEFLECTION OF A THIN HOLLOW CIRCULAR DISK UNDER AXISYMMETRIC HEAT SOURCE

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ABSTRACT. A Green's function approach is adopted to solve the two-dimensional thermoelastic problem of a thin hollow circular disk. Initially, the disk is kept at temperature $T_0(r, z)$. For times $t > 0$, the inner and outer circular edges are thermally insulated and the upper and lower surfaces of the disk are subjected to convection heat transfer with convection coefficient h_c and fluid temperature T_∞, while the disk is also subjected to the axisymmetric heat source. As a special case, different metallic disks have been considered. The results for temperature and thermal deflection has been computed numerically and illustrated graphically.

1. INTRODUCTION

Roy Choudhury [1] discussed the normal deflection of a thin clamped circular plate due to ramp-type heating of a concentric circular region of the upper face and the lower face of the plate kept at zero temperature while the circular edge is thermally insulated. Grysa and Kozlowski [2] investigated an inverse one-dimensional transient thermoelastic problem and obtained the temperature and heat flux on the surface of an isotropic infinite slab. Ootao et al. [3, 4] studied the theoretical analysis of a three-dimensional transient thermal stress problem for a nonhomogeneous/functionally graded hollow circular cylinder due to a moving heat source in the axial direction from the inner and outer surfaces. Tanigawa et al. [5] discussed the theoretical analysis of thermoelastoplastic deformation of a circular plate due to a partially distributed heat supply. Noda et al. [6] discussed the transient thermoelastoplastic bending problems, making use of the strain increment theorem, and determined the temperature field and the thermoelastic deformation for the heating and cooling processes in a thin circular plate subjected to a partially distributed and axisymmetric heat supply on the upper surface.

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Chakraborty et al. [7] solved the deflection of a circular plate due to the heating of a concentric circular region.

Recently, some cases of thermal deflection [7, 8, 9, 10, 11, 12], thermal stresses [13, 14, 15, 16, 17], or both of them [18, 19, 20] have been investigated on thin circular plates of solid [7, 8, 9, 12, 13, 15, 16, 18, 20] and annular disk [10, 11, 14, 19] under different initial and boundary conditions/input heat source. Moreover, most of these studies on the thermoelastic problem of thin-wall plates analysis involving integral transform [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20] methods were used to obtain the analytical solution.

In this article, we extended the work of Gaikwad [14] for a two-dimensional Green's function approach to the transient thermoelastic problem of a thin hollow circular disk under the axisymmetric heat source. The convection boundary condition is assumed in this study for both the upper and lower surfaces of the disk with its inner and outer edges are thermally insulated and subjected to an axisymmetric heat source. The analytical method of Green's function is employed with the help of integral transform technique to determine the temperature distribution function. The thermal deflection is also obtained based on the calculated temperature distribution considering the state of plane stresses.

The remainder of this study offers the following:

- The governing transient heat conduction equation with the thermoelastic equation of the thin hollow circular disk understudy is formulated as a boundary value problem.
- The Green's function method is used to solve the transient heat conduction equation.
- The finite Hankel and Fourier integral transform technique is used to derive Green's function.
- Based on the temperature distribution, the thermal deflection in a thin hollow circular disk is also obtained.
- The mathematical model is prepared for different metallic disks and the results for temperature, and thermal deflection has been computed numerically and illustrated graphically with the help of Mathcad software.

It is believed that this particular problem has not been considered by anyone. This is a new and novel contribution to the field of thermoelasticity. The results presented here will be more useful in engineering problems particularly, in the determination of the state of strain in a thin hollow circular disk constituting foundations of containers for hot gases or liquids, in the foundations for furnaces, etc.

2. ANALYSIS

2.1. **Transient Heat Conduction Problem.** :

We consider a thin hollow circular disk as shown in Fig. 1, of radius a and thickness by h, the occupying space D is $a \leq r \leq b$, $-h/2 \leq z \leq h/2$ and is initially at temperature $T_0(r, z)$. For $t > 0$, the fixed circular edges ($r = a, r = b$) are thermally insulated and the upper and lower surfaces ($z = \pm h/2$) of the disk are subjected to convection heat transfer with convection coefficient h_c and fluid temperature T_∞, while the disk is subjected to the

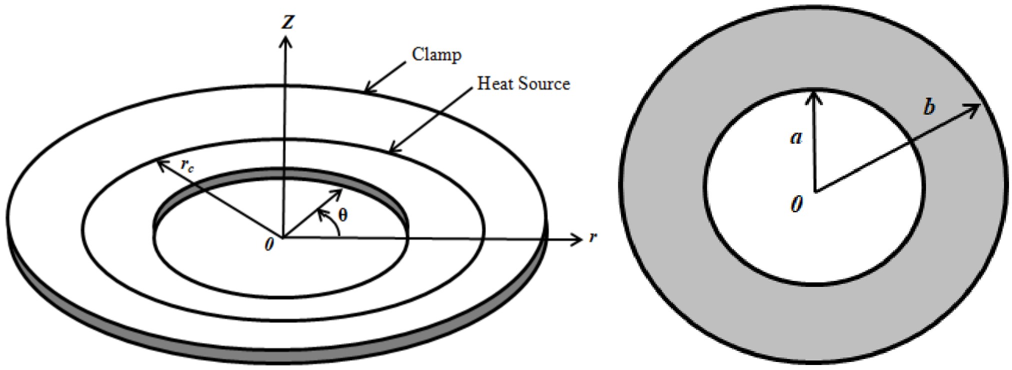

FIGURE 1. The Hollow circular disk with an axisymmetric heat source.

axisymmetric heat source g_0 (W.m^{-3}). Under these realistic prescribed conditions temperature, and thermal deflection/stresses in a thin hollow disk due to the axisymmetric heat source are required to be determined.

The temperature of the hollow circular disk $T(r, z, t)$ at time t satisfies the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{g_0}{k_t} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad \text{in} \quad a \le r \le b,\ -h/2 \le z \le h/2,\ t > 0 \tag{2.1}$$

with the boundary conditions,

$$T\mid_{r=0} < \infty \tag{2.2}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=a} = 0 \tag{2.3}$$

$$\left.\frac{\partial T}{\partial r}\right|_{r=b} = 0 \tag{2.4}$$

$$-k_t\left.\frac{\partial T}{\partial z}\right|_{h/2} = h_c(\,T\mid_{z=h/2} - T_\infty) \tag{2.5}$$

$$k_t\left.\frac{\partial T}{\partial z}\right|_{-h/2} = h_c(\,T\mid_{z=-h/2} - T_\infty) \tag{2.6}$$

and the initial condition,

$$T\mid_{t=0} = T_0(r, z) \tag{2.7}$$

where k_t is the thermal conductivity, the thermal diffusivity is defined as $\alpha = k_t/\rho c$ with ρ and c_p denoting the density and specific heat of the material of the hollow disk respectively. $g(r, z, t) = g_i(t)\delta(r - r_c)\delta(z - z_c)$ represents an axisymmetric heat source, where g_i is an instantaneous line heat source, and δ is a Dirac delta function that characterizes the location of the line heat source at r_c and z_c.
Here the hollow disk is assumed sufficiently thin.

2.2. **Determination of the Temperature.** :

First, we modified the formulated boundary value problem with homogeneous boundary conditions. The temperature field $T(r,z,t)$ is divided into two components, $T(r,z,t) = \psi(r,z,t) + T_\infty$, where the constant ambient component T_∞ satisfies Eq. (2.1) and the dynamic component ψ satisfies the following equation:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} + \frac{g_0}{k_t} = \frac{1}{\alpha}\frac{\partial\psi}{\partial t} \tag{2.8}$$

with the boundary conditions,

$$\psi\mid_{r=0} < \infty$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = 0$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=b} = 0$$

$$\left.\left(\frac{\partial\psi}{\partial z} + h_{s1}\psi\right)\right|_{h/2} = 0$$

$$\left.\left(\frac{\partial\psi}{\partial z} - h_{s2}\psi\right)\right|_{-h/2} = 0$$

and the initial condition,

$$\psi\mid_{t=0} = T_0 - T_\infty$$

where $h_{s1} = h_c/k_t$ and $h_{s2} = h_c/k_t$ be the relative heat transfer coefficients on the upper and lower surface of the thin hollow circular disk.

To determine the Green's function, we consider the homogeneous form of Eq. (2.8) with $g_0(r,z,t) = 0$:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{\alpha}\frac{\partial\psi}{\partial t} \tag{2.9}$$

with the boundary conditions,

$$\psi\mid_{r=0} < \infty \tag{2.10}$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = 0 \tag{2.11}$$

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=b} = 0 \tag{2.12}$$

$$\left.\left(\frac{\partial\psi}{\partial z} + h_{s1}\psi\right)\right|_{h/2} = 0 \tag{2.13}$$

$$\left.\left(\frac{\partial\psi}{\partial z} - h_{s2}\psi\right)\right|_{-h/2} = 0 \tag{2.14}$$

and the initial condition,

$$\psi\mid_{t=0} = T_0 - T_\infty = A(r,z) \tag{2.15}$$

Secondly, for the temperature function $\psi(r,z,t)$, we introduce the finite Hankel transform **H** over the variable r and its inverse transform defined in [21] as:

$$\overline{T}(\beta_m,z,t)=\int_{r'=a}^{b} r' R_0(\beta_m,r')T(r',z,t).dr' \tag{2.16}$$

$$T(r,z,t)=\sum_{m=1}^{\infty}\frac{R_0(\beta_m,r)}{N(\beta_m)}\overline{T}(\beta_m,z,t) \tag{2.17}$$

where

$$R_0(\beta_m,r)=\left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)}-\frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)}\right]$$

$$\frac{1}{N(\beta_m)}=\frac{\pi}{\sqrt{2}}\frac{\beta_m J_0'(\beta_m b).Y_0'(\beta_m b)}{\left[1-\dfrac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)}\right]^{1/2}}$$

and β_1, β_2, $\beta_3, \dots$ are the positive roots of transcendental equation

$$\frac{J_0'(\beta_m a)}{J_0'(\beta_m b)}-\frac{Y_0'(\beta_m a)}{Y_0'(\beta_m b)}=0.$$

This transform satisfies the relation

$$H\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right]=-\beta_m^2\overline{T}(\beta_m,z,t) \tag{2.18}$$

Thirdly, for the Hankel-transformed function $\overline{T}(\beta_m,z,t)$, we introduce the finite Fourier transform over the variable z and its inverse transform defined in [21] as

$$\tilde{\overline{T}}(\beta_m,\eta_p,t)=\int_{z=-h/2}^{h/2} Z(\eta_p,z').\overline{T}(\beta_m,z',t).dz' \tag{2.19}$$

$$\overline{T}(\beta_m,z,t)=\sum_{p=1}^{\infty}\frac{Z(\eta_p,z)}{N(\eta_p)}\tilde{\overline{T}}(\beta_m,\eta_p,t) \tag{2.20}$$

where

$$Z(\eta_p,z)=\eta_p\cos(\eta_p z)+h_{s1}\sin(\eta_p z)$$

$$\frac{1}{N(\eta_p)}=\sqrt{2}\left[(\eta_p^2+h_{s1}^2)\left(\frac{h}{2}+\frac{h_{s2}}{\eta_p^2+h_{s2}^2}\right)+h_{s1}\right]^{-1}$$

and $\eta_1,\eta_2,\dots$ are the positive roots of the transcendental equation

$$\tan\left(\frac{\eta_p h}{2}\right)=\frac{\eta_p(h_{s1}+h_{s2})}{\eta_p^2-h_{s1}h_{s2}},\quad p=1,2,3,\dots$$

Applying the finite Hankel transform and finite Fourier transform defined in Eqs. (2.16) and (2.19) and their respective inverses defined in Eqs. (2.17) and (2.20) and operate them on Eqs. (2.9)-–(2.15):

$$\psi(r,z,t)=\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\frac{R_0(\beta_m,r)Z(\eta_p,z)}{N(\beta_m)N(\eta_p)}.e^{-\alpha(\beta_m^2+\eta_p^2)t}.\tilde{\bar{A}}(\beta_m,\eta_p)$$

where

$$\tilde{\bar{A}}(\beta_m,\eta_p)=\int_{r'=a}^{b}\int_{z'=-h/2}^{h/2} r'\ R_0(\beta_m,r')\ Z(\eta_p,z')\ (T_0(r',z')-T_\infty)\ dr'\ dz'$$

$\psi(r,z,t)$ can also be given from the Green's function approach [21]:

$$\psi(r,z,t)=\int_{r'=a}^{b}\int_{z'=-h/2}^{h/2} G(r,z,t|r',z',\tau)|_{\tau=0}(T_0(r',z')-T_\infty)\ r'\ dr'\ dz'$$

From the above, the Green's function can be obtained as:

$$G(r,z,t|r',z',\tau)=\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\frac{R_0(\beta_m,r)Z(\eta_p,z)R_0(\beta_m,r')Z(\eta_p,z')}{N(\beta_m)N(\eta_p)}.e^{-\alpha(\beta_m^2+\eta_p^2)(t-\tau)}$$

Finally, the solution of the nonhomogeneous problem of Eqs. (2.1) and (2.2)–(2.7) in terms of the above Green's function is given as:

$$\begin{aligned}T(r,z,t)=T_\infty&+\int_{r'=a}^{b}\int_{z'=-h/2}^{h/2} G(r,z,t|r',z',\tau)|_{\tau=0}\ (T_0(r',z')-T_\infty)\ dr'\ dz'\\&+\frac{\alpha}{k\,t}\int_{\tau=0}^{t}\int_{r'=a}^{b}\int_{z'=-h/2}^{h/2} r'\ G(r,z,t|r',z',\tau)\ g_0(r',z',\tau)\ dr'\ dz'\ dt'\end{aligned}\tag{2.21}$$

2.3. **Special Case.** :

Setting, $T_0(r,z)=T_\infty$ and $g(r,z,t)=1(t)\delta(r-r_c)(z-z_c)$ with $1(t)$ denoting a unit step function in Eq. (2.21), the transient temperature field is given as follows:

$$T(r,z,t)=T_0+\frac{r_c}{k_t}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}R_0(\beta_m,r)R_0(\beta_m,r_c)\left[\frac{Z(\eta_p,z)Z(\eta_p,z_c)(1-e^{-\alpha(\beta_m^2+\eta_p^2)t})}{N(\beta_m)N(\eta_p)(\beta_m^2+\eta_p^2)}\right]\tag{2.22}$$

3. Determination of thermal deflection

The thermal bending problem of a thin disk with a thickness h, it will be assumed that the deflection, which means a deformation in the out-of-plane direction of the disk, is small. By Kirchhoff-Love hypothesis that the plane initially perpendicular to the neutral plane of the disk remains a plane after deformation and is perpendicular to the deformed neutral plane.

The differential equation satisfied the deflection function $\omega(r,t)$ as defined in [22] as

$$\nabla^2\nabla^2\omega = -\frac{1}{(1-\nu)D}\nabla^2 M_T \tag{3.1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

and M_T is the thermal moment of the disk, ν is the Poisson's ratio of the plate material, D is the flexural rigidity of the disk denoted by

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

The term M_T is defined as

$$M_T = \alpha_t E\int_{-h/2}^{h/2}(T(r,z,t)-T_0)zdz \tag{3.2}$$

where α_t and E are the coefficients of the linear thermal expansion and the Young's modulus, respectively.

For out-of-plane deformation, the boundary conditions are given as

$$\left.\frac{\partial\omega}{\partial r}\right|_{r=a} = \left.\frac{\partial\omega}{\partial r}\right|_{r=b} = 0 \tag{3.3}$$

Initially

$$T\mid_{t=0} = \omega\mid_{t=0} = T_0(r,z) \tag{3.4}$$

Assume the solution of Eq. (3.1) satisfying conditions (3.3) as

$$\omega(r,t) = \sum_{m=1}^{\infty} C_m(t)\left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)}\right] \tag{3.5}$$

where $\beta_m' s$ are the positive roots of transcendental equation,

$$\frac{J_0'(\beta_m a)}{J_0'(\beta_m b)} - \frac{Y_0'(\beta_m a)}{Y_0'(\beta_m b)} = 0.$$

It can be easily shown that

$$\frac{\partial w}{\partial r} = \sum_{m=1}^{\infty} C_m(t)\left[\frac{J_0'(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0'(\beta_m r)}{Y_0'(\beta_m b)}\right]$$

$$\left.\frac{\partial w}{\partial r}\right|_{r=a} = \left.\frac{\partial w}{\partial r}\right|_{r=b} = 0$$

Hence, the solution of Eq. (3.5) satisfies the condition of Eq. (3.3).

Now

$$\nabla^2\nabla^2 w = \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right)^2 \sum_{m=1}^{\infty} C_m(t)\left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)}\right] \tag{3.6}$$

Using the well-known result:

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) J_0(\beta_m r) = -\beta_m^2 J_0(\beta_m r)$$

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) Y_0(\beta_m r) = -\beta_m^2 Y_0(\beta_m r)$$

in Eq. (3.6), one obtains

$$\nabla^2\nabla^2 w = \sum_{m=1}^{\infty} C_m(t)\beta_m^4 \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)}\right] \tag{3.7}$$

The thermal moment could be obtained by substituting Eq. (2.22) into Eq. (3.2):

$$M_T = D_m \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} R_0(\beta_m, r)R_0(\beta_m, r_c)\left[\frac{\sin(\eta_p h/2)Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2+\eta_p^2)t})}{N(\beta_m)N(\eta_p)(\beta_m^2 + \eta_p^2)}\right]$$

where $D_m = \dfrac{\alpha_t r_c E[h\eta_p^2 + 2h_{s1}]}{k_t \eta_p^2}$

Now

$$\nabla^2 M_T = \nabla^2 D_m \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} R_0(\beta_m, r)R_0(\beta_m, r_c)\left[\frac{\sin(\eta_p h/2)Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2+\eta_p^2)t})}{N(\beta_m)N(\eta_p)(\beta_m^2 + \eta_p^2)}\right] \tag{3.8}$$

solving Eq. (3.8), one obtains

$$\nabla^2 M_T = -D_m \sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \beta_m^2 R_0(\beta_m, r)R_0(\beta_m, r_c)\left[\frac{\sin(\eta_p h/2)Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2+\eta_p^2)t})}{N(\beta_m)N(\eta_p)(\beta_m^2 + \eta_p^2)}\right] \tag{3.9}$$

Substituting Eqs. (3.7) and (3.9) into Eq. (3.1) yields

$$\begin{aligned}&\sum_{m=1}^{\infty} C_m(t)\beta_m^4 \left[\frac{J_0(\beta_m r)}{J_0'(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0'(\beta_m b)}\right] \\ &= \frac{D_m}{(1-\nu)D}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \beta_m^2 R_0(\beta_m, r)R_0(\beta_m, r_c)\left[\frac{\sin(\eta_p h/2)Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2+\eta_p^2)t})}{N(\beta_m)N(\eta_p)(\beta_m^2 + \eta_p^2)}\right]\end{aligned} \tag{3.10}$$

Solving Eq. (3.10), one obtains

$$C_m(t) = \frac{D_m}{(1-\nu)D}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{\beta_m^2} R_0(\beta_m, r_c)\left[\frac{\sin(\eta_p h/2)Z(\eta_p, z_c)(1 - e^{-\alpha(\beta_m^2+\eta_p^2)t})}{N(\beta_m)N(\eta_p)(\beta_m^2 + \eta_p^2)}\right] \tag{3.11}$$

Finally, substituting Eq. (3.11) in Eq. (3.5), one obtains the expression for the quasi-static thermal deflection $\omega(r,t)$ as:

$$\omega(r,t)=\frac{D_m}{(1-\nu)D}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty}\frac{1}{\beta_m^2}R_0(\beta_m,r)R_0(\beta_m,r_c)\left[\frac{\sin(\eta_p h/2)Z(\eta_p,z_c)(1-e^{-\alpha(\beta_m^2+\eta_p^2)t})}{N(\beta_m)N(\eta_p)(\beta_m^2+\eta_p^2)}\right] \tag{3.12}$$

4. Numerical results and discussion

4.1. Dimension. :

The constants associated with the numerical calculation are taken as:
Inner radius of a circular disk a = 1 m,
Outer radius of a circular disk b = 5 m,
Thickness of circular disk h = 0.5 m,
Constant line heat source g_i = 200 W/m,
Initial temperature $T_0(r,z)$=0,
Relative heat transfer coefficients $h_{s1}=10$, $h_{s2}=0$.

4.2. Material Properties. :

The four different materials was chosen for the purpose of numerical evaluation, for which we take the following values of the physical constants as[23]:

TABLE 1. Thermal properties of materials.

Materials	k_t (W/m-K)	c_p (J/kg-K)	ρ (kg/m^3)	α (10^{-6} m^2/s)	α_t (10^{-6} $1/K$)	E (GP$_a$)	ν
Aluminum (Al)	204	896	2727	84.18	22.2	70	0.35
Copper (Cu)	386	383	8954	112.34	16.6	117	0.36
Iron (Fe)	72.7	452	7897	20.34	12	193	0.21
Steel 0.5% carbon (St)	53.6	465	7833	14.74	13	200	0.26

4.3. Roots of the Transcendental Equation. :

Here $\beta_1=3.1965$, $\beta_2=6.3123$, $\beta_3=9.4445$, $\beta_4=12.5812$, $\beta_5=15.7199$ are the positive root of the transcendental equation

$$\left[\frac{J_0'(\beta a)}{J_0'(\beta b)}-\frac{Y_0'(\beta a)}{Y_0'(\beta b)}\right]=0$$

and $\eta_1 = 1.4289$, $\eta_2 = 4.3058$, $\eta_3 = 7.2281$, $\eta_4 = 10.2003$, $\eta_5 = 13.2142$ are positive roots of the transcendental equation

$$\tan\left(\frac{\eta h}{2}\right) = \frac{\eta(h_{s1} + h_{s2})}{\eta^2 - h_{s1}h_{s2}}$$

which takes the form $\eta \tan\left(\frac{\eta}{4}\right) = c$, for $h = 0.5$, $h_{s1} = c = 10$, $h_{s2} = 0$ in [21]. The numerical calculations have been presented by the PTC MATHCAD (Prime-3.1) and the results are depicted graphically.

The constants λ_0 and μ_0 are given as:

$$\lambda_0 = \frac{D_m.10^4}{(1-\nu)D}, \qquad \mu_0 = \frac{2\mu(1+\nu)\alpha_t}{10^7}$$

i.e. elastic material constants.

The obtained expressions for the temperature field and thermal deflection provide important intuition into the role of the thermomechanical material properties in elastic behaviors of the thin hollow circular disks under the axisymmetric heat source. The temperature distribution in the disk is only dependent on its thermal properties, on the other hand, the disk deflection is dependent on both thermal and mechanical properties. We have used the first 50 terms (p=1–50) for the inner series summation, as given by Eq. (2.22), and have used the first 10 terms (m=1–10) of the outer series summation. However, for very small times ($t \tilde{<} 1s$), the convergence of the inner series, particularly at $r \simeq 0$, was closer to 2%; hence exploration of very small times would require additional terms for the outer series summation (i.e. $m >$10) to achieve greater accuracy. The numerical calculations have been carried out for four different materials (Aluminum, Copper, Iron, and Steel), which have mechanical and thermal properties as shown in Table 1. Assume that the disk is subjected to a constant heat line source of g_i= 200 W/m with initial temperature $T_0(r, z) = 0$.

Figure 2 shows the temperature distribution for the iron disk along with the disk radius at the lower surface ($z = -h/2$), when the heat source is located at (r_c=4 m and z_c=0.5 m) at times different times ranged from 10s to 5000 s. It should be noted that the disk temperature gradually increases in the range $1 \leq r \leq 4$ with increases time, attaining maximum value at the heat source location, and goes on decreasing towards the outer circular edge. It is clear that the disk temperature the rate slows down with respect to time as it approaches the steady-state.

Figure 3 shows the temperature distribution for the iron disk along with the disk radius at the lower surface ($z = -h/2$), when the heat source is specified with a constant intensity of g_i = 200 W/m, as the heating location with different radii's on the upper surface ($z = h/2$) for steady-state temperature t=3000 s. It can be observed that the maximum temperature stays around 85^0 for r_c=1.5, 2.5, 3, and 3.5 m, and it becomes 100 and 102 ^{0}C for r_c=4.5, 4.7 m respectively. It should be noted that the heat source approaches toward the insulated boundary, the heat tends to be accumulated locally rather than being dissipated in all directions as it does at the median area of the disk.

Figure 4 shows the temperature distribution at the lower surface due to axisymmetric heat source (z_c=0.5 m, and t=3000 s) at different radii with increasing heat source intensity r_c, that

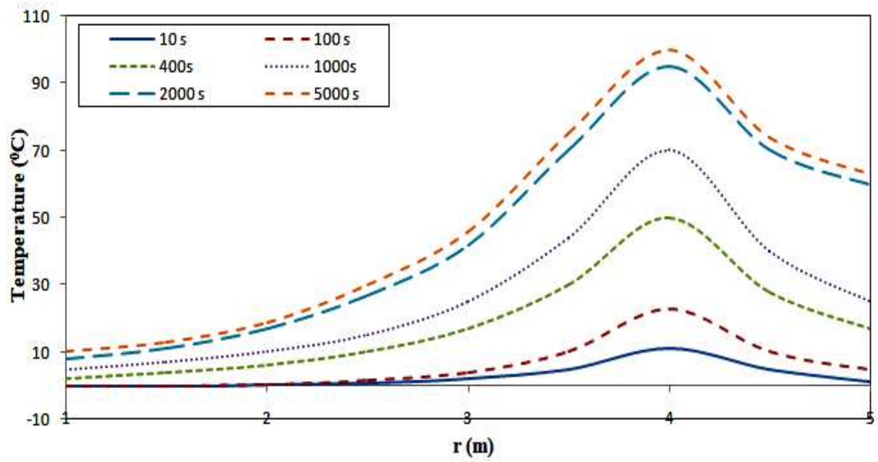

FIGURE 2. The temperature distribution at the lower surface due to axisymmetric heat source (r_c=4 m, z_c=0.5 m) at different times parameters.

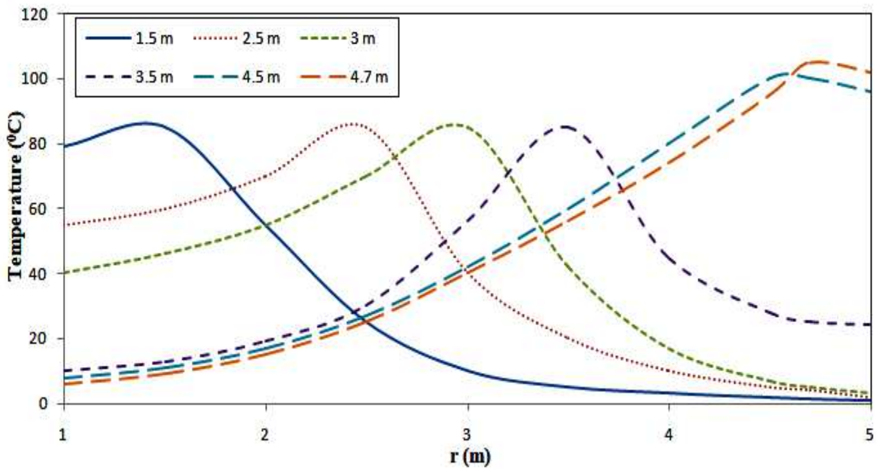

FIGURE 3. The temperature distribution at the lower surface due to axisymmetric heat source (z_c=0.5 m, and t=3000 s) at different radii with constant heat source intensity (g_i = 200 W/m).

is g_i=40, 80, 120, 160, 200, and 240 W/m for r_c=1.5, 2.5, 3, 3.5, 4.5, and 4.7 m respectively. It should be noted that the temperature increases monotonically mainly due to the increasing heat source intensity with r_c, because less amount of heat is dissipated as the heat source approaches closer to the insulated boundary.

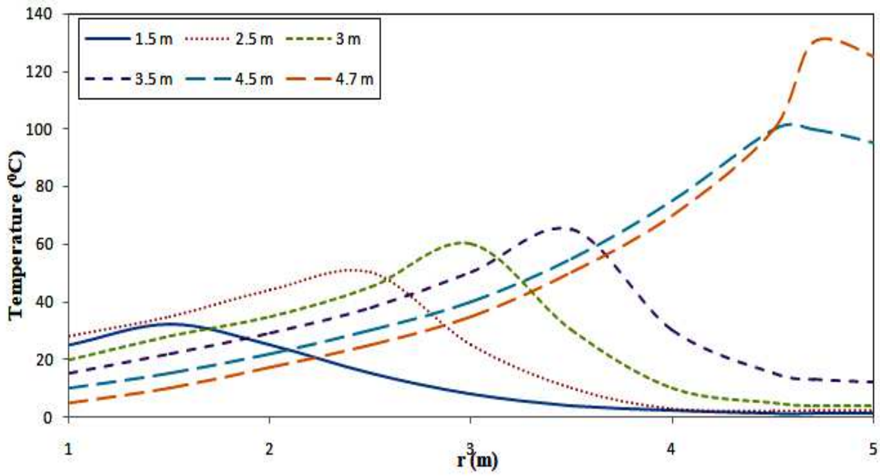

FIGURE 4. The temperature distribution at the lower surface due to axisymmetric heat source (z_c=0.5 m, and t=3000 s) at different radii with increasing heat source intensity r_c.

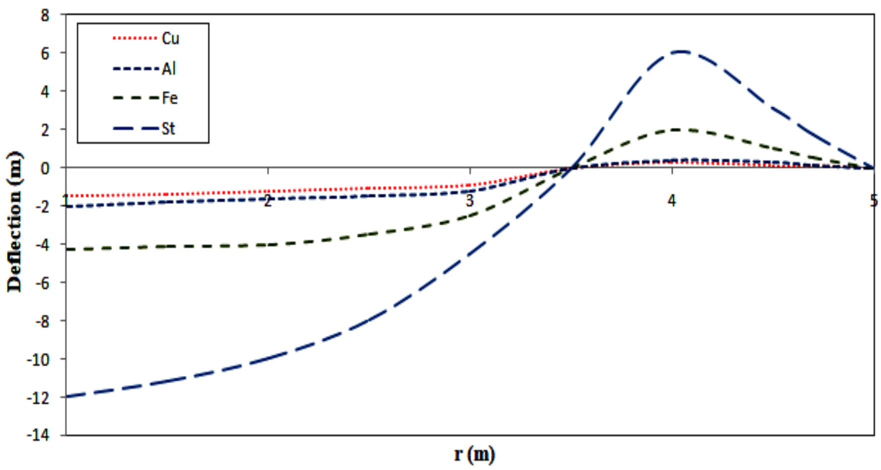

FIGURE 5. The thermal deflection at the mid-plane due to axisymmetric heat source (r_c=4 m, z_c=0.5 m, and t=3000 s).

Figure 5 shows the thermal deflection at the mid-plane ($z = 0$) of the disk for four different materials under the same conditions used to obtain Fig. 2. It can be observed that for all materials, the deflection is maximum at the heat source location, and it decreases with increasing r to reach to zeros. The obtained results show good agreement with boundary conditions (3.3). The steel (53.6 W/m-K) and iron (72.7 W/m-K) disks have smaller thermal conductivity compared with the aluminum and copper disks, so the steel and iron disks have larger deflection than the aluminum (204 W/m-K) and silver (386 W/m-K) disks. It is due to the larger temperature gradients induced in the steel and iron disks which results from the lower thermal conductivity of these materials. However the steel and iron disks have larger Young's moduli (200 and 193 GPa) compared with the aluminum (70 GPa) and copper (117 GPa) disks, so the steel and iron disks have larger deflection than the aluminum and copper disks. This conclusion gives us the knowledge that the material thermal properties have a dominant effect on the thermal deflection compared to the mechanical properties.

5. Conclusions

Green's function approach to analyzing the two-dimensional transient thermoelastic problem of a thin hollow circular disk under the axisymmetric heat source proposed. The convection boundary condition is assumed in this study for both the upper and lower surfaces of the disk with its inner and outer edges are thermally insulated and subjected to an axisymmetric heat source. The analytical method of Green's function is employed with the help of an integral transform technique to determine the temperature distribution function. The thermal deflection is also obtained based on the calculated temperature distribution considering the state of plane stresses. We introduce the two-dimensional treatment based on the so-called Kirchhoff-Love's hypothesis; thereafter basic equations are derived from the problem. For the thermoelastic deformation, the analytical solution is obtained and some important conclusions have been drawn as follows:

- The temperature distribution per Eq. (2.22) shows the several interesting changes are rooted in the physics of the boundary condition. The temperature and thermal deflection occur near the heat source, due to the axisymmetric heat source.
- The convergence of the series summation is rapid for a large time.
- The numerical values of the temperature and thermal deflection for the disk of materials Steel, Iron, Aluminum, and Copper are in the proportion and follow the relation $Steel \leq Iron \leq Aluminum \leq Copper$. We conclude that the thermal conductivity of material decreases its deflection increases. Hence, these values are inversely proportional to their thermal conductivity.
- It is observed that the material thermal properties have a dominant effect on the thermal deflection compared to the mechanical properties.
- It should be noted that a high strength level is needed for a good thermal resistance material.

The rotating disk has applications in aerospace engineering, particularly in gas turbines and gears. The rotating disk represents work under thermo-mechanical loads. Also, any particular case of special interest can be derived by assigning suitable values to the parameters and functions in the expressions (2.22)–(3.12).

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