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BOOLEAN MULTIPLICATIVE CONVOLUTION AND CAUCHY-STIELTJES KERNEL FAMILIES

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Abstract. Denote by $\mathcal{M}_+$ the set of probability measures supported on $\mathbb{R}_+$. Suppose V_ν is the variance function of the Cauchy-Stieltjes Kernel (CSK) family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure $\nu \in \mathcal{M}_+$. We determine the formula for variance function under boolean multiplicative convolution power. This formula is used to identify the relation between variance functions under the map $\nu \mapsto \mathbb{M}_t(\nu) = \left(\nu^{\boxtimes(t+1)}\right)^{\cup\!\!\!\!\times \frac{1}{t+1}}$ from $\mathcal{M}_+$ onto itself.

1. Introduction

According to Wesołowski [18], the kernel families generated by a kernel $k(x,\theta)$ with generating measure ν is the set of probability measures

$$\{k(x,\theta)/L(\theta) : \theta \in \Theta\},$$

where $L(\theta) = \int k(x,\theta)\nu(dx)$ is the normalizing constant and ν is the generating measure. The theory of natural exponential families (NEFs) is based on the exponential kernel $k(x,\theta) = \exp(\theta x)$. The theory of Cauchy-Stieltjes Kernel (CSK) families is recently introduced and it arises from a procedure analogous to the definition NEFs by using the Cauchy-Stieltjes kernel $1/(1-\theta x)$ instead of the exponential kernel. Bryc [7] initiated the study of CSK families for compactly supported probability measures ν. He has shown that such families can be parameterized by the mean and under this parametrization, the family (and measure ν) is uniquely determined by the variance function $V(m)$ and the mean m_0 of ν. He also described the class of quadratic CSK families. This class consists of the free Meixner distributions. A formula for variance function under the power of free additive convolution $\boxplus$ is also given. In [10], Bryc and Hassairi extend the results in [7] to allow measures ν with unbounded support, providing the method to determine the domain of means, introducing the "pseudo-variance" function that has no direct probabilistic interpretation

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but has similar properties to the variance function. They have also introduced the notion of reciprocity between two CSK families by defining a relation between the R-transforms of the corresponding generating probability measure. This leads to describing a class of cubic CSK families (with support bounded from one side) which is related to quadratic class by a relation of reciprocity. A general description of polynomial variance function with arbitrary degree is given in [9]. In particular, a complete resolution of cubic compactly supported CSK families is given. Other properties and characterizations in CSK families regarding the mean of the reciprocal and orthogonal polynomials are also given in [11] and [12].

On the other hand, the authors in [17] introduce a new kind of convolution between probability measures in the context of non-commutative probability theory with boolean independence: the boolean additive convolution $\uplus$. It plays an important role in free probability theory. However, it gained prominence in works by Biane, Belinschi and their collaborators ([5]), as it turned out to be useful in analyzing properties of the much more important free convolution. Reference [13] studies boolean additive convolution from a point of view related to CSK families. The author in [13] determines the formula for pseudo-variance function (or variance function V_ν in case of existence) under boolean additive convolution power. This formula is used to identify the relation between variance functions under Boolean Bercovici-Pata Bijection. The connection between boolean cumulants and variance function is also given. In particular, this allows to relate boolean cumulants of some probability measures to Catalan numbers and Fuss Catalan numbers. Pursuing the study of CSK families, the authors in [15] determine the effect on the variance function of the free multiplicative convolution $\boxtimes$. They use the variance functions to re-derive in a easy way the limit theorems given in [16] related to this type of convolution and involving the free additive convolution and the boolean additive convolution.

A multiplicative counterpart of boolean additive convolution was introduced by Franz [14], who also showed how to calculate it using moment generating series. Bercovici [4] also introduced another version of the boolean multiplicative convolution $\Cup\!\!\!\!\times$, and showed how to calculate it in terms of moment generating functions. He has proved that the boolean multiplicative convolution does not preserve $\mathcal{M}_+$: the set of probabilities measures supported on $\mathbb{R}_+$. However, there still exists a Boolean power $\mu^{\Cup\!\!\!\!\times t} \in \mathcal{M}_+$ for $0 \leq t \leq 1$.

In this paper, we are interested in the study of boolean multiplicative convolution from the perspective of CSK families. The variance function is the fundamental concept for CSK families. So, after a review of some facts regarding CSK families in Section 2, we determine in Section 3 the formula for variance function under boolean multiplicative convolution power. We also identify the relation between variance functions under the multiplicative analog of Belinschi-Nica type semigroup and we give the probabilistic interpretation of the construction of polynomial variance functions with arbitrary degree given in [9].

2. CSK families: Preliminaries and notations

Our notations are the ones used in [8], [11], [12] and [13]. Let ν be a non-degenerate probability measure with support bounded from above. Then

$$M_\nu(\theta) = \int \frac{1}{1-\theta x}\nu(dx) \tag{1}$$

is defined for all $\theta \in [0, \theta_+)$ with $1/\theta_+ = \max\{0, \sup \operatorname{supp}(\nu)\}$.

For $\theta \in [0, \theta_+)$, we set

$$P_{(\theta,\nu)}(dx) = \frac{1}{M_\nu(\theta)(1-\theta x)}\nu(dx).$$

The set

$$\mathcal{K}_+(\nu) = \{P_{(\theta,\nu)}(dx); \theta \in (0, \theta_+)\}$$

is called the one-sided Cauchy-Stieltjes kernel family generated by ν.

Let $k_\nu(\theta) = \int x P_{(\theta,\nu)}(dx)$ denote the mean of $P_{(\theta,\nu)}$. According to [10, pp. 579–580] the map $\theta \mapsto k_\nu(\theta)$ is strictly increasing on $(0, \theta_+)$ and is given by the formula

$$k_\nu(\theta) = \frac{M_\nu(\theta) - 1}{\theta M_\nu(\theta)}. \tag{2}$$

The image of $(0, \theta_+)$ by k_ν is called the (one sided) domain of means of the family $\mathcal{K}_+(\nu)$, it is denoted $(m_0(\nu), m_+(\nu))$. This leads to a parametrization of the family $\mathcal{K}_+(\nu)$ by the mean. In fact, denoting by ψ_ν the reciprocal of k_ν, and writing for $m \in (m_0(\nu), m_+(\nu))$, $Q_{(m,\nu)}(dx) = P_{(\psi_\nu(m),\nu)}(dx)$, we have that

$$\mathcal{K}_+(\nu) = \{Q_{(m,\nu)}(dx) : m \in (m_0(\nu), m_+(\nu))\}.$$

Now let

$$B = B(\nu) = \max\{0, \sup \operatorname{supp}(\nu)\} = 1/\theta_+ \in [0, \infty). \tag{3}$$

Then it is shown in [10] that the bounds $m_0(\nu)$ and $m_+(\nu)$ of the one-sided domain of means $(m_0(\nu), m_+(\nu))$ are given by

$$m_0(\nu) = \lim_{\theta \to 0^+} k_\nu(\theta) \tag{4}$$

and with $B = B(\nu)$,

$$m_+(\nu) = B - \lim_{z \to B^+} \frac{1}{G_\nu(z)}, \tag{5}$$

where $G_\nu(z)$ is the Cauchy transform of ν given by

$$G_\nu(z) = \int \frac{1}{z-x}\nu(dx). \tag{6}$$

It is worth mentioning here that one may define the one-sided CSK family for a measure ν with support bounded from below. This family is usually denoted $\mathcal{K}_-(\nu)$ and parameterized by θ such that $\theta_- < \theta < 0$, where θ_- is either $1/A(\nu)$ or $-\infty$ with $A = A(\nu) = \min\{0, \inf \operatorname{supp}(\nu)\}$. The domain of the mean for $\mathcal{K}_-(\nu)$ is the interval $(m_-(\nu), m_0(\nu))$ with $m_-(\nu) = A - 1/G_\nu(A)$.

If ν has a compact support, the natural domain for the parameter θ of the two-sided CSK family $\mathcal{K}(\nu) = \mathcal{K}_+(\nu) \cup \mathcal{K}_-(\nu) \cup \{\nu\}$ is $\theta_- < \theta < \theta_+$.

We come now to the notions of variance and pseudo-variance functions. The variance function

$$m \mapsto V_\nu(m) = \int (x-m)^2 Q_{(m,\nu)}(dx) \tag{7}$$

is a fundamental concept both in the theory of NEFs and in the theory of CSK families as presented in [7]. Unfortunately, if ν hasn't a first moment which is for example the case for a 1/2-stable law, all the distributions in the CSK family generated by ν have infinite variance. This fact has led the authors in [10] to introduce a notion of pseudo-variance function defined by

$$\frac{\mathbb{V}_\nu(m)}{m} = \frac{1}{\psi_\nu(m)} - m. \tag{8}$$

If $m_0(\nu) = \int x d\nu$ is finite, then (see [10]) the pseudo-variance function is related to the variance function by

$$\frac{\mathbb{V}_\nu(m)}{m} = \frac{V_\nu(m)}{m - m_0}. \tag{9}$$

In particular, $\mathbb{V}_\nu = V_\nu$ when $m_0(\nu) = 0$.

The generating measure ν is uniquely determined by the pseudo-variance function $\mathbb{V}_\nu$. In fact, according to [10, Proposition 3.5], if we set

$$z = z(m) = m + \frac{\mathbb{V}_\nu(m)}{m}, \tag{10}$$

then the Cauchy transform satisfies

$$G_\nu(z) = \frac{m}{\mathbb{V}_\nu(m)}. \tag{11}$$

We now recall the effect on a CSK family of applying an affine transformation to the generating measure. Consider the affine transformation

$$\varphi : x \mapsto (x - \lambda)/\beta$$

where $\beta \neq 0$ and $\lambda \in \mathbb{R}$ and let $\varphi(\nu)$ be the image of ν by φ. In other words, if X is a random variable with law ν, then $\varphi(\nu)$ is the law of $(X - \lambda)/\beta$, or $\varphi(\nu) = D_{1/\beta}(\nu \boxplus \delta_{-\lambda})$, where $D_r(\mu)$ denotes the dilation of measure μ by a number $r \neq 0$, that is $D_r(\mu)(U) = \mu(U/r)$. The point m_0 is transformed to $(m_0 - \lambda)/\beta$. In particular, if $\beta < 0$ the support of the measure $\varphi(\nu)$ is bounded from below so that it generates the left-sided family $\mathcal{K}_-(\varphi(\nu))$. For m close enough to $(m_0-\lambda)/\beta$, the pseudo-variance function is

$$\mathbb{V}_{\varphi(\nu)}(m) = \frac{m}{\beta(m\beta+\lambda)}\mathbb{V}_\nu(\beta m+\lambda). \tag{12}$$

In particular, if the variance function exists, then

$$V_{\varphi(\nu)}(m) = \frac{1}{\beta^2}V_\nu(\beta m+\lambda).$$

Note that using the special case where φ is the reflection $\varphi(x)=-x$, one can transform a right-sided CSK family to a left-sided family. If ν has a support bounded from above and its right-sided CSK family $\mathcal{K}_+(\nu)$ has domain of means (m_0,m_+) and pseudo-variance function $\mathbb{V}_\nu(m)$, then $\varphi(\nu)$ generates the left-sided CSK family $\mathcal{K}_-(\varphi(\nu))$ with domain of means $(-m_+,-m_0)$ and pseudo-variance function $\mathbb{V}_{\varphi(\nu)}(m)=\mathbb{V}_\nu(-m)$.

Let ν be a probability measure on the real line. The K-transform K_ν of ν is defined by

$$K_\nu(z) = z - \frac{1}{G_\nu(z)} \quad \text{for } \ z\in\mathbb{C}^+. \tag{13}$$

The function K_ν is usually called a self energy and it represents the analytic backbone of boolean additive convolution. The following technical result lists the properties of K-transform that we need. In fact this result is proved for probability measure ν with support bounded from above with $b=\sup\operatorname{supp}(\nu)<\infty$ (see [13, Proposition 2.2]). If ν is a probability measure with support bounded from below with $a=\inf\operatorname{supp}(\nu)>-\infty$, we deal with left sided CSK families and we have the following result when its proof is analogous to the proof of right sided case.

Proposition 2.1. *Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure ν with support bounded from below with $a=\inf\operatorname{supp}(\nu)>-\infty$. Then*

(i) *K_ν is strictly decreasing on $(-\infty,a)$.*
(ii) *For $m\in(m_-(\nu),m_0(\nu))$*

$$K_\nu\left(m+\mathbb{V}_\nu(m)/m\right) = m. \tag{14}$$

(iii) *$\lim\limits_{z\to A^-} K_\nu(z) = m_-(\nu)$, with $A=A(\nu)=\min\{0,a\}$.*
(iv) *$\lim\limits_{z\to-\infty} K_\nu(z) = m_0(\nu) \le +\infty$.*

3. Variance function and boolean multiplicative convolution

In this section we study the effect of boolean multiplicative convolution power on a CSK family. We also identify the relation between variance functions under the multiplicative analog of Belinschi-Nica type semigroup.

3.1. Variance function under boolean multiplicative convolution power

According to [6], for $\nu \in \mathcal{M}_+$, the function

$$\Psi_\nu(z) = \int_0^{+\infty} \frac{zx}{1-zx}\nu(dx), \qquad z \in \mathbb{C} \setminus \mathbb{R}_+ \tag{15}$$

is univalent in the left plane $i\mathbb{C}_+$ and $\Psi_\nu(i\mathbb{C}_+)$ is a region contained in the circle with diameter $(\nu(\{0\})-1, 0)$. Moreover $\Psi_\nu(i\mathbb{C}_+) \cap \mathbb{R} = (\nu(\{0\})-1, 0)$. It is well known that

$$\Psi_\nu(z^{-1}) = zG_\nu(z) - 1,$$

where G_ν is the Cauchy transform defined by (6).

We now recall the following from [1]. For $\nu \in \mathcal{M}_+$, the η-transform of ν is defined by:

$$\eta_\nu : \mathbb{C}\backslash\mathbb{R}_+ \to \mathbb{C}\backslash\mathbb{R}_+; \quad z \mapsto \eta_\nu(z) = \frac{\Psi_\nu(z)}{1+\Psi_\nu(z)}. \tag{16}$$

It is clear that ν is determined uniquely from the function η_ν. For $\nu \in \mathcal{M}_+$, it is known that $\eta_\nu((-\infty,0)) \subset (-\infty,0)$, $\lim\limits_{x\to 0,\, x<0} \eta_\nu(x) = \eta_\nu(0^-) = 0$ and $\eta_\nu(\overline{z}) = \overline{\eta_\nu(z)}$ for $z \in \mathbb{C}\backslash\mathbb{R}_+$. Also $\arg(z) \leq \arg(\eta_\nu(z)) < \pi$ for $z \in \mathbb{C}^+$.

The analytic function

$$B_\nu(z) = \frac{z}{\eta_\nu(z)} \tag{17}$$

is well defined in the region $z \in \mathbb{C}\backslash\mathbb{R}_+$. Now for $\mu,\ \nu \in \mathcal{M}_+$, their multiplicative boolean convolution $\mu \mathbin{\rlap{$\cup$}\times} \nu$ is defined as the unique probability measure in $\mathcal{M}_+$ that satisfies

$$B_{\mu \mathbin{\rlap{$\cup$}\times} \nu}(z) = B_\mu(z)B_\nu(z) \qquad \text{for } \ z \in \mathbb{C}\backslash\mathbb{R}_+. \tag{18}$$

Note that for $\mu,\ \nu \in \mathcal{M}_+$ which satisfies

(i) $\arg(\eta_\mu(z)) + \arg(\eta_\nu(z)) - \arg(z) < \pi$ for $z \in \mathbb{C}^+ \cup (-\infty, 0)$,
(ii) at least one of the first moments of one of the measures μ or ν exists finitely,

then $\mu \mathbin{\rlap{$\cup$}\times} \nu \in \mathcal{M}_+$ is well defined.

The following technical result will be used in the proof of Theorem 3.2.

Proposition 3.1. *Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure $\nu \in \mathcal{M}_+$ with mean $m_0(\nu) \leq +\infty$ and $0 \leq a = \inf \operatorname{supp}(\nu) < \infty$. Then, with $A = \min\{0, a\} = 0$, we have that*

(i) *B_ν is strictly decreasing on $(-\infty, 0)$.*
(ii) *$\lim\limits_{z\to 0^-} B_\nu(z) = 1/m_0(\nu)$ and $\lim\limits_{z\to -\infty} B_\nu(z) = 1/m_-(\nu)$.*

(iii) *For* $m \in (m_-(\nu), m_0(\nu))$ *and* $z = z(m) = \psi_\nu(m) = \frac{1}{m+\mathbb{V}_\nu(m)/m} \in (\theta_-, 0) = (-\infty, 0)$*, the* B*-transform of* ν *satisfies*

$$B_\nu(z) = \frac{1}{m}. \tag{19}$$

Proof. (i) The B-transform B_ν is related to the self energy function K_ν by $K_\nu(z^{-1}) = 1/B_\nu(z)$. From the fact that K_ν is strictly decreasing on $(-\infty, 0)$, it is easy to see that $B_\nu(\cdot)$ is strictly decreasing on $(-\infty, 0)$.

(ii) We have that

$$\lim_{z\to 0^-} B_\nu(z) = \lim_{z\to 0^-} 1/K_\nu(z^{-1}) = \lim_{t\to -\infty} 1/K_\nu(t) = 1/m_0(\nu)$$

and

$$\lim_{z\to -\infty} B_\nu(z) = \lim_{z\to -\infty} 1/K_\nu(z^{-1}) = \lim_{t\to 0^-} 1/K_\nu(t) = 1/m_-(\nu).$$

(iii) From [15, Proposition 2.1], for $z = z(m) = \psi_\nu(m) = \frac{1}{m+\mathbb{V}_\nu(m)/m}$, we have

$$\Psi_\nu(z) = \frac{m^2}{\mathbb{V}_\nu(m)}. \tag{20}$$

Combining (17) with (16) and (20), we get

$$\begin{aligned} B_\nu(\psi_\nu(m)) = \frac{\psi_\nu(m)}{\eta_\nu(\psi_\nu(m))} &= \frac{\psi_\nu(m)[1+\Psi_\nu(\psi_\nu(m))]}{\Psi_\nu(\psi_\nu(m))} \\ &= \frac{\frac{m}{m^2+\mathbb{V}_\nu(m)}\left[\frac{m^2+\mathbb{V}_\nu(m)}{\mathbb{V}_\nu(m)}\right]}{m^2/\mathbb{V}_\nu(m)} = \frac{1}{m}, \end{aligned}$$

which is (19). □

We now state and prove our main result concerning the effect of the boolean multiplicative convolution on a CSK family. According to [4], the boolean multiplicative boolean convolution power $\nu^{\boxtimes\alpha}$ is defined for $0 \leq \alpha \leq 1$ by $B_{\nu^{\boxtimes\alpha}}(z) = B_\nu(z)^\alpha$.

Theorem 3.2. *Suppose* $\mathbb{V}_\nu$ *is the pseudo-variance function of the CSK family* $\mathcal{K}_-(\nu)$ *generated by a non degenerate probability measure* $\nu \in \mathcal{M}_+$ *with mean* $m_0(\nu) < +\infty$*. For* $\alpha \in [0,1]$*, we have*

(i) $m_-(\nu^{\boxtimes\alpha}) = (m_-(\nu))^\alpha$ *and* $m_0(\nu^{\boxtimes\alpha}) = (m_0(\nu))^\alpha$*, and for* $m \in (m_-(\nu^{\boxtimes\alpha}), m_0(\nu^{\boxtimes\alpha}))$

$$\mathbb{V}_{\nu^{\boxtimes\alpha}}(m) = m^{1+1/\alpha} - m^2 + m^{1-1/\alpha}\mathbb{V}_\nu\left(m^{1/\alpha}\right). \tag{21}$$

(ii) *The variance functions of the CSK families generated by* ν *and* $\nu^{\boxtimes\alpha}$ *exists and*

$$V_{\nu^{\boxtimes\alpha}}(m) = \frac{m-m_0^\alpha}{m^{1/\alpha}-m_0}V_\nu\left(m^{1/\alpha}\right) + (m-m_0^\alpha)(m^{1/\alpha}-m). \tag{22}$$

Proof. We see that for the domain of means of the CSK family $\mathcal{K}_-(\nu^{\uplus\alpha})$ we have $m_0(\nu^{\uplus\alpha}) = (m_0(\nu))^\alpha$ and $m_-(\nu^{\uplus\alpha}) = (m_-(\nu))^\alpha$. This follows from Proposition 3.1(ii) and the multiplicative property of the B-transform. So for $m \in ((m_-(\nu))^\alpha, (m_0(\nu))^\alpha)$ we have that $m^{1/\alpha} \in (m_-(\nu), m_0(\nu))$ and $\frac{1}{m+\mathbb{V}_{\nu^{\uplus\alpha}}(m)/m} \in (-\infty, 0)$. We can apply (19) and the multiplicative property of the B-transform to see that

$$\begin{aligned} B_\nu\left(\frac{1}{m+\mathbb{V}_{\nu^{\uplus\alpha}}(m)/m}\right) &= \left[B_{\nu^{\uplus\alpha}}\left(\frac{1}{m+\mathbb{V}_{\nu^{\uplus\alpha}}(m)/m}\right)\right]^{1/\alpha} \\ &= \frac{1}{m^{1/\alpha}} = B_\nu\left(\frac{1}{m^{1/\alpha}+\mathbb{V}_\nu(m^{1/\alpha})/m^{1/\alpha}}\right). \end{aligned}$$

This implies that

$$m + \mathbb{V}_{\nu^{\uplus\alpha}}(m)/m = m^{1/\alpha} + \mathbb{V}_\nu(m^{1/\alpha})/m^{1/\alpha},$$

which is nothing but (24).

(ii) We have that $m_0 < +\infty$, then the variance functions of the CSK families $\mathcal{K}_-(\nu)$ and $\mathcal{K}_-(\nu^{\uplus\alpha})$ exists and relation (22) follows from (9) and (21). $\square$

3.2. Relation with Belinschi-Nica type semigroup for multiplicative convolutions

Let $\nu \in \mathcal{M}_+$ such that $\delta = \nu(\{0\}) < 1$, and consider the function $\Psi_\nu(\cdot)$ given by (15). Denoting $\mathbb{C}^+ = \{x+iy \in \mathbb{C} : y > 0\}$, the function Ψ_ν is univalent in the left half-plane $i\mathbb{C}^+$ and its image $\Psi_\nu(i\mathbb{C}^+)$ is contained in the disc with diameter $(\nu(\{0\})-1, 0)$. Moreover $\Psi_\nu(i\mathbb{C}^+) \cap \mathbb{R} = (\nu(\{0\})-1, 0)$. Let $\chi_\nu : \Psi_\nu(i\mathbb{C}^+) \to i\mathbb{C}^+$ be the inverse function of Ψ_ν. Then the S-transform of ν is the function

$$S_\nu(z) = \chi_\nu(z)\frac{1+z}{z}. \tag{23}$$

The product of S-transforms is an S-transform, and the multiplicative free convolution $\nu_1 \boxtimes \nu_2$ of the measures ν_1 and ν_2 is defined by

$$S_{\nu_1\boxtimes\nu_2}(z) = S_{\nu_1}(z)S_{\nu_2}(z).$$

We say that the probability measure $\nu \in \mathcal{M}_+$ is infinitely divisible with respect to $\boxtimes$ if for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}_+$ such that

$$\nu = \underbrace{\nu_n \boxtimes \cdots \boxtimes \nu_n}_{n\text{-times}}.$$

The multiplicative free convolution power $\nu^{\boxtimes\alpha}$ is defined at least for $\alpha \geq 1$ by $S_{\nu^{\boxtimes\alpha}}(z) = S_\nu(z)^\alpha$. For more details about the S-transform, see [6]. The effect of the free multiplicative convolution on a CSK family is given in [15], in fact under the action of power of free multiplicative convolution we have that for all $m \in \left(m_-(\nu^{\boxtimes\alpha}), m_0(\nu^{\boxtimes\alpha})\right) = \left((m_-(\nu))^\alpha, (m_0(\nu))^\alpha\right)$,

$$\mathbb{V}_{\nu^{\boxtimes\alpha}}(m) = m^{2-2/\alpha}\mathbb{V}_\nu\left(m^{1/\alpha}\right). \tag{24}$$

Furthermore, if $m_0 < +\infty$, then the variance functions of the CSK families generated by ν and $\nu^{\boxtimes\alpha}$ exist and

$$V_{\nu^{\boxtimes\alpha}}(m) = \frac{m - m_0^\alpha}{m^{1/\alpha} - m_0} m^{1-1/\alpha} V_\nu\left(m^{1/\alpha}\right). \tag{25}$$

The following result shows how the permutation of power between free and boolean multiplicative convolutions affect variance functions.

Theorem 3.3. *Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure $\nu \in \mathcal{M}_+$ with mean $m_0(\nu) < +\infty$. For $\alpha > 0$ such that $\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}$ and $\left(\nu^{\uplus\alpha}\right)^{\boxtimes 1/\alpha}$ are defined, they generate respectively CSK families with pseudo-variance functions given by*

$$\mathbb{V}_{\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}}(m) = m^{1+\alpha} - m^2 + m^{\alpha-1}\mathbb{V}_\nu(m) \tag{26}$$

and

$$\mathbb{V}_{\left(\nu^{\uplus\alpha}\right)^{\boxtimes 1/\alpha}}(m) = m^{-\alpha+3} - m^2 + m^{1-\alpha}\mathbb{V}_\nu(m) \tag{27}$$

for $m \in (m_-(\nu), m_0(\nu))$. Furthermore, the variance function of the CSK families generated by ν, $\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}$ and $\left(\nu^{\uplus\alpha}\right)^{\boxtimes 1/\alpha}$ exists and

$$V_{\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}}(m) = (m - m_0)(m^\alpha - m) + m^{\alpha-1}V_\nu(m) \tag{28}$$

and

$$V_{\left(\nu^{\uplus\alpha}\right)^{\boxtimes 1/\alpha}}(m) = (m - m_0)(m^{-\alpha+2} - m) + m^{1-\alpha}V_\nu(m). \tag{29}$$

Proof. For $\alpha > 0$ such that $\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}$ is defined, one sees that

$$m_0\left(\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}\right) = m_0\left(\nu^{\boxtimes\alpha}\right)/\alpha = m_0(\nu), \text{ and}$$
$$m_-\left(\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}\right) = m_-\left(\nu^{\boxtimes\alpha}\right)/\alpha = m_-(\nu).$$

For $m \in (m_-(\nu), m_0(\nu))$, combining formulas (24) and (21), we get:

$$\begin{aligned}\mathbb{V}_{\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}}(m) &= m^{1+\alpha} - m^2 + m^{1-\alpha}\mathbb{V}_{\nu^{\boxtimes\alpha}}(m^\alpha)\\ &= m^{1+\alpha} - m^2 + m^{1-\alpha}[m^{2\alpha-2}\mathbb{V}_\nu(m)]\\ &= m^{1+\alpha} - m^2 + m^{\alpha-1}\mathbb{V}_\nu(m).\end{aligned}$$

Furthermore, we have that $m_0 < +\infty$, then the variance function of the CSK families $\mathcal{K}_-(\nu)$ and $\mathcal{K}_-\left(\left(\nu^{\boxtimes\alpha}\right)^{\uplus 1/\alpha}\right)$ exists and relation (28) follows from (9) and (26).

The same arguments are used for probability measure $\left(\nu^{\uplus\alpha}\right)^{\boxtimes 1/\alpha}$ to get formulas (27) and (29). $\square$

The authors in [2] introduce the analogue of Belinschi-Nica type semigroup ([3]) for the multiplicative convolutions: that is for $t \geq 0$,

$$\begin{aligned} \mathbb{M}_t : \mathcal{M}_+ &\to \mathcal{M}_+ \\ \nu &\mapsto \left(\nu^{\boxtimes(t+1)}\right)^{\uplus\frac{1}{t+1}}. \end{aligned}$$

$\nu^{\uplus t} \in \mathcal{M}_+$ is defined for any probability measure $\nu \in \mathcal{M}_+$ and $0 \leq t \leq 1$. The following result gives the pseudo-variance function and variance function of the CSK family generated by $\mathbb{M}_t(\nu)$. In fact this easily follows from (26) and (28) by choosing $\alpha = 1 + t$.

Proposition 3.4. *Suppose $\mathbb{V}_\nu$ is the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure $\nu \in \mathcal{M}_+$ with mean $m_0(\nu) < +\infty$. For $t \geq 0$, the probability measure*

$$\mathbb{M}_t(\nu) = \left(\nu^{\boxtimes(t+1)}\right)^{\uplus\frac{1}{t+1}} \tag{30}$$

generates the CSK family with pseudo-variance function given by

$$\mathbb{V}_{\mathbb{M}_t(\nu)}(m) = m^t\ \mathbb{V}_\nu(m) + m^2(m^t - 1) \tag{31}$$

for $m \in (m_-(\nu), m_0(\nu))$. Furthermore, the variance function of the CSK families generated by ν and $\mathbb{M}_t(\nu)$ exists and

$$V_{\mathbb{M}_t(\nu)}(m) = m^t\ V_\nu(m) + m^{t+2} - m_0 m^{t+1} - m^2 + m m_0. \tag{32}$$

The authors in [9] construct a class of examples which exhausts all cubic variance functions, and provide examples of polynomial variance functions of arbitrary degree. In fact, they use some algebraic operations that allow to build new variance functions from known ones. Formula (32) gives a probabilistic interpretation of the relation between variance functions and proves that polynomial variance functions can be obtained from known ones by applying the multiplicative analog of Belinschi-Nica type semigroup to the generating probability measure.

Proposition 3.5. *Let $\nu \in \mathcal{M}_+$ be a non degenerate probability measure with mean $m_0(\nu) < +\infty$. Then*

$$\left(\nu^{\uplus\frac{1}{\alpha}}\right)^{\boxtimes\alpha} \xrightarrow{\alpha\to+\infty} \mathbb{M}_1(\nu), \qquad \textit{in distribution.}$$

Proof. We have that $m_0\left(\left(\nu^{\uplus\frac{1}{\alpha}}\right)^{\boxtimes\alpha}\right) = m_0(\nu) = m_0(\mathbb{M}_1(\nu))$. Furthermore,

$$V_{\left(\nu^{\uplus\frac{1}{\alpha}}\right)^{\boxtimes\alpha}}(m) \xrightarrow{\alpha\to+\infty} mV_\nu(m) + m^3 - (m_0+1)m^2 + mm_0 = V_{\mathbb{M}_1(\nu)}(m).$$

This implies that $\left(\nu^{\uplus\frac{1}{\alpha}}\right)^{\boxtimes\alpha} \xrightarrow{\alpha\to+\infty} \mathbb{M}_1(\nu)$, in distribution, by [7, Proposition 4.2]. □

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