

## INVERTIBILITY OF GENERALIZED BESSEL MULTIPLIERS IN HILBERT $C^*$ -MODULES

GHOLAMREZA ABBASPOUR TABADKAN AND HESSAM HOSSEINNEZHAD

**ABSTRACT.** This paper includes a general version of Bessel multipliers in Hilbert  $C^*$ -modules. In fact, by combining analysis, an operator on the standard Hilbert  $C^*$ -module and synthesis, we reach so-called generalized Bessel multipliers. Because of their importance for applications, we are interested to determine cases when generalized multipliers are invertible. We investigate some necessary or sufficient conditions for the invertibility of such operators and also we look at which perturbation of parameters preserve the invertibility of them. Subsequently, our attention is on how to express the inverse of an invertible generalized frame multiplier as a multiplier. In fact, we show that for all frames, the inverse of any invertible frame multiplier with an invertible symbol can always be represented as a multiplier with an invertible symbol and appropriate dual frames of the given ones.

### 1. Introduction

Frames in Hilbert spaces were originally introduced by Duffin and Schaeffer [13] to deal with some problems in nonharmonic Fourier analysis. Many generalizations of frames were introduced, e.g. pseudo-frames [21], g-frames [26] and fusion frames (frames of subspaces) [11].

Frank and Larson [14] extended the frame theory for the elements of  $C^*$ -algebras and (finitely or countably generated) Hilbert  $C^*$ -modules. Extending the results to this more general framework is not a routine generalization, as there are essential differences between Hilbert  $C^*$ -modules and Hilbert spaces. For example, we know that the Riesz representation theorem for continuous linear functionals on Hilbert spaces does not extend to Hilbert  $C^*$ -modules and there exist closed subspaces in Hilbert  $C^*$ -modules that have no orthogonal complement. Moreover, we know that every bounded operator on a Hilbert space has an adjoint, while there are bounded operators on Hilbert  $C^*$ -modules which do not have one. It should be mentioned that, contrasting to the Hilbert

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space situation, an arbitrary Hilbert  $C^*$ -module need not possess an orthonormal basis.

Bessel multipliers in Hilbert spaces were introduced by Balazs in [5]. Bessel multipliers are operators that are defined by a fixed multiplication pattern which is inserted between the analysis and synthesis operators. This class of operators is not only of interest for applications in modern life, for example in acoustics [27], psychoacoustics [9] and denoising [22], but also it is important in different branches of functional analysis [7]. In this respect, it is important to find the inverse of a multiplier if it exists. Recently, M. Mirzaee Azandaryani and A. Khosravi generalized multipliers to Hilbert  $C^*$ -modules [19].

The standard matrix description of operators on Hilbert spaces, using an orthonormal basis, was presented in [12]. This idea was developed for Bessel sequences, frames and Riesz sequences by Balazs [6]. In the last paper, the author also studied the dual function, which assigns an operator to a matrix. Using this approach, a generalization of Bessel multipliers is obtained, as introduced in [8]. In the present paper, the concept of generalized multipliers is extended for Hilbert modules and then some properties of these operators are investigated. In particular, special attention is devoted to the study of invertible generalized multipliers. The paper is organized as follows.

In Section 2, some notations and preliminary results of Hilbert modules, their frames and Bessel multipliers are given. Section 3 is devoted to the generalization of Bessel multipliers in Hilbert  $C^*$ -modules and then some conditions for invertibility of such operators are obtained. In the last section, we extend the results from [10] in more general cases, i.e., for generalized multipliers in Hilbert  $C^*$ -modules. In more details, our attention is on how to express the inverse of an invertible generalized multiplier as a multiplier. In fact, we show that for all frames, the inverse of any invertible frame multiplier with an invertible symbol can always be represented as a multiplier with the invertible symbol and appropriate dual frames of the given ones.

## 2. Notation and preliminaries

In this section, we recall some definitions and basic properties of Hilbert  $C^*$ -modules and their frames. Throughout this paper,  $A$  is a unital  $C^*$ -algebra,  $E$  and  $F$  are finitely or countably generated Hilbert  $A$ -modules and  $\mathbb{I}$  is an at most countable index set.

A (left) Hilbert  $C^*$ -module over the  $C^*$ -algebra  $A$  is a left  $A$ -module  $E$  equipped with an  $A$ -valued inner product  $\langle \cdot, \cdot \rangle : E \times E \rightarrow A$  satisfying the following conditions:

- (1)  $\langle x, x \rangle \geq 0$  for every  $x \in E$  and  $\langle x, x \rangle = 0$  if and only if  $x = 0$ ,
- (2)  $\langle x, y \rangle = \langle y, x \rangle^*$  for every  $x, y \in E$ ,
- (3)  $\langle \cdot, \cdot \rangle$  is  $A$ -linear in the first argument,
- (4)  $E$  is complete with respect to the norm  $\|x\|^2 = \|\langle x, x \rangle\|_A$ .

Given Hilbert  $C^*$ -modules  $E$  and  $F$ , we denote by  $L(E, F)$  the set of all adjointable operators from  $E$  to  $F$  (i.e., of all maps  $T : E \rightarrow F$  such that there exists  $T^* : F \rightarrow E$  with the property  $\langle Tx, y \rangle = \langle x, T^*y \rangle$  for all  $x \in E, y \in F$ ). It is well-known that each adjointable operator is necessarily bounded and  $A$ -linear in the sense  $T(ax) = aT(x)$  for all  $a \in A, x \in E$ .

For each elements  $x \in E, y \in F$ , we define the operator  $\Theta_{x,y} : E \rightarrow F$  by  $\Theta_{x,y}(z) = \langle z, x \rangle y$  for each  $z \in E$ . It is easy to check that  $\Theta_{x,y} \in L(E, F)$  and  $(\Theta_{x,y})^* = \Theta_{y,x}$ . Operators of this form are called elementary operators. Each finite linear combination of elementary operators is said to be a finite rank operator. The closed linear span of the set  $\{\Theta_{x,y} : x \in F, y \in E\}$  in  $L(E, F)$  is denoted by  $K(E, F)$  and its elements will be called compact operators. Specially, if  $E = F$ , we write  $L(E)$  and  $K(E)$ , respectively. It is well-known that  $L(E)$  is a  $C^*$ -algebra and  $K(E)$  is the closed two-sided ideal in  $L(E)$ . Recall that the center of a Banach algebra  $A$ , denoted  $Z(A)$ , is defined as  $Z(A) = \{a \in A : ab = ba, \forall b \in A\}$ . It is clear that if  $a \in Z(A)$ , then  $a^* \in Z(A)$ . Also if  $a$  is a positive element of  $Z(A)$ , then  $a^{\frac{1}{2}} \in Z(A)$ .

Let  $A$  be a  $C^*$ -algebra. Consider

$$\ell^2(A, \mathbb{I}) := \left\{ \{a_i\}_{i \in \mathbb{I}} \subseteq A : \sum_{i \in \mathbb{I}} a_i a_i^* \text{ converges in norm in } A \right\}.$$

It is easy to see that  $\ell^2(A, \mathbb{I})$  with pointwise operations and the inner product

$$\langle \{a_i\}_{i \in \mathbb{I}}, \{b_i\}_{i \in \mathbb{I}} \rangle = \sum_{i \in \mathbb{I}} a_i b_i^*,$$

becomes a Hilbert  $C^*$ -module which is called the *standard Hilbert  $C^*$ -module* over  $A$ . A Hilbert  $A$ -module  $E$  is called *finitely generated* (resp. *countably generated*) if there exists a finite subset  $\{x_1, \dots, x_n\}$  (resp. countable subset  $\{x_i\}_{i \in \mathbb{I}}$ ) of  $E$  such that  $E$  equals the closed  $A$ -linear hull of this set.

Let  $E$  and  $F$  be Hilbert  $A$ -modules. An  $A$ -linear operator  $t : \text{Dom}(t) \subseteq E \rightarrow F$  is called *densely defined* if  $\text{Dom}(t)$  is a dense submodule of  $E$  (not necessarily identical with  $E$ ) and whose range is in  $F$ . A densely defined operator  $t : \text{Dom}(t) \subseteq E \rightarrow F$  is called *closed* if its graph  $G(t) = \{(x, tx) : x \in \text{Dom}(t)\}$  is a closed submodule of the Hilbert  $A$ -module  $E \oplus F$ . A densely defined operator  $t : \text{Dom}(t) \subseteq E \rightarrow F$  is called *adjointable* if it possesses a densely defined map  $t^* : \text{Dom}(t^*) \subseteq F \rightarrow E$  with the domain

$$\begin{aligned} \text{Dom}(t^*) = \{y \in F : \text{there is } z \in E \text{ such that } \langle tx, y \rangle_F = \langle x, z \rangle_E \\ \text{for any } x \in \text{Dom}(t)\}. \end{aligned}$$

The above property implies that  $t^*$  is a closed  $A$ -linear map. For more details about Hilbert  $C^*$ -modules, we refer the interested reader to the books [20, 23].

Now, we recall the concept of frame in Hilbert  $C^*$ -modules which is defined in [14]. Let  $E$  be a countably generated Hilbert module over a unital  $C^*$ -algebra  $A$ . A sequence  $\{x_i\}_{i \in \mathbb{I}} \subset E$  is said to be a *frame* if there exist two constant

$C, D > 0$  such that

$$(2.1) \quad C \langle x, x \rangle \leq \sum_{i \in \mathbb{I}} \langle x, x_i \rangle \langle x_i, x \rangle \leq D \langle x, x \rangle$$

for every  $x \in E$ . The optimal constants (i.e., maximal for  $C$  and minimal for  $D$ ) are called *frame bounds*. If the sum in (2.1) converges in norm, the frame is called a *standard frame*. The sequence  $\{x_i\}_{i \in \mathbb{I}}$  is called a *Bessel sequence* with bound  $D$  if the upper inequality in (2.1) holds for every  $x \in E$ .

A *Riesz basis* in a Hilbert  $C^*$ -module  $E$  is a frame  $\{x_i\}_{i \in \mathbb{I}}$  such that for each  $i \in \mathbb{I}$ ,  $x_i \neq 0$  and if an  $A$ -linear combination  $\sum_{i \in \mathbb{J} \subseteq \mathbb{I}} a_i x_i$  is equal to zero, then every summand  $a_i x_i$  is equal to zero.

Suppose that  $X = \{x_i\}_{i \in \mathbb{I}}$  is a Bessel sequence in Hilbert  $A$ -module  $E$  with bound  $D$ . The operator  $T_X : \ell^2(A, \mathbb{I}) \rightarrow E$  defined by

$$T_X \{a_i\}_{i \in \mathbb{I}} = \sum_{i \in \mathbb{I}} a_i x_i,$$

is called the *synthesis operator*. The adjoint operator  $T_X^* : E \rightarrow \ell^2(A, \mathbb{I})$  which is given by

$$T_X^* x = \{\langle x, x_i \rangle\}_{i \in \mathbb{I}},$$

is called the *analysis operator*. Composing  $T_X$  and  $T_X^*$ , we obtain the *frame operator*  $S_X : E \rightarrow E$  as

$$S_X x = T_X T_X^* x = \sum_{i \in \mathbb{I}} \langle x, x_i \rangle x_i.$$

If  $X = \{x_i\}_{i \in \mathbb{I}}$  is a standard frame with bounds  $C, D$ , the frame operator  $S_X$  is well-defined, positive, invertible and adjointable. Moreover, it satisfies  $C \leq S_X \leq D$  and  $D^{-1} \leq S_X^{-1} \leq C^{-1}$ . Also, for each  $x \in E$ , we have the *reconstruction formula* as follows,

$$(2.2) \quad x = \sum_{i \in \mathbb{I}} \langle x, S_X^{-1} x_i \rangle x_i = \sum_{i \in \mathbb{I}} \langle x, x_i \rangle S_X^{-1} x_i.$$

The sequence  $\{\tilde{x}_i\}_{i \in \mathbb{I}} = \{S_X^{-1} x_i\}_{i \in \mathbb{I}}$ , which is a standard frame with bounds  $D^{-1}$  and  $C^{-1}$ , is called the *canonical dual frame* of  $X = \{x_i\}_{i \in \mathbb{I}}$ . Sometimes the reconstruction formula of standard frames is valid with other (standard) frames  $\{y_i\}_{i \in \mathbb{I}}$  instead of  $\{S_X^{-1} x_i\}_{i \in \mathbb{I}}$ . They are said to be *alternative dual frames* of  $X = \{x_i\}_{i \in \mathbb{I}}$ .

Now, let us take a brief review of the definition of Bessel multipliers in Hilbert  $C^*$ -modules.

Let  $E$  and  $F$  be two Hilbert modules over a unital  $C^*$ -algebra  $A$ , and let  $X = \{x_i\}_{i \in \mathbb{I}} \subseteq E$  and  $Y = \{y_i\}_{i \in \mathbb{I}} \subseteq F$  be standard Bessel sequences. Moreover, let  $m = \{m_i\}_{i \in \mathbb{I}} \in \ell^\infty(A, \mathbb{I})$  be such that  $m_i \in Z(A)$ , for each  $i \in \mathbb{I}$ , and  $\mathcal{M}_m$  defined on  $\ell^2(A, \mathbb{I})$  as  $\mathcal{M}_m \{a_i\}_{i \in \mathbb{I}} = \{m_i a_i\}_{i \in \mathbb{I}}$ .

The operator  $\mathbf{M}_{m,Y,X} : E \rightarrow F$  which is defined by

$$(2.3) \quad \mathbf{M}_{m,Y,X} = T_Y \mathcal{M}_m T_X^*,$$

is called the *Bessel multiplier* for the Bessel sequences  $\{x_i\}_{i \in \mathbb{I}}$  and  $\{y_i\}_{i \in \mathbb{I}}$ . It is easy to see that  $\mathbf{M}_{m,Y,X}(x) = \sum_{i \in \mathbb{I}} m_i \langle x, x_i \rangle y_i$ . For more details about the Bessel multipliers in Hilbert  $C^*$ -modules, one can see [19].

### 3. Generalized Bessel multipliers in Hilbert $C^*$ -modules

The matrix representation of operators in Hilbert spaces using an orthonormal basis [12], Gabor frames [15] and linear independent Gabor systems [25] led Balazs to develop this idea in full generality for Bessel sequences, frames and Riesz sequences [6]. In the same paper, the author also established the function which assigns an operator in  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$  to an infinite matrix in  $\mathcal{B}(\ell^2(\mathbb{I}))$ . The last concept is a generalization of Bessel multiplier as introduced in [6]. The following essential definition is recalled from [6, 8].

**Definition.** Let  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be Hilbert spaces and  $X = \{x_i\}_{i \in \mathbb{I}} \subset \mathcal{H}_1$  and  $Y = \{y_i\}_{i \in \mathbb{I}} \subset \mathcal{H}_2$  be Bessel sequences. Moreover, let  $M$  be an infinite matrix defining a bounded operator from  $\ell^2(\mathbb{I})$  to  $\ell^2(\mathbb{I})$ ,  $(Mc)_i = \sum_{k \in \mathbb{I}} M_{i,k} c_k$ . Then the operator  $\mathcal{O}^{(X,Y)}(M) : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  defined by

$$\left(\mathcal{O}^{(X,Y)}(M)\right)h = T_Y M T_X^* h = \sum_{k \in \mathbb{I}} \sum_{j \in \mathbb{I}} M_{k,j} \langle h, x_j \rangle y_k, \quad (h \in \mathcal{H}_1),$$

is called the *generalized Bessel multiplier* for the Bessel sequences  $X$  and  $Y$ .

In the sequel, first, we introduce the concept of Generalized Bessel multipliers for countably generated Hilbert  $C^*$ -modules and then, we will discuss some properties of such operators.

**Definition.** Let  $E$  and  $F$  be two Hilbert  $C^*$ -modules over a unital  $C^*$ -algebra  $A$  and  $X = \{x_i\}_{i \in \mathbb{I}} \subset E$  and  $Y = \{y_i\}_{i \in \mathbb{I}} \subset F$  be standard Bessel sequences. Also, let  $U \in L(\ell^2(A, \mathbb{I}))$  be an arbitrary non-zero operator. The operator  $\mathbf{M}_{U,Y,X} : E \rightarrow F$  which is defined as

$$(3.1) \quad \mathbf{M}_{U,Y,X}(x) = T_Y U T_X^*(x), \quad (x \in E),$$

is called the *Generalized Bessel multiplier* associated with  $X$  and  $Y$  with symbol  $U$ . Some of the main properties of the generalized Bessel multipliers are summarized in the next proposition.

**Proposition 3.1.** *For the generalized Bessel multiplier  $\mathbf{M}_{U,Y,X}$ , the following assertions hold:*

- (1)  $\mathbf{M}_{U,Y,X} \in L(E, F)$  and  $\mathbf{M}_{U,Y,X}^* = \mathbf{M}_{U^*,X,Y}$ .
- (2) If  $U$  is a compact operator on  $\ell^2(A, \mathbb{I})$ , then  $\mathbf{M}_{U,Y,X} \in K(E, F)$ .
- (3) If  $U$  is a positive operator on  $\ell^2(A, \mathbb{I})$ , then  $\mathbf{M}_{U,Y,X} \in L(E)$  is a positive operator.

*Proof.* (1) It is clear that  $\mathbf{M}_{U,Y,X} \in L(E, F)$ . Also

$$\mathbf{M}_{U,Y,X}^* = (T_Y U T_X^*)^* = T_X U^* T_Y^* = \mathbf{M}_{U^*,X,Y}.$$

(2) At first, let us prove that  $\mathbf{M}_{U,Y,X}$  is a finite rank operator if  $U$  is one. If  $U$  is a finite rank operator, then  $U = \sum_{i=1}^n \Theta_{a_i, b_i}$  for some  $a_i, b_i \in \ell^2(A, \mathbb{I})$ , ( $i = 1, \dots, n$ ). Hence,

$$\mathbf{M}_{U,Y,X} = T_Y U T_X^* = T_Y \left( \sum_{i=1}^n \Theta_{a_i, b_i} \right) T_X^* = \sum_{i=1}^n \Theta_{T_X a_i, T_Y b_i}.$$

Therefore,  $\mathbf{M}_{U,Y,X}$  is a finite rank operator from  $E$  to  $F$ . Now, let  $U$  be a compact operator on  $\ell^2(A, \mathbb{I})$ . Thus there exists a sequence of finite rank operators on  $\ell^2(A, \mathbb{I})$ , say  $\{U_\alpha\}_\alpha$ , such that  $\|U_\alpha - U\| \rightarrow 0$ . So

$$\|\mathbf{M}_{U_\alpha, Y, X} - \mathbf{M}_{U, Y, X}\| \leq \|T_Y^*\| \|U_\alpha - U\| \|T_X\|.$$

As seen above,  $\mathbf{M}_{U_\alpha, Y, X}$  are finite rank operators. From this fact, we conclude that  $\mathbf{M}_{U, Y, X}$  is a compact operator.

(3) Since  $U$  is positive, by [20, Lemma 4.1],  $\langle a, Ua \rangle \geq 0$  for all  $a = \{a_i\}_{i \in \mathbb{I}} \in \ell^2(A, \mathbb{I})$ . So

$$\langle x, \mathbf{M}_{U, X, X} x \rangle = \langle x, T_X U T_X^* x \rangle = \langle T_X^* x, U T_X^* x \rangle \geq 0.$$

Again by [20, Lemma 4.1], it follows that  $\mathbf{M}_{U, X, X}$  is positive.  $\square$

In the following, we investigate sufficient conditions for being standard frames by properties of the generalized Bessel multipliers.

**Proposition 3.2.** Assume that  $X = \{x_i\}_{i \in \mathbb{I}} \subset E$  and  $Y = \{y_i\}_{i \in \mathbb{I}} \subset F$  are Bessel sequences.

- (1) If  $\mathbf{M}_{U, Y, X}$  has a left inverse, then  $X$  is a standard frame for  $E$ .
- (2) If  $\mathbf{M}_{U, Y, X}$  has a right inverse, then  $Y$  is a standard frame for  $F$ .

*Proof.* Let  $\mathbf{L}$  be the left inverse of  $\mathbf{M}_{U, Y, X}$ . Then for every  $x \in E$ ,

$$\begin{aligned} \|x\|^2 &= \|\langle \mathbf{L} \mathbf{M}_{U, Y, X} x, x \rangle\| \\ &= \|\langle \mathbf{L} T_Y U T_X^* x, x \rangle\| \\ &\leq \sqrt{B_Y} \|\mathbf{L}\| \|U\| \|x\| \|T_X^* x\|. \end{aligned}$$

It follows that

$$\frac{\|x\|}{\sqrt{B_Y} \|\mathbf{L}\| \|U\|} \leq \left\| \sum_{i \in \mathbb{I}} \langle x, x_i \rangle \langle x_i, x \rangle \right\|^{1/2},$$

and so by [17, Proposition 3.8],  $X$  is a standard frame for  $E$ . The second part is similar.  $\square$

Similar to the case of operators on Hilbert spaces, we also have the following criterion for the invertibility of operators on Hilbert modules.

**Lemma 3.3.** *Let  $E$  be a Hilbert  $A$ -module and  $U : E \rightarrow E$  be an invertible operator. Also let  $W \in L(E)$  be such that for each  $x \in E$ ,  $\|Ux - Wx\| \leq \lambda\|x\|$  where  $\lambda \in [0, \|U^{-1}\|^{-1})$ . Then  $W$  is invertible and*

$$\frac{1}{\lambda + \|U\|} \|x\| \leq \|W^{-1}x\| \leq \frac{1}{\|U^{-1}\|^{-1} - \lambda} \|x\|.$$

*Proof.* It follows directly from the proofs of [4, Theorem 3.2.3] and [24, Proposition 2.2].  $\square$

The next proposition investigates some sufficient conditions for invertibility of generalized frame multipliers.

**Proposition 3.4.** *Let  $E$  be a Hilbert  $A$ -module and  $X = \{x_i\}_{i \in \mathbb{I}}$  be a standard frame for  $E$  with bounds  $C$  and  $D$ . Suppose that  $Y = \{y_i\}_{i \in \mathbb{I}}$  is a sequence of  $E$  and there exists a positive constant  $\lambda < \frac{1}{D} \left( \frac{CD^2 - C^2D}{C^2 + D^2} \right)^2$  such that*

$$(3.2) \quad \left\| \sum_{i \in \mathbb{I}} \langle x, x_i - y_i \rangle \langle x_i - y_i, x \rangle \right\| \leq \lambda \|x\|^2.$$

*Moreover, suppose that  $U$  is a non-zero adjointable operator on  $\ell^2(A, \mathbb{I})$  with  $\|U - I\| < \frac{C^2}{D^2}$ . Then  $Y$  is a standard frame and  $\mathbf{M}_{U,X,Y}$  is invertible.*

*Proof.* The first part follows from [16, Theorem 3.2]. Now, let us deal with the second claim. Suppose  $S_X$  is the frame operator associated to  $X$ . For each  $x \in E$ :

$$\begin{aligned} \|\mathbf{M}_{U,X,X}(x) - S_X(x)\| &= \|\mathbf{M}_{U,X,X}(x) - \mathbf{M}_{I,X,X}(x)\| \\ &= \|\mathbf{M}_{U-I,X,X}(x)\| \\ &= \|T_X(U - I)T_X^*(x)\| \\ &\leq \|T_X\| \|U - I\| \|T_X^*(x)\| \\ &< (C^2/D) \|x\|. \end{aligned}$$

So by Lemma 3.3,  $\mathbf{M}_{U,X,X}$  is an invertible operator with

$$\frac{1}{\|S_X\| + C^2/D} \leq \|\mathbf{M}_{U,X,X}^{-1}\| \leq \frac{1}{\|S_X^{-1}\|^{-1} - C^2/D}.$$

Now for every  $x \in E$ ,

$$\begin{aligned} \|\mathbf{M}_{U,X,Y}(x) - \mathbf{M}_{U,X,X}(x)\| &= \|\mathbf{M}_{U,X,Y-X}(x)\| \\ &= \|T_X U T_{Y-X}^*(x)\| \\ &\leq \|T_X\| \|U\| \|T_{Y-X}^*(x)\| \\ &\leq \|U\| \sqrt{D} \sqrt{\lambda} \|x\|. \end{aligned}$$

If we show that  $\|U\|\sqrt{D}\sqrt{\lambda} < \frac{1}{\|\mathbf{M}_{U,X,X}^{-1}\|}$ , then the proof will be completed. But

$$\|U\|\sqrt{D}\sqrt{\lambda} \leq \frac{C^2 + D^2}{D^2} \sqrt{D}\sqrt{\lambda} < C - \frac{C^2}{D} \leq \|S_X^{-1}\|^{-1} - \frac{C^2}{D} \leq \frac{1}{\|\mathbf{M}_{U,X,X}^{-1}\|},$$

and so by Lemma 3.3 the result holds.  $\square$

The following two propositions contain sufficient conditions for the invertibility of frame multipliers.

**Proposition 3.5.** *Let  $Y = \{y_i\}_{i \in \mathbb{I}}$  be a standard frame for Hilbert  $A$ -module  $E$  with bounds  $C, D$  and  $W : E \rightarrow E$  be an adjointable and bijective operator such that  $x_i = Wy_i$  for each  $i \in \mathbb{I}$ . Moreover, let  $U$  be a bounded operator on  $\ell^2(A, \mathbb{I})$  such that  $\|U - I\| < \frac{C}{D}$ . Then the following statements hold:*

- (1)  $X = \{x_i\}_{i \in \mathbb{I}}$  is a standard frame for  $E$ .
- (2)  $\mathbf{M}_{U,Y,X}$  (resp.  $\mathbf{M}_{U,X,Y}$ ) is invertible and  $\mathbf{M}_{U,Y,X}^{-1} = (W^{-1})^* \mathbf{M}_{U,Y,Y}^{-1}$  (resp.  $\mathbf{M}_{U,X,Y}^{-1} = \mathbf{M}_{U,Y,Y}^{-1}(W^{-1})$ ).

*Proof.* (1) It follows from [3, Theorem 2.5].

(2) First note that  $\mathbf{M}_{U,Y,X} = \mathbf{M}_{U,Y,Y} W^*$ . Indeed

$$\begin{aligned} \mathbf{M}_{U,Y,Y} W^*(x) &= T_Y U T_Y^* W^*(x) = T_Y U \{\langle W^* x, y_i \rangle\}_{i \in \mathbb{I}} \\ &= T_Y U \{\langle x, x_i \rangle\}_{i \in \mathbb{I}} \\ &= T_Y U T_X^*(x) \\ &= \mathbf{M}_{U,Y,X}(x). \end{aligned}$$

So it is enough to prove that  $\mathbf{M}_{U,Y,Y}$  is invertible. For every  $x \in E$ ,

$$\|\mathbf{M}_{U,Y,Y}(x) - S_Y(x)\| = \|\mathbf{M}_{U-I,Y,Y}(x)\| \leq D\|U - I\|\|x\| < C\|x\|.$$

Since  $C \leq \frac{1}{\|S_Y^{-1}\|}$ , it follows from Lemma 3.3 that  $\mathbf{M}_{U,Y,Y}$  is invertible. The invertibility of  $\mathbf{M}_{U,X,Y}$  is obtained with the same argument.  $\square$

**Proposition 3.6.** *Let  $X = \{x_i\}_{i \in \mathbb{I}}$  be a standard frame for Hilbert  $A$ -module  $E$  with upper bound  $D$  and  $X^d = \{x_i^d\}_{i \in \mathbb{I}}$  be a dual frame of  $X$  with upper bound  $D'$ . Also, let  $U$  be a bounded operator on  $\ell^2(A, \mathbb{I})$  such that  $\|U - I\| < \frac{1}{\sqrt{DD'}}$ . Then the multiplier  $\mathbf{M}_{U,X,X^d}$  (resp.  $\mathbf{M}_{U,X^d,X}$ ) is invertible.*

*Proof.* For every  $x \in E$ ,

$$\|\mathbf{M}_{U,X,X^d}(x) - x\| = \|\mathbf{M}_{U-I,X,X^d}\| \leq \sqrt{DD'}\|U - I\|\|x\| < \|x\|.$$

So  $\mathbf{M}_{U,X,X^d}$  is invertible.  $\square$

**Proposition 3.7.** *Let  $Y = \{y_i\}_{i \in I}$  be a standard frame for Hilbert  $A$ -module  $E$  with bounds  $C$  and  $D$  and  $\tilde{Y} = \{\tilde{y}_i\}_{i \in I}$  be its canonical dual frame.*



(1) If  $X = \{x_i\}_{i \in \mathbb{I}}$  is a standard Bessel sequence such that

$$(3.3) \quad \sum_{i \in \mathbb{I}} \|x_i - \tilde{y}_i\|^2 < \frac{1}{4D},$$

then  $\mathbf{M}_{I,Y,X}$  is invertible.

(2) Let  $X = \{x_i\}_{i \in \mathbb{I}}$  be a standard Bessel sequence and (3.3) hold. Also, let  $U$  be a bounded operator on  $\ell^2(A, \mathbb{I})$  with  $\|U\| < 1$  and  $\|U - I\| < \frac{\sqrt{C}}{2\sqrt{D}}$ . Then  $\mathbf{M}_{U,Y,X}$  is invertible.

*Proof.* (1) For every  $x \in E$ ,

$$\begin{aligned} \|\mathbf{M}_{I,Y,X}(x) - x\| &= \|T_Y T_X^*(x) - T_Y T_{\tilde{Y}}^*(x)\| \\ &= \|T_Y T_{X-\tilde{Y}}^*(x)\| \\ &\leq \sqrt{D} \|\{ \langle x, x_i - \tilde{y}_i \rangle \}_{i \in \mathbb{I}}\|_{\ell^2(A, \mathbb{I})} \\ &= \sqrt{D} \left\| \sum_{i \in \mathbb{I}} \langle x, x_i - \tilde{y}_i \rangle \langle x_i - \tilde{y}_i, x \rangle \right\|^{1/2} \\ &\leq \sqrt{D} \left( \sum_{i \in \mathbb{I}} \|x\|^2 \|x_i - \tilde{y}_i\|^2 \right)^{1/2} \\ &= \sqrt{D} \|x\| \left( \sum_{i \in \mathbb{I}} \|x_i - \tilde{y}_i\|^2 \right)^{1/2} \\ &< \|x\|, \end{aligned}$$

and so  $\mathbf{M}_{I,Y,X}$  is invertible.

(2) For every  $x \in E$  we have:

$$\begin{aligned} \|\mathbf{M}_{U,Y,X}(x) - x\| &\leq \|\mathbf{M}_{U,Y,X}(x) - \mathbf{M}_{U,Y,\tilde{Y}}(x)\| + \|\mathbf{M}_{U,Y,\tilde{Y}}(x) - \mathbf{M}_{I,Y,\tilde{Y}}(x)\| \\ &= \|T_Y U T_{X-\tilde{Y}}^*(x)\| + \|T_Y (U - I) T_{\tilde{Y}}^*(x)\| \\ &\leq \sqrt{D} \|U\| \|T_{X-\tilde{Y}}^*(x)\| + (\sqrt{D/C}) \|U - I\| \|x\| \\ &\leq \sqrt{D} \|U\| \|x\| \left( \sum_{i \in I} \|x_i - \tilde{y}_i\|^2 \right)^{1/2} + (\sqrt{D/C}) \|U - I\| \|x\| \\ &< \|x\|. \end{aligned}$$

Hence we conclude that  $\mathbf{M}_{U,Y,X}$  is invertible.  $\square$

We are now in a position to state and prove our main result about the stability of invertible generalized multipliers.

**Proposition 3.8.** Let  $X = \{x_i\}_{i \in \mathbb{I}}$ ,  $Y = \{y_i\}_{i \in \mathbb{I}}$ ,  $Z = \{z_i\}_{i \in \mathbb{I}}$  and  $F = \{f_i\}_{i \in \mathbb{I}}$  be Bessel sequences for Hilbert module  $E$  with bounds  $D_X, D_Y, D_Z$  and  $D_F$ , respectively. Moreover, let  $U, V \in L(\ell^2(A, \mathbb{I}))$  with  $\|U - V\| < \epsilon$  for some  $\epsilon > 0$ , and

$$(3.4) \quad D\epsilon + 5D\|U\| < D^{-1}\|U\|^{-1},$$

where  $D = \max\{D_X, D_Y, D_Z, D_F\}$ . If  $\mathbf{M}_{U,Y,X}$  is invertible, then  $\mathbf{M}_{V,Z,F}$  is also invertible and  $Z, F$  are standard frames for  $E$ .

*Proof.* First, note that

$$\frac{1}{D\|U\|} \leq \frac{1}{\sqrt{D_X D_Y}\|U\|} \leq \frac{1}{\|\mathbf{M}_{U,Y,X}\|}.$$

Furthermore,

$$\begin{aligned} & \|\mathbf{M}_{V,Z,F} - \mathbf{M}_{U,Y,X}\| \\ &= \|\mathbf{M}_{V,Z,F} - \mathbf{M}_{U,Z,F} + \mathbf{M}_{U,Z,F} - \mathbf{M}_{U,Z,X} + \mathbf{M}_{U,Z,X} - \mathbf{M}_{U,Y,X}\| \\ &\leq \|T_Z(V - U)T_F^*\| + \|T_Z U(T_F^* - T_X^*)\| + \|(T_Z - T_Y)UT_X^*\| \\ &\leq \sqrt{D_Z D_F}\|V - U\| + \sqrt{D_Z}\|U\|\|T_F^* - T_X^*\| + \sqrt{D_X}\|U\|\|T_Z - T_Y\| \\ &\leq \sqrt{D_Z D_F}\epsilon + \sqrt{D_Z}\|U\|(\sqrt{D_F} + \sqrt{D_X}) + \sqrt{D_X}\|U\|(\sqrt{D_Z} + \sqrt{D_Y}) \\ &\leq D\epsilon + 5D\|U\| \\ &< \frac{1}{\|\mathbf{M}_{U,Y,X}\|}. \end{aligned}$$

Now, Lemma 3.3 implies that  $\mathbf{M}_{V,Z,F}$  is invertible. The second part is obtained by Proposition 3.2.  $\square$

Riesz bases in Hilbert  $C^*$ -modules are much more different than the Hilbert space cases. For instant, as it was proved in [1, Proposition 4.1], the generalized Riesz multiplier  $\mathbf{M}_{U,Y,X}$  is invertible if and only if  $U$  is invertible. However, this result is no longer true for Riesz multipliers in the Hilbert  $C^*$ -modules setting. Consider the following example.

**Example 3.9.** Let  $A = M_{2 \times 2}(\mathbb{C})$  denote the  $C^*$ -algebra of all  $2 \times 2$  complex matrices. Let  $E = A$  and for any  $B, C \in E$  define

$$\langle B, C \rangle = BC^*.$$

Then  $E$  is a Hilbert  $A$ -module. Let  $A_{ij}$  be the  $2 \times 2$  matrix with 1 in the  $ij$ -th entry and 0 elsewhere, where  $1 \leq i, j \leq 2$ . Then  $X = \{A_{21}, A_{12}\}$  and  $Y = \{A_{11}, A_{22}\}$  are Riesz bases for  $E$ . If  $U = I_{\ell^2(A, \mathbb{I})}$ , then  $\mathbf{M}_{U,Y,X} = 0$  and therefore is not invertible, while  $U$  is invertible.

Hence, we can use the concept of modular Riesz bases, which share many properties with Riesz bases in Hilbert spaces. First, we recall the following definition from [18].

**Definition.** Let  $A$  be a unital  $C^*$ -algebra and  $E$  be a finitely or countably generated Hilbert  $A$ -module. A sequence  $\{x_i\}_{i \in \mathbb{I}}$  is a *modular Riesz basis* for  $E$  if there exists an adjointable and invertible operator  $U : \ell^2(A, \mathbb{I}) \rightarrow E$  such that  $U\delta_i = x_i$  for each  $i \in \mathbb{I}$ , where  $\{\delta_i\}_{i \in \mathbb{I}}$  is the standard orthonormal basis of  $\ell^2(A, \mathbb{I})$ .

The next two propositions give some necessary and sufficient conditions for invertibility of generalized multipliers associated to modular Riesz bases.

**Proposition 3.10.** *Let  $U$  be a bounded linear operator on  $\ell^2(A, \mathbb{I})$  and  $X = \{x_i\}_{i \in \mathbb{I}}$  and  $Y = \{y_i\}_{i \in \mathbb{I}}$  be two modular Riesz bases for Hilbert  $A$ -module  $E$ . Then  $U$  is invertible if and only if the generalized multiplier  $\mathbf{M}_{U,Y,X}$  is invertible.*

*Proof.* Assume that  $\tilde{X}$  and  $\tilde{Y}$  are the dual modular Riesz bases of  $X$  and  $Y$ , respectively. First, note that due to [19, Lemma 4.1], we have  $T_X^* T_{\tilde{X}} = I$ . Now, if  $U$  is invertible, then

$$(\mathbf{M}_{U,Y,X}) (\mathbf{M}_{U^{-1},\tilde{X},\tilde{Y}}) = (T_Y U T_X^*) (T_{\tilde{X}} U^{-1} T_{\tilde{Y}}^*) = I,$$

and similarly  $(\mathbf{M}_{U^{-1},\tilde{X},\tilde{Y}}) (\mathbf{M}_{U,Y,X}) = I$ . Therefore,  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},\tilde{X},\tilde{Y}}$ .

Conversely, suppose  $\mathbf{M}_{U,Y,X}$  is an invertible operator. Then

$$\begin{aligned} U (T_X^* \mathbf{M}_{U,Y,X}^{-1} T_Y) &= T_{\tilde{Y}}^* T_Y (U T_X^* \mathbf{M}_{U,Y,X}^{-1}) T_Y \\ &= T_{\tilde{Y}}^* (T_Y U T_X^* \mathbf{M}_{U,Y,X}^{-1}) T_Y \\ &= T_{\tilde{Y}}^* T_Y \\ &= I, \end{aligned}$$

also  $(T_X^* \mathbf{M}_{U,Y,X}^{-1} T_Y) U = I$ . So  $U$  is invertible.  $\square$

**Proposition 3.11.** *Let  $U$  be a bounded invertible operator on  $\ell^2(A, \mathbb{I})$  and  $Y = \{y_i\}_{i \in \mathbb{I}} \subset E$  be a modular Riesz basis. Moreover, let  $X = \{x_i\}_{i \in \mathbb{I}}$  be a standard frame for  $E$ . Then the following assertions are equivalent.*

- (1)  $X$  has a unique dual frame.
- (2)  $\mathbf{M}_{U,Y,X}$  is invertible.

*Proof.* (1) $\Rightarrow$ (2) To obtain the second statement from the first one, suppose that  $X$  has a unique dual frame. Then by [17, Theorem 4.9], the associated analysis operator  $T_X^*$  is surjective. Also by using the reconstruction formula (2.2), we conclude that  $T_X^*$  is injective and so  $T_X^*$  is bijective. Due to the fact that  $T_Y$  and  $U$  are bijective, we deduce  $\mathbf{M}_{U,Y,X}$  is invertible.

(2) $\Rightarrow$ (1) Now, to drive the first statement from the second one, we assume  $\mathbf{M}_{U,Y,X}$  is invertible. Then  $T_X^*$  is surjective and so by [17, Theorem 4.9],  $X$  has a unique dual frame.  $\square$

#### 4. Representation of the inverse of a multiplier

As we have seen in Proposition 3.10, for modular Riesz bases  $X = \{x_i\}_{i \in \mathbb{I}}$  and  $Y = \{y_i\}_{i \in \mathbb{I}}$ , if  $U \in L(\ell^2(A, \mathbb{I}))$  is invertible, then the generalized multiplier  $\mathbf{M}_{U,Y,X}$  is automatically invertible and vice versa. Moreover,

$$(4.1) \quad \mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},\tilde{X},\tilde{Y}}.$$

This result motivates us to generalize this idea for frames and even non-Bessel sequences. In more details, we will show that there are other invertible frame multipliers  $\mathbf{M}_{U,Y,X}$  whose inverses can be represented as multipliers using the inverted symbol and suitable dual frames of  $X$  and  $Y$ .

The following proposition gives a representation of the inverse of an invertible frame multiplier with an invertible symbol.

**Proposition 4.1.** *Let  $X = \{x_i\}_{i \in \mathbb{I}}$  and  $Y = \{y_i\}_{i \in \mathbb{I}}$  be two standard frames for Hilbert  $A$ -module  $E$  and  $U$  be an invertible operator on  $\ell^2(A, \mathbb{I})$ . Assume that  $\mathbf{M}_{U,Y,X}$  is invertible. Then the following hold.*

- (1) *There exists a dual frame  $Y^\dagger$  of  $Y$  such that for any dual frame  $X^d$  of  $X$  we have*

$$(4.2) \quad \mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},X^d,Y^\dagger}.$$

- (2) *There exists a dual frame  $X^\dagger$  of  $X$  such that for any dual frame  $Y^d$  of  $Y$  we have*

$$(4.3) \quad \mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},X^\dagger,Y^d}.$$

- (3) *If  $F = \{f_i\}_{i \in \mathbb{I}}$  is a Bessel sequence in  $E$  such that  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},X^\dagger,F}$  (resp.  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},F,Y^\dagger}$ ), then  $F$  must be a dual of  $Y$  (resp.  $X$ ).*

*Proof.* (1) Denote  $\mathbf{M} = \mathbf{M}_{U,Y,X}$ . Let  $\{\delta_i\}_{i \in \mathbb{I}}$  be the standard orthonormal basis of  $\ell^2(A, \mathbb{I})$ . The sequence  $\{(\mathbf{M}^{-1})^* T_X U^* \delta_i\}_{i \in \mathbb{I}}$  is a dual of  $Y$ , since

$$\begin{aligned} \sum_{i \in \mathbb{I}} \langle x, (\mathbf{M}^{-1})^* T_X U^* \delta_i \rangle y_i &= \sum_{i \in \mathbb{I}} \langle U T_X^* \mathbf{M}^{-1}(x), \delta_i \rangle y_i \\ &= T_Y U T_X^* \mathbf{M}^{-1}(x) \\ &= \mathbf{M} \mathbf{M}^{-1}(x) = x. \end{aligned}$$

Put  $Y^\dagger = \{(\mathbf{M}^{-1})^* T_X U^* \delta_i\}_{i \in \mathbb{I}}$ . We have

$$(\mathbf{M}^{-1})^* T_X (U^* \delta_i) = T_{Y^\dagger} (U^{-1})^* (U^* \delta_i).$$

By the boundedness of operators and surjectivity of  $U$ , we conclude that

$$(4.4) \quad (\mathbf{M}^{-1})^* T_X = T_{Y^\dagger} (U^{-1})^*.$$

Using any dual  $X^d$  of  $X$ , we obtain  $(\mathbf{M}^{-1})^* = T_{Y^\dagger} (U^{-1})^* T_{X^d}^*$  and hence  $\mathbf{M}^{-1} = T_{X^d} U^{-1} T_{Y^\dagger}^* = \mathbf{M}_{U^{-1},X^d,Y^\dagger}$ .

- (2) The sequence  $\{\mathbf{M}^{-1} T_Y U \delta_i\}_{i \in \mathbb{I}}$  is a dual of  $X$ , because

$$\begin{aligned} \sum_{i \in \mathbb{I}} \langle x, x_i \rangle \mathbf{M}^{-1} T_Y U \delta_i &= \mathbf{M}^{-1} T_Y U \left( \sum_{i \in \mathbb{I}} \langle x, x_i \rangle \delta_i \right) \\ &= \mathbf{M}^{-1} T_Y U T_X^*(x) \\ &= \mathbf{M}^{-1} \mathbf{M}(x) = x. \end{aligned}$$

Put  $X^\dagger = \{\mathbf{M}^{-1}T_Y U\delta_i\}_{i \in \mathbb{I}}$ . Now, we have:

$$\mathbf{M}^{-1}T_Y(U\delta_i) = T_{X^\dagger}U^{-1}(U\delta_i).$$

The boundedness of operators and surjectivity of  $U$  imply that

$$(4.5) \quad \mathbf{M}^{-1}T_Y = T_{X^\dagger}U^{-1}.$$

Using any dual frame  $Y^d$  of  $Y$ , we get  $\mathbf{M}^{-1} = T_{X^\dagger}U^{-1}T_{Y^d}^* = \mathbf{M}_{U^{-1},X^\dagger,Y^d}$ .

(3) Let  $F = \{f_i\}_{i \in \mathbb{I}}$  be a Bessel sequence in  $E$  such that  $\mathbf{M}^{-1} = \mathbf{M}_{U^{-1},X^\dagger,F}$ . Then, by (4.5),

$$T_Y T_F^* = \mathbf{M} T_{X^\dagger} U^{-1} T_F^* = \mathbf{M} \mathbf{M}^{-1} = I,$$

which implies that  $F$  is a dual frame of  $Y$ . In a similar way, every Bessel sequence  $F$  in  $E$  which satisfies  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},F,Y^\dagger}$  must be a dual of  $X$ .  $\square$

*Remark 4.2.* It is worth mentioning that in Proposition 4.1, if  $E = \ell^2(A, \mathbb{I})$ , one can show that  $X^\dagger$  and  $Y^\dagger$  are unique. For instance, assume that there exist dual frames  $Y^\dagger$  and  $Y^\ddagger$  of  $Y$  such that for any dual frame  $X^d$  of  $X$ ,

$$(4.6) \quad \mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},X^d,Y^\dagger} = \mathbf{M}_{U^{-1},X^d,Y^\ddagger}.$$

We show that  $Y^\dagger = Y^\ddagger$ . According to (4.6), for every  $a \in E$  and dual frame  $X^d$  of  $X$ ,

$$(4.7) \quad (\mathbf{M}_{U^{-1},X^d,Y^\dagger} - \mathbf{M}_{U^{-1},X^d,Y^\ddagger})(a) = (T_{Y^\dagger - Y^\ddagger} U^{-1} T_{X^d}^*)(a) = 0.$$

On the other hand, as a consequence of [2, Theorem 3.4], the dual frames of  $X$  are characterized as the sequences  $\left\{ \tilde{x}_i + h_i - \sum_{j \in \mathbb{I}} \langle \tilde{x}_i, x_j \rangle h_j \right\}_{i \in \mathbb{I}}$ , where  $H = \{h_i\}_{i \in \mathbb{I}}$  runs through the Bessel sequences in  $E$ . Using this characterization, for every  $a \in E$  we have

$$\begin{aligned} & T_{Y^\dagger - Y^\ddagger} U^{-1} T_{\{\tilde{x}_i + h_i - \sum_{j \in \mathbb{I}} \langle \tilde{x}_i, x_j \rangle h_j\}}^*(a) \\ &= T_{Y^\dagger - Y^\ddagger} U^{-1} T_X^*(a) + T_{Y^\dagger - Y^\ddagger} U^{-1} T_{\{h_i - \sum_{j \in \mathbb{I}} \langle \tilde{x}_i, x_j \rangle h_j\}}^*(a) = 0 \end{aligned}$$

for every Bessel sequence  $H = \{h_i\}_{i \in \mathbb{I}}$  in  $E$ . By (4.7),  $T_{Y^\dagger - Y^\ddagger} U^{-1} T_X^*(a) = 0$  and so

$$(4.8) \quad \begin{aligned} & T_{Y^\dagger - Y^\ddagger} U^{-1} T_{\{h_i - \sum_{j \in \mathbb{I}} \langle \tilde{x}_i, x_j \rangle h_j\}}^*(a) \\ &= T_{Y^\dagger - Y^\ddagger} U^{-1} T_H^*(a) - T_{Y^\dagger - Y^\ddagger} U^{-1} T_{\{\sum_{j \in \mathbb{I}} \langle \tilde{x}_i, x_j \rangle h_j\}}^*(a) = 0 \end{aligned}$$

for every Bessel sequence  $H = \{h_i\}_{i \in \mathbb{I}}$  in  $E$ . Apply (4.8) for the Bessel sequence  $H_i = \{0, \dots, 0, \delta_i, 0, \dots\}$ , we obtain

$$(4.9) \quad T_{Y^\dagger - Y^\ddagger} U^{-1} T_{H_i}^*(a) - T_{Y^\dagger - Y^\ddagger} U^{-1} T_{\{\langle \tilde{x}_i, x_i \rangle \delta_i\}}^*(a) = 0.$$

Using (4.7), it is concluded that

$$T_{Y^\dagger - Y^\ddagger} U^{-1} T_{\{\langle \tilde{x}_i, x_i \rangle \delta_i\}}^*(a) = \langle a, \delta_i \rangle T_{Y^\dagger - Y^\ddagger} U^{-1} T_X^*(x_i) = 0.$$

Therefore,  $T_{Y^\dagger - Y^\ddagger} U^{-1} T_{H_i}^*(a) = 0$  for every  $a \in E$ . By choosing  $a = \delta_i$ , we have  $T_{Y^\dagger - Y^\ddagger} U^{-1}(\{0, \dots, 0, 1_A, 0, \dots\}) = 0$ , with the unit being the  $i$ -th entry.

Since  $i \in \mathbb{I}$  is chosen arbitrary and  $U$  is invertible, it follows that  $T_{Y^\dagger - Y^\ddagger}$  is a null-operator on  $\ell^2(A, \mathbb{I})$ . So  $Y^\dagger - Y^\ddagger = 0$  and therefore  $Y^\dagger = Y^\ddagger$ .

*Remark 4.3.* Recall that two frames  $X$  and  $Y$  are called *equivalent* if there exists an invertible operator  $W : E \rightarrow E$  so that  $x_i = Wy_i$  for all  $i \in \mathbb{I}$ . Due to [14, Theorem 6.1], when  $Y$  is a frame for  $E$ , then a dual frame  $Y^d$  of  $Y$  is equivalent to  $Y$  if and only if  $Y^d = \tilde{Y}$ . Now, regarding Proposition 4.1, it is natural to ask whether the frame  $X^\dagger$  (resp.  $Y^\dagger$ ) is the canonical dual of  $X$  (resp.  $Y$ ). It is easy to check that  $X^\dagger = \tilde{X}$  (resp.  $Y^\dagger = \tilde{Y}$ ) if and only if  $X$  is equivalent to  $\{T_Y U \delta_i\}_{i \in \mathbb{I}}$  (resp.  $Y$  is equivalent to  $\{T_X U^* \delta_i\}_{i \in \mathbb{I}}$ ). In this case,  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},\tilde{X},Y^d}$  (resp.  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},X^d,\tilde{Y}}$ ).

For the more general case of invertible symbols, we have the following result.

**Proposition 4.4.** *Let  $X = \{x_i\}_{i \in \mathbb{I}}$  and  $Y = \{y_i\}_{i \in \mathbb{I}}$  be standard frames for Hilbert  $A$ -module  $E$  and  $U \in L(\ell^2(A, \mathbb{I}))$  be invertible. If  $\mathbf{M}_{U,Y,X}$  is invertible, then there exists a bounded operator  $\Gamma_{U,Y,X} : E \rightarrow \ell^2(A, \mathbb{I})$  such that*

$$(4.10) \quad \mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},\tilde{X},Y^d} + (\Gamma_{U,Y,X})^* T_{Y^d}^*$$

for all dual frames  $Y^d = \{y_i^d\}_{i \in \mathbb{I}}$  of  $Y$ .

*Proof.* Define  $\Gamma_{U,Y,X} : E \rightarrow \ell^2(A, \mathbb{I})$  by

$$(4.11) \quad \Gamma_{U,Y,X}(x) := T_Y^* \left( \mathbf{M}_{U,Y,X}^{-1} \right)^* (x) - (U^{-1})^* T_X^* S_X^{-1}(x), \quad (x \in E).$$

Then, the operator  $\Gamma_{U,Y,X}$  is bounded and

$$\mathbf{M}_{U,Y,X}^{-1} T_Y = S_X^{-1} T_X U^{-1} + (\Gamma_{U,Y,X})^*.$$

Using any dual frame  $Y^d$  of  $Y$ , we get

$$(4.12) \quad \mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},\tilde{X},Y^d} + (\Gamma_{U,Y,X})^* T_{Y^d}^*. \quad \square$$

*Remark 4.5.* Using similar arguments like the above ones can show that if  $\mathbf{M}_{U,Y,X}$  is invertible, then there exists a bounded operator  $\Gamma_{U,Y,X} : E \rightarrow \ell^2(A, \mathbb{I})$  such that

$$\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1},X^d,\tilde{Y}} + T_{X^d} \Gamma_{U,Y,X}.$$

In fact, it is enough to set  $\Gamma_{U,Y,X} = T_X^* \mathbf{M}_{U,Y,X}^{-1} - U^{-1} T_Y^* S_Y^{-1}$ . Moreover, if  $E = \ell^2(A, \mathbb{I})$ , then the operator  $\Gamma_{U,Y,X}$  can be obtained uniquely. In fact, suppose on the contrary that the equation (4.12) holds for two operators  $\Gamma_1$  and  $\Gamma_2$ . It follows that

$$(4.13) \quad (\Gamma_1 - \Gamma_2)^* T_{Y^d}^* = 0$$

for all duals  $Y^d$  of  $Y$ . As is mentioned the dual frames of  $Y$  are characterized as the sequences

$$\left\{ \tilde{y}_i + h_i - \sum_{j \in \mathbb{I}} \langle \tilde{y}_i, y_j \rangle h_j \right\}_{i \in \mathbb{I}},$$

where  $\{h_i\}_{i \in \mathbb{I}}$  runs through the Bessel sequences in  $E$ . So

$$(\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2)^* T_{\{\tilde{y}_i + h_i - \sum_{j \in \mathbb{I}} \langle \tilde{y}_i, y_j \rangle h_j\}_{i \in \mathbb{I}}}^* = (\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2)^* \left( T_{\tilde{Y}}^* + T_{\{h_i - \sum_{j \in \mathbb{I}} \langle \tilde{y}_i, y_j \rangle h_j\}}^* \right) = 0.$$

Using (4.13) with  $\tilde{Y}$  in the role of  $Y^d$ , it then follows that

$$(\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2)^* T_{\{h_i - \sum_{j \in \mathbb{I}} \langle \tilde{y}_i, y_j \rangle h_j\}}^* = 0.$$

Take  $\{h_i\}_{i \in \mathbb{I}} = \{\delta_1, 0, 0, \dots\}$ , where  $\{\delta_i\}_{i \in \mathbb{I}}$  is the canonical orthonormal basis for  $E = \ell^2(A, \mathbb{I})$ . Then, for every  $a \in E$ ,

$$\langle a, \delta_1 \rangle (\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2)^* (\delta_1 - \{\langle y_1, \tilde{y}_i \rangle\}_{i \in \mathbb{I}}) = 0.$$

Using (4.13) it is concluded that  $(\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2)^* (\{\langle y_1, \tilde{y}_i \rangle\}_{i \in \mathbb{I}}) = 0$  and hence

$$\langle a, \delta_1 \rangle (\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2)^* (\delta_1) = 0.$$

Choosing  $a = \delta_1$ , one comes to the conclusion that  $(\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2)^* (\delta_1) = 0$ . In a similar way, taking  $\{h_i\}_{i \in \mathbb{I}} = \{0, 0, \dots, 0, \delta_i, 0, \dots\}$ , with  $\delta_i$  being at the  $i$ -th place,  $i \geq 2$ , we get  $(\mathbf{\Gamma}_1 - \mathbf{\Gamma}_2)^* (\delta_i) = 0$  for all  $i \in \mathbb{I}$ . Therefore,  $\mathbf{\Gamma}_1 = \mathbf{\Gamma}_2$ .

The next proposition determines a class of multipliers which are invertible and whose inverses can be written as a multiplier. While in Proposition 4.1, it is assumed that the frame multiplier is invertible, in the following we investigate a sufficient condition for invertibility of frame multipliers. Moreover, in the special case  $E = \ell^2(A, \mathbb{I})$ , we can find an equivalent condition for invertibility of frame multipliers.

**Proposition 4.6.** *Let  $X = \{x_i\}_{i \in \mathbb{I}}$  and  $Y = \{y_i\}_{i \in \mathbb{I}}$  be standard frames for Hilbert  $A$ -module  $E$  and  $U \in L(\ell^2(A, \mathbb{I}))$  be invertible. If  $X$  is equivalent to  $\{T_Y U \delta_i\}_{i \in \mathbb{I}}$  (resp.  $Y$  is equivalent to  $\{T_X U^* \delta_i\}_{i \in \mathbb{I}}$ ), then  $\mathbf{M}_{U,Y,X}$  is invertible and  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1}, \tilde{X}, Y^d}$  (resp.  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1}, X^d, \tilde{Y}}$ ) for all dual frame  $Y^d$  (resp.  $X^d$ ) of  $Y$  (resp.  $X$ ). Furthermore, the proposition also holds in the opposite side if  $E = \ell^2(A, \mathbb{I})$ .*

*Proof.* Suppose that  $X$  is equivalent to  $\{T_Y U \delta_i\}_{i \in \mathbb{I}}$ . So there exists an invertible operator  $W$  in  $L(E)$  such that  $T_Y U \delta_i = W x_i$  for every  $i \in \mathbb{I}$ . Since

$$\mathbf{M}_{U,Y,X}(x) = T_Y U T_X^*(x) = \sum_{i \in \mathbb{I}} \langle x, x_i \rangle T_Y U \delta_i = \sum_{i \in \mathbb{I}} \langle x, x_i \rangle W x_i = W S_X(x),$$

so  $\mathbf{M}_{U,Y,X}$  is invertible. Moreover, by the fact that  $T_X U^{-1} = W^{-1} T_Y = T_{W^{-1}Y}$ , we have

$$\mathbf{M}_{U^{-1}, \tilde{X}, Y^d} = T_{\tilde{X}} U^{-1} T_{Y^d}^* = S_X^{-1} T_X U^{-1} T_{Y^d}^* = S_X^{-1} T_{W^{-1}Y} T_{Y^d}^* = (W S_X)^{-1}.$$

Conversely, for  $E = \ell^2(A, \mathbb{I})$ , if  $\mathbf{M}_{U,Y,X}$  is invertible and  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1}, \tilde{X}, Y^d}$  for all dual frame  $Y^d$  of  $Y$ , then Proposition 4.4 implies that  $(\mathbf{\Gamma}_{U,Y,X})^* T_{Y^d}^* = 0$ . Now, with an argument similar to Remark 4.5, one can conclude that  $\mathbf{\Gamma}_{U,Y,X} = 0$ . Therefore, by the equation (4.11), we have

$$U^* T_Y^* = T_X^* S_X^{-1} \mathbf{M}_{U,Y,X}^*.$$

Hence, by taking the adjoint, we obtain

$$T_Y U \delta_i = \mathbf{M}_{U,Y,X} S_X^{-1} T_X \delta_i = \mathbf{M}_{U,Y,X} S_X^{-1} x_i.$$

The second part is similar.  $\square$

Now, we extend the results of Proposition 4.1 to non-Bessel sequences. First, consider the following definition.

**Definition.** Let  $X = \{x_i\}_{i \in \mathbb{I}}$  be a standard frame for Hilbert  $A$ -module  $E$ . The sequence  $Y = \{y_i\}_{i \in \mathbb{I}}$  with elements from  $E$  is called

- (i) an *analysis pseudo-dual* (in short, *a-pseudo-dual*) of  $X$ , if for every  $x \in E$ ,

$$(4.14) \quad x = \sum_{i \in \mathbb{I}} \langle x, y_i \rangle x_i;$$

- (ii) a *synthesis pseudo-dual* (in short, *s-pseudo-dual*) of  $X$ , if for every  $x \in E$ ,

$$(4.15) \quad x = \sum_{i \in \mathbb{I}} \langle x, x_i \rangle y_i.$$

*Remark 4.7.* In [17, Proposition 3.9], it was shown that if  $X = \{x_i\}_{i \in \mathbb{I}}$  and  $Y = \{y_i\}_{i \in \mathbb{I}}$  are two standard Bessel sequences for Hilbert  $A$ -module  $E$  and for every  $x \in E$ ,  $x = \sum_{i \in \mathbb{I}} \langle x, y_i \rangle x_i$ , then both  $X$  and  $Y$  are standard frames of  $E$  and  $x = \sum_{i \in \mathbb{I}} \langle x, x_i \rangle y_i$  holds for any  $x \in E$ .

Consider a sequence  $X = \{x_i\}_{i \in \mathbb{I}}$  in a Hilbert  $A$ -module  $E$  and define the possibly unbounded synthesis operator  $T_X : \text{Dom}(T_X) \subseteq \ell^2(A, \mathbb{I}) \rightarrow E$  by

$$T_X \{a_i\}_{i \in \mathbb{I}} = \sum_{i \in \mathbb{I}} a_i x_i, \quad \text{Dom}(T_X) := \left\{ \{a_i\}_{i \in \mathbb{I}} \in \ell^2(A, \mathbb{I}) \mid \sum_{i \in \mathbb{I}} a_i x_i \text{ is convergent} \right\}.$$

Note that the finite sequences are dense in  $\ell^2(A, \mathbb{I})$  and contained in  $\text{Dom}(T_X)$ . Thus, the operator  $T_X$  is automatically densely defined.

The next propositions determine how to represent the inverse of an invertible generalized multiplier for non-Bessel sequences and invertible symbol. In following, the synthesis operators are assumed to be closed operators. Moreover,  $\{\delta_i\}_{i \in \mathbb{I}}$  is the standard orthonormal basis of  $\ell^2(A, \mathbb{I})$ .

**Proposition 4.8.** Let  $X = \{x_i\}_{i \in \mathbb{I}}$  be a standard frame for Hilbert  $A$ -module  $E$ ,  $Y = \{y_i\}_{i \in \mathbb{I}}$  be a sequence with elements from  $E$  and  $U \in L(\ell^2(A, \mathbb{I}))$  be invertible such that  $U \delta_i \in \text{Dom}(T_Y)$  for every  $i \in \mathbb{I}$ . Moreover, assume that  $\mathbf{M}_{U,Y,X}$  is invertible. Then, there exists a dual frame  $X^\dagger$  of  $X$  such that for any *a-pseudo-duals*  $Y^{ad}$  of  $Y$ ,  $\mathbf{M}_{U,Y,X}^{-1} = \mathbf{M}_{U^{-1}, X^\dagger, Y^{ad}}$ .



*Proof.* Denote  $\mathbf{M}_{U,Y,X} = \mathbf{M}$ . Consider the sequence  $X^\dagger = \{\mathbf{M}^{-1}T_Y U \delta_i\}_{i \in \mathbb{I}}$ . Note that for every  $a \in \ell^2(A, \mathbb{I})$ , we have

$$\begin{aligned} T_{X^\dagger} U^{-1} a &= \sum_{i \in \mathbb{I}} \langle U^{-1} a, \delta_i \rangle \mathbf{M}^{-1} T_Y U \delta_i \\ &= \mathbf{M}^{-1} T_Y U \sum_{i \in \mathbb{I}} \langle U^{-1} a, \delta_i \rangle \delta_i \\ &= \mathbf{M}^{-1} T_Y (a). \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{M}_{U^{-1}, X^\dagger, Y^{ad}}(x) &= T_{X^\dagger} U^{-1} T_{Y^{ad}}^*(x) \\ &= \mathbf{M}^{-1} T_Y \{ \langle x, y_i^{ad} \rangle \}_{i \in \mathbb{I}} \\ &= \mathbf{M}^{-1} \left( \sum_{i \in \mathbb{I}} \langle x, y_i^{ad} \rangle y_i \right) = \mathbf{M}^{-1}(x). \end{aligned} \quad \square$$

**Proposition 4.9.** *Let  $Y = \{y_i\}_{i \in \mathbb{I}}$  be a standard frame for Hilbert  $A$ -module  $E$ ,  $X = \{x_i\}_{i \in \mathbb{I}}$  be a sequence with elements from  $E$  and  $U \in L(\ell^2(A, \mathbb{I}))$  be invertible such that  $U^* \delta_i \in \text{Dom}(T_X)$  for every  $i \in \mathbb{I}$ . Moreover, assume that  $\mathbf{M}_{U,Y,X}$  is invertible. Then, there exists a dual frame  $Y^\dagger$  of  $Y$  such that for any  $s$ -pseudo-duals  $X^{sd}$  of  $X$ ,  $\mathbf{M}_{U^{-1}, X^{sd}, Y^\dagger} = \mathbf{M}_{U^{-1}, X^{sd}, Y^\dagger}$ .*

*Proof.* Put  $Y^\dagger = \{(\mathbf{M}^{-1})^* T_X U^* \delta_i\}_{i \in \mathbb{I}}$ . Then, for every  $x \in E$

$$\begin{aligned} (U^{-1} T_{Y^\dagger}^*(x))_i &= \left\langle U^{-1} \left\{ \langle x, (\mathbf{M}^{-1})^* T_X U^* \delta_i \rangle \right\}, \delta_i \right\rangle \\ &= \sum_{i \in \mathbb{I}} \langle U T_X^* \mathbf{M}^{-1}(x), \delta_i \rangle \langle \delta_i, (U^{-1})^* \delta_i \rangle \\ &= \left\langle U T_X^* \mathbf{M}^{-1}(x), (U^{-1})^* \delta_i \right\rangle \\ &= \langle \mathbf{M}^{-1}(x), T_X \delta_i \rangle = \langle x, (\mathbf{M}^{-1})^* x_i \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{M}_{U^{-1}, X^{sd}, Y^\dagger}(x) &= T_{X^{sd}} U^{-1} T_{Y^\dagger}^*(x) \\ &= T_{X^{sd}} \left\{ \langle x, (\mathbf{M}^{-1})^* x_i \rangle \right\}_{i \in \mathbb{I}} \\ &= \sum_{i \in \mathbb{I}} \langle \mathbf{M}^{-1}(x), x_i \rangle x_i^{sd} = \mathbf{M}^{-1}(x). \end{aligned} \quad \square$$

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GHOLAMREZA ABBASPOUR TABADKAN  
SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE  
DAMGHAN UNIVERSITY  
DAMGHAN, IRAN  
Email address: [abbaspour@du.ac.ir](mailto:abbaspour@du.ac.ir)

HESSAM HOSSEINNEZHAD  
SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE  
DAMGHAN UNIVERSITY  
DAMGHAN, IRAN  
Email address: [hosseinnezhad.h@yahoo.com](mailto:hosseinnezhad.h@yahoo.com)