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RANDOM CHORD IN A CIRCLE AND BERTRAND'S PARADOX: NEW GENERATION METHOD, EXTREME BEHAVIOUR AND LENGTH MOMENTS

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Abstract. In this paper a new generating procedure of a random chord is presented. This problem has its roots in the Bertrand's paradox. A study of the limit behaviour of its maximum length and the rate of convergence is conducted. In addition, moments of record values of random chord length are obtained for this case, as well as other cases of solutions of Bertrand's paradox.

1. Introduction

Let $\{X_n\}$ be a sequence of independent and identically distributed (iid) random variables with random chord length cumulative distribution function (cdf) presented as in [7]. The cdf $F(x)$ for this sequence depends upon the procedure of choosing a random chord on a circle. This problem has its beginning in the Bertrand's paradox. This paradox was developed as a simple question that raised doubt on the principle of indifference for cases with infinitely many possibilities at hand, see [17]. The question was: "What is the probability that a chord selected "at random" in a circle is larger than a side of the inscribed equilateral triangle?"

In [5], Bertrand himself obtained probabilities 1/3, 1/2 and 1/4 by different random chord generation procedures: by choosing a chord with one end at a vertex of the inscribed equilateral triangle in a circle; by choosing a chord perpendicular to the diameter which is the right bisector of the equilateral triangle; and selecting a point inside a circle and denoting it as a chord midpoint, respectively. More details related to this problem could be found in many subsequent papers, see e.g. [1, 4, 10–12, 21–23]. According to [1], when the sample space and probability measure are uniquely defined, this paradox becomes a solvable problem. Bertrand's second solution was considered by Poincaré as a natural choice based on the fact that the right Haar measure is invariant under rigid transformation group, see e.g. [13, p. 16].

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The limiting distribution of the maximum of the sequence $\{X_i\}$, for various choices of the chord generation procedures, was studied in [24]. Comparing to the non-randomness property of the solution found in [23], in this paper we provided a new generating method of random chords that naturally excludes this restriction. Also, we extended previous results on asymptotic behaviour of the maximum and added new results concerning records of random chord lengths.

Record values were introduced by Chandler [6] as a model of successful extremes. If $\{T(n),\, n \geq 1\}$ is defined by $T(1) = 1$, $T(n) = \min\{j \,|\, j > T(n-1), X_j > X_{T(n-1)}\}$ for $n \geq 2$, then $\{R_n,\, n \geq 1\} = \{X_{T(n)},\, n \geq 1\}$ is said to be a sequence of record values. The sequence $\{T(n),\, n \geq 1\}$ is called the sequence of record times. An analogous definition can be given for lower record values. We refer to [2, 3, 18], and reference therein, for more information about record values. The pdf of R_n, $n \geq 1$, is given by

$$f_{R_n}(x) = \frac{1}{\Gamma(n)}\{-\log(1 - F(x))\}^{n-1} f(x),\ -\infty < x < \infty. \tag{1}$$

Resnick in [20] obtained three possible limit distributions for record values, and a theorem that presents a direct connection between non-degenerate limit laws for sample maxima and record values, known as the Duality theorem.

The rest of the paper is organized as follows. In Section 2, we propose a new procedure for generating a random chord in a circle and examine the limiting properties of its maximum length. In Section 3 we study the record chord length distributions, for various generating procedures, in terms of the asymptotic behaviour and their lth moments.

2. New solution

In [23], a new continuous family of planar probabilistic models for Bertrand's paradox is introduced. This family contains classical models for generating random chords as its limits. The chord constructing model consists of fixing the thrower at a distance $h > 1$ from a unit circle and constructing lines that intersect the circle with the origin at the thrower. However, this model requires to "manually" manipulate the distance h. This contradicts the randomness selection property of h and by that this model of constructing chords is incomplete with respect to Bertrand's paradox. This paradox is complete if the randomness selection property of chords is fully satisfied. With this idea in mind, we now present a new solution where the choice of the chords is fully random.

The solution is obtained as follows.

Step 1. Select at random an angle θ, $\theta \sim U(0, \pi)$, formed by two tangents of the circle, say t and s, and denote A as its vertex;

Step 2. Choose at random an angle ϕ, $\phi \sim U(0, \theta)$, that lies on one tangent, say s, and A as its vertex;

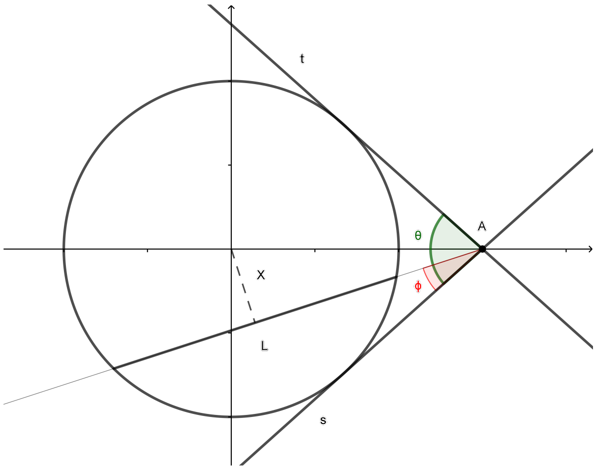

FIGURE 1. New random chord generation procedure.

Step 3. Select a semi-line which is directed by an angle ϕ, with A as its starting point. A chord is formed by its intersection with the circle (Figure 1).

Let X be the distance from the center of the circle and the chord, let L be the corresponding chord length and let h be the distance between the point A and the center of the circle. There are two cases: $\phi \in (0, \frac{\theta}{2})$ and $\phi \in (\frac{\theta}{2}, \theta)$. Because they are symmetric, we can consider only the case $\phi \in (0, \frac{\theta}{2})$. It then follows that $X = \sqrt{1 - \frac{L^2}{4}}$, $\sin(\frac{\theta}{2} - \phi) = \frac{X}{h}$ and $\sin\frac{\theta}{2} = \frac{1}{h}$. Combining them we get

$$\phi = \frac{\theta}{2} - \arcsin\left(\sqrt{1 - \frac{L^2}{4}} \sin\frac{\theta}{2}\right). \tag{2}$$

Transforming (2), the cdf of L can be found as

$$\begin{aligned} F(l) &= 2\int_0^{\pi}\int_0^{l} \frac{x \sin\frac{\theta}{2}}{4\theta\sqrt{1 - \frac{x^2}{4}}\sqrt{1 - (1 - \frac{x^2}{4})\sin^2\frac{\theta}{2}}}\,dx\,d\theta \\ &= \int_0^{\pi} \frac{t - 2\operatorname{arccsc}\left(\frac{2\csc(t/2)}{\sqrt{4 - l^2}}\right)}{t\pi}\,dt. \end{aligned} \tag{3}$$

This integral can't be obtained explicitly, so we can only provide numerical solutions. For the Bertrand's case $l = \sqrt{3}$ we have

$$P\left\{L > \sqrt{3}\right\} = 1 - F_L(\sqrt{3}) = 0.4454, \tag{4}$$

With Monte Carlo simulations, with n =100, 500, 1000 and 10000 replications, we can confirm our theoretical results. The simulation results are presented in Table 1.

TABLE 1. Monte Carlo simulations for $P\{L > \sqrt{3}\}$.

n	100	500	1000	10000
Estimated prob.	0.4455	0.4457	0.4452	0.4455

2.1. Maximal chord length convergence

We can now find the limiting behaviour of the maximum of the chord length.

Theorem 2.1. *Let $\{X_n\}$ be iid with cdf (3) and let $M_n = \max_{1\le i\le n} X_i$. Then, the normalized maximum $\frac{M_n-b_n}{a_n}$ converges in distribution to the Weibull distribution with the shape parameter $\frac{1}{2}$, i.e., $P\{\frac{M_n-b_n}{a_n} \le x\} \to e^{-(-x)^{\frac{1}{2}}} I\{x < 0\}$. The normalizing constants are $a_n = \frac{\pi^2}{4C^2n^2}$, where $C = \int_0^\pi \frac{1}{t\csc\frac{t}{2}}\,dt$, and $b_n = 2$ for $n \ge 1$. The rate of convergence is $\mathcal{O}(\frac{1}{n})$.*

Proof. The limiting Weibull distribution and its shape parameter $\alpha = 1/2$ follow from [15, Theorem 1.6.1]. Let $a_n > 0$ and b_n be such that $P\{\frac{M_n-b_n}{a_n} \le x\} \underset{n\to\infty}{\to} e^{-(-x)^{1/2}}$ for $x < 0$. This is equivalent to

$$(F_L(a_nx + b_n))^n \underset{n\to\infty}{\to} e^{-(-x)^{1/2}}, \tag{5}$$

which can be stated as

$$\ln F_L(a_nx + b_n) = -\frac{(-x)^{1/2}}{n} + o\left(\frac{1}{n}\right),\ n \to \infty, \tag{6}$$

for $x < 0$. Denote $\{u_n\} = \{a_nx + b_n\}$ and the sequence $\{f_n(t)\}_{n\ge1}$ as

$$f_n(t) = \frac{1 - \frac{2}{t}\operatorname{arccsc}\left(\frac{2\csc(\frac{t}{2})}{\sqrt{4-u_n^2}}\right)}{\pi},\ n \ge 1. \tag{7}$$

It is evident that

$$|f_n(t)| \le 1 := g(t) \tag{8}$$

for all $t \in (0, \pi)$ and $n \ge 1$. Moreover, $\int_0^\pi g(t)\,dt = \pi < \infty$. The sequence $\{f_n\}$ converges pointwise as $n \to \infty$ ($u_n \to 2$ as $n \to \infty$). Therefore, all conditions for implementing the theorem of dominant convergence (TDC) are satisfied. Implementing TDC and using the relations $\operatorname{arccsc}(x) = \frac{1}{x} + o\left(\frac{1}{x}\right)$ as $x \to \infty$ and $e^x = 1 + x + o(x)$ as $x \to 0$ we get

$$-\ln\pi + \ln\int_0^\pi \left(1 - 2\sqrt{1 - \left(\frac{u_n}{2}\right)^2}\frac{1}{t\csc\frac{t}{2}}\right)dt = -\frac{(-x)^{1/2}}{n} + o\left(\frac{1}{n}\right)$$

for $n \to \infty$ and $x < 0$. Denote $C = \int_0^\pi \frac{1}{t \csc \frac{t}{2}}\, dt$. Further, we have

$$\ln\left(1 - \frac{2C}{\pi}\sqrt{1 - \left(\frac{u_n}{2}\right)^2}\right) = -\frac{(-x)^{1/2}}{n} + o\left(\frac{1}{n}\right),$$

i.e.,

$$\frac{2C}{\pi}\sqrt{1 - \left(\frac{u_n}{2}\right)^2} = -\frac{(-x)^{1/2}}{n} + o\left(\frac{1}{n}\right) \tag{9}$$

for $n \to \infty$ and $x < 0$. Hence, the asymptotic normalizing constants are $a_n = \frac{\pi^2}{4C^2 n^2}$ and $b_n = 2$ for $n \geq 1$. The above procedure was motivated by [15, Corollary 1.6.3].

Denote $\tau_n = n(1 - F_L(u_n))$ and $\tau = (-x)^{1/2}$. It follows that $\tau - \tau_n \sim \frac{\pi^2}{32C^2 n^2}(-x)^{3/2}$ and, according to [15, Theorem 2.4.2], the convergence speed is $\mathcal{O}(\frac{1}{n})$. □

We conducted a comparative simulation study in [19] to illustrate the rate of convergence of the limiting maxima for the chord length with cdf's presented in [24], listed below, together with cdf (3).

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{2}{\pi}\arcsin\frac{x}{2}, & 0 \leq x < 2 \\ 1, & x \geq 2, \end{cases} \tag{10}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \leq x < 2 \\ 1, & x \geq 2, \end{cases} \tag{11}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - \frac{\sqrt{4 - x^2}}{2}, & 0 \leq x < 2 \\ 1, & x \geq 2, \end{cases} \tag{12}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{2}{\pi}\left(\arcsin\frac{x}{2} - \frac{x\sqrt{4 - x^2}}{4}\right), & 0 \leq x < 2 \\ 1, & x \geq 2, \end{cases} \tag{13}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{2}{\pi}\arccos\frac{\sqrt{4 - x^2}}{2} - \frac{x(6 + x^2)\sqrt{4 - x^2}}{12\pi}, & 0 \leq x < 2 \\ 1, & x \geq 2. \end{cases} \tag{14}$$

We have presented the rate of convergence of the limiting maxima for each case with respect to Bertand's probabilities associated with appropriate chord length distributions (3) and (10)-(14). For clarity, these probabilities are sorted in increasing order. We generated 10000 replications of appropriately normalized maxima with $n = 10, 25, 50$ and 100 for each case. Kolmogorov-Smirnov

(K-S) test statistic was used to compare the empirical cdfs with the theoretical ones. The values of K-S test statistic are presented in Table 2. In general, we noticed that the rate of convergence performance increases as Bertrand's probabilities increases. These results are intuitively reasonable and expected.

Table 2. Rate of convergence for normalized M_n with cdf's (3) and (10)-(14) with respect to Bertrand's probability.

Bertrand prob.	n	K-S
	10	0.01649
0.25	25	0.01024
	50	0.00887
	100	0.00842
	10	0.03127
0.3333	25	0.0139
	50	0.01045
	100	0.00905
	10	0.00384
0.4454	25	0.00131
	50	0.00093
	100	0.00054
	10	0.02871
0.5	25	0.01403
	50	0.01029
	100	0.00896
	10	0.00348
0.609	25	0.00146
	50	0.00076
	100	0.00055
	10	0.00342
0.7468	25	0.00154
	50	0.00064
	100	0.00042

3. Record chord length

3.1. Convergence results

In this section, we study the asymptotic behaviour of the corresponding records, together with the case of the cdf (3).

Proposition 3.1. (i) *Upper record values following cdf's* (3) *and* (10)-(14) *converge to* 2 *with probability one as* $n \to \infty$.

(ii) *For upper record values following cdf's* (3) *and* (10)-(14) *non degenerate limit laws under linear or power normalization do not hold.*

Proof. The first part follows directly from [20].

Using the duality theorem, in order to prove the second statement it is sufficient to show that the associated cdf $F_a(x) = 1-\exp\{-\sqrt{-\log(1-F(x))}\}$ does not belong to the domain of attraction for maxima of Weibull distribution, where F takes the form (3) or (10)-(14). To do so, we follow several steps. First, it is evident that $x_F = \sup\{x : F_a(x) < 1\} = 2$ and that $F_a(x_F-) < 1$. Let $\{u_n = a_n x + b_n : a_n > 0, b_n \in \mathbf{R}\}$ be a sequence such that $P\{M_n^a \le u_n\} \to e^{-\tau}$ as $n \to \infty$ for some $\tau > 0$, where $M_n^a = \max\{\xi_1, \xi_2, \ldots, \xi_n\}$ and where $\{\xi_n\}$ is an iid sequence with cdf F_a. According to [15, Theorem 1.5.1, Corollary 1.5.2], the last relation is equivalent to $n(1-F_a(u_n)) \to \tau$ as $n \to \infty$. It then follows that $F(u_n) \to 1-\exp\{-\ln^2 \frac{\tau}{n}\}$. From [24, Theorem 3] and Theorem 2.1, we know that F belongs to the domain of attraction for maxima of Weibull distribution with normalizing constants a_n and b_n such that $u_n \to 2$ as $n \to \infty$. With this in mind, we find that $P\{M_n \le u_n\} = (1-(1-F(u_n))^n \to (1-\exp\{-\ln^2 \frac{\tau}{n}\})^n \to 1$ as $n \to \infty$. This contradicts the fact that F belongs to the domain of attraction for maxima of Weibull distribution. Hence, $\tau = 0$. Finally, [15, Corollary 1.5.2] proves (ii) for the linear normalization case. For the case of power normalization, associated function is of the form $\widehat{A}_1(x) = F_a(e^x) I_{[\ln x_0, \ln 2)}$ for some $x_0 \in (0,2)$ (see [9]). Further, for this case we have that $x_F = \sup\{x : F_a(e^x) < 1\} = \ln 2$ and the sequence $\{u_n\}$ is such that $u_n \to \ln 2$ as $n \to \infty$. The rest of the proof follows the same steps as above. □

3.2. Moments

In the following theorem we provide analytical expressions for the lth moments of records of random chords.

Theorem 3.2. *Let $\phi_1(j) = \sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i} (-1)^i \frac{x^j}{4^i}$ and $\phi_2(j) = \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (-1)^i \frac{x^j}{4^i}$. For $n \ge 2$ and $l \ge 1$, the lth moment of record chord length with cdf's* (10)-(14) *and* (3)*, respectively, are given by*

$$\mu_{n,l}^{(1)} = \frac{1}{\pi\Gamma(n)} \int_0^2 \left\{ \sum_{m=0}^{\infty} \frac{2^{m+1} x^{m+1}}{(m+1)\pi^{m+1}} \sum_{k=0}^{\infty} c_k x^{2k} \right\}^{n-1} \phi_1(2i+l)\, dx, \tag{15}$$

where $c_0 = b_0^{m+1}$, $c_j = \frac{1}{jb_0} \sum_{k=1}^{j} (k(m+1)-j+k) b_k c_{j-k}$ for $j \ge 1$ and $\{b_k, k \ge 0\} = \left\{ \binom{2k}{k} \frac{1}{2^{4k+1}(2k+1)}, k \ge 0 \right\}$;

$$\mu_{n,l}^{(2)} = \frac{2^{l+1}}{\Gamma(n)} \sum_{m=0}^{\infty} \frac{c_m}{2m+2n+l}, \tag{16}$$

where $c_0 = 1$, $c_m = \frac{1}{m} \sum_{k=1}^{m} (k(n-1)-m+k) \frac{c_{m-k}}{k+1}$ for $m \ge 1$;

$$\mu_{n,l}^{(3)} = \frac{2^{l-n}\sqrt{\pi}}{\Gamma(n)} \sum_{m=0}^{\infty} c_m \frac{\Gamma(\frac{l}{2}+m+u)}{\Gamma(\frac{1}{2}+\frac{l}{2}+m+u)}, \tag{17}$$

where $c_0 = 1, c_m = \frac{1}{m}\sum_{k=1}^{m}(k(n-1)-m+k)\frac{c_{m-k}}{k+1}$ *for* $m \geq 1$;

$$\mu_{n,l}^{(4)} = \frac{1}{2\pi\Gamma(n)}\int_0^2 \left\{\sum_{m=0}^{\infty}\frac{2^{m+1}x^{m+1}}{\pi^{m+1}(m+1)}\sum_{k=0}^{\infty}c_k x^{2k}\right\}^{n-1}\phi_1(2(i+1)+l)\,dx, \tag{18}$$

where $\{d_k, k \geq 0\} = \left\{\frac{\binom{2k}{k}}{2^{4k+1}(2k+1)} + \frac{\binom{\frac{1}{2}}{k}(-1)^{k+1}}{2^{2k+1}}, k \geq 0\right\}$ *and* $c_0 = d_0^{m+1}, c_j = \frac{1}{jd_0}\sum_{k=1}^{j}(k(m+1)-j+k)d_k c_{j-k}$ *for* $j \geq 1$;

$$\begin{aligned}\mu_{n,l}^{(5)} = \frac{1}{6\pi\Gamma(n)}\int_0^2 &\left[-\log\left(\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{2^{2k-1}(2k+1)}\phi_2(2i)^{2k+1} + \frac{(x^2+6)\phi_2(2i+1)}{6\pi}\right)\right]^{n-1}\\ &\times \phi_1(2(i+2)+l)\,dx;\end{aligned} \tag{19}$$

$$\begin{aligned}\mu_{n,l}^{(6)} = \frac{1}{\pi\Gamma(n)}\int_0^2 &\left[-\log\left(1-\int_0^{\pi}\frac{t-2\operatorname{arccsc}\left(\csc(t/2)\phi_1(2i)\right)}{t\pi}\,dt\right)\right]^{n-1}\\ &\times\left(\int_0^{\pi}\frac{1}{t}\frac{\phi_1(2i+1+l)}{\sqrt{4(\csc^2\frac{t}{2}-1)+x^2}}\,dt\right)dx.\end{aligned} \tag{20}$$

Proof. Throughout the proof we use the following lemma which can be found in [25, (0.316)].

Lemma. *We have*

$$\left(\sum_{k=0}^{\infty}a_k x^k\right)^n = \sum_{k=0}^{\infty}c_k x^k, \tag{21}$$

where

$$c_0 = a_0^n, \quad c_m = \frac{1}{ma_0}\sum_{k=1}^{m}(kn-m+k)a_k c_{m-k}$$

for $m \geq 1$ *and* $n \in \mathbf{N}$.

First, consider the case (15).

$$\begin{aligned}\mu_{n,l}^{(1)} &= \frac{1}{\Gamma(n)}\int_0^2 x^l\{-\log(1-F(x))\}^{n-1}f(x)\,dx\\ &= \frac{1}{\pi\Gamma(n)}\int_0^2 x^l\left\{-\log\left(1-\frac{2}{\pi}\arcsin\frac{x}{2}\right)\right\}^{n-1}\frac{dx}{\sqrt{1-\frac{x^2}{4}}}.\end{aligned} \tag{22}$$

Expanding $\arcsin\frac{x}{2}$ in power series, we obtain

$$\mu_{n,l}^{(1)} = \frac{1}{\pi\Gamma(n)}\int_0^2 x^l\left\{-\log\left(1-\frac{2}{\pi}\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^{2k+1}}{2^{4k+1}(2k+1)}\right)\right\}^{n-1}\frac{dx}{\sqrt{1-\frac{x^2}{4}}}$$

$$= \frac{1}{\pi\Gamma(n)} \int_0^2 x^l \left\{ \sum_{m=0}^{\infty} \frac{2^{m+1} x^{m+1}}{(m+1)\pi^{m+1}} \left(\sum_{k=0}^{\infty} \binom{2k}{k} \frac{x^{2k}}{2^{4k+1}(2k+1)} \right)^{m+1} \right\}^{n-1}$$

$$\times \frac{dx}{\sqrt{1 - \frac{x^2}{4}}}. \tag{23}$$

Denote $\{b_k, k \geq 0\} = \left\{ \binom{2k}{k} \frac{1}{2^{4k+1}(2k+1)}, k \geq 0 \right\}$. Using Lemma 1 we have

$$\left(\sum_{k=0}^{\infty} b_k (x^2)^k \right)^{m+1} = \sum_{k=0}^{\infty} c_k (x^2)^k, \tag{24}$$

where $c_0 = b_0^{m+1}$ and $c_j = \frac{1}{j b_0} \sum_{k=1}^{j} (k(m+1) - j + k) b_k c_{j-k}$ for $j \geq 1$. Therefore

$$\mu_{n,l}^{(1)} = \frac{1}{\pi\Gamma(n)} \int_0^2 x^l \left\{ \sum_{m=0}^{\infty} \frac{2^{m+1} x^{m+1}}{(m+1)\pi^{m+1}} \sum_{k=0}^{\infty} c_k x^{2k} \right\}^{n-1} \frac{dx}{\sqrt{1 - \frac{x^2}{4}}}$$

$$= \frac{1}{\pi\Gamma(n)} \int_0^2 \left\{ \sum_{m=0}^{\infty} \frac{2^{m+1} x^{m+1}}{(m+1)\pi^{m+1}} \sum_{k=0}^{\infty} c_k x^{2k} \right\}^{n-1} \sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i} (-1)^i \frac{x^{2i+l}}{4^i} \, dx. \tag{25}$$

Other cases are omitted due to their similarity. □

Further, we have conducted here a comparative study among various parameters of a chord length distributions (3) and (10)-(14) based on records. As above, we have provided Bertand's probabilities associated with each cdf's (3) and (10)-(14) and presented them in increasing order. The parameters under investigation are the mean, variance, skewness and kurtosis. We summarized all comparisons in Table 3. Briefly, we may highlight that the variance is decreasing for all record values as n increases. It was also noted that as n increases the kurtosis also increases for all cases indicating a high level of heavy tailed phenomena. On the contrary, skewness is negative for all cases and decreases as n increases. Therefore, we can conclude that all record values have tails on the left side of their distributions.

It was observed that the variance is negatively correlated with respect to Bertrand's probabilities as intuitively expected. Generally, the same holds true for skewness while for the case of the mean and kurtosis positive correlation is realized. Also, these results are realistic and expected. Overall, these results are in concordance with those obtained for limiting maxima.

4. Conclusion

In this paper we proposed a new "solution" to Bertrand's paradox. The randomization procedure of this solution takes place outside the circle (see [17]) and follows the recommendation from [8, p. 305]. We may find this solution to be a randomized version of Jaynes's experiment, see [11].

TABLE 3. Moments of R_n with cdf's (3) and (10)-(14) with respect to Bertrand's probability.

Bertrand prob.	n	Mean	Variance	Skewness	Kurtosis
	1	1.3282	0.22461	−0.55059	2.36322
	2	1.71108	0.0839	−1.46609	5.03734
0.25	3	1.86238	0.03196	−2.34663	9.86038
	4	1.93375	0.01113	−3.22618	17.36272
	5	1.96718	0.00383	−4.2452	29.27783
	1	1.27261	0.37444	−0.4847	1.92304
	2	1.75222	0.12952	−1.99237	6.74093
0.3333	3	1.912	0.03535	−3.71715	20.0899
	4	1.97143	0.00733	−6.32481	57.29845
	5	1.98967	0.00191	−10.38128	169.0421
	1	1.49742	0.2415	−0.97726	2.98127
	2	1.84485	0.06043	−2.46571	9.82857
0.4454	3	1.9503	0.01247	−4.27357	27.77369
	4	1.98385	0.00263	−7.19977	76.92249
	5	1.99445	0.00058	−9.89047	138.2683
	1	1.57146	0.19927	−1.15787	3.52091
	2	1.8714	0.04254	−2.60536	11.20524
0.5	3	1.96075	0.00888	−5.00608	38.18515
	4	1.98699	0.00162	−6.36067	58.35393
	5	1.99586	0.00029	−9.60456	142.1021
	1	1.69224	0.11895	−1.45026	4.69214
	2	1.91669	0.02052	−2.94325	13.80732
0.609	3	1.97493	0.00362	−4.93059	36.62403
	4	1.9919	0.00078	−8.81611	121.7533
	5	1.99746	0.00012	−9.8181	137.3913
	1	1.81384	0.05175	−1.88891	7.03427
	2	1.951	0.00782	−3.50046	19.81464
0.7468	3	1.9857	0.00128	−5.51952	48.13205
	4	1.99528	0.00031	−11.45692	234.9656
	5	1.99847	0.00006	−15.37753	358.8772

We concluded that the maximum $M_n = \max_{1 \leq i \leq n} X_i$ of an iid sequence $\{X_n\}$ with cdf (3) has limiting Weibull distribution, which is in concordance with [14], and we derived the appropriate convergence results. Moreover, we obtained the moments of records of random chord lengths that can be useful for characterization purposes.

Further perspectives may concern new generating methods of random chords with possible extreme behaviour results, as well as undergoing inferential procedures on the dependence structure of various statistics of chord length record values and the limiting rate of convergence for chord length maxima with Bertand's probabilities. It would be also interesting to see if moments (15)-(20) will satisfy some recurrence relations. This might eventually simplify their evaluations.

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References

[1] D. Aerts and M. Sassoli de Bianchi, *Solving the hard problem of Bertrand's paradox*, J. Math. Phys. **55** (2014), no. 8, 083503, 17 pp. https://doi.org/10.1063/1.4890291
[2] M. Ahsanullah and V. B. Nevzorov, *Records via Probability Theory*, Atlantis Studies in Probability and Statistics, 6, Atlantis Press, Paris, 2015. https://doi.org/10.2991/978-94-6239-136-9
[3] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, *Records*, Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons, Inc., New York, 1998. https://doi.org/10.1002/9781118150412
[4] S. Bangu, *On Bertrand's paradox*, Analysis (Oxford) **70** (2010), no. 1, 30–35. https://doi.org/10.1093/analys/anp124
[5] J. Bertrand, *Calcul des probabilites*, Paris: Gautier-Villars et Fils, 1889.
[6] K. N. Chandler, *The distribution and frequency of record values*, J. Roy. Statist. Soc. Ser. B **14** (1952), 220–228.
[7] S. S. Chiu and R. C. Larson, *Bertrand's paradox revisited: More lessons about that ambiguous word, Random*, J. Industrial Systems Engineering (JISE) **3** (2009), no.1, 1–26.
[8] B. C. van Fraassen, *Laws and Symmetry*, The Clarendon Press, Oxford University Press, New York, 1989. https://doi.org/10.1093/0198248601.001.0001
[9] B. Grigelionis, *Limit theorems for record values using power normalization*, Lithuanian Math. J. **46** (2006), no. 4, 398–405; translated from Liet. Mat. Rink. **46** (2006), no. 4, 492–500. https://doi.org/10.1007/s10986-006-0037-8
[10] J. Holbrook and S. S. Kim, *Bertrand's paradox revisited*, Math. Intelligencer **22** (2000), no. 4, 16–19. https://doi.org/10.1007/BF03026762
[11] E. T. Jaynes, *The well-posed problem*, Found. Phys. **3** (1973), 477–492. https://doi.org/10.1007/BF00709116
[12] V. Jevremovic and M. Obradovic, *Bertrand's paradox: Is there anything else?*, Quality & Quantity **46** (2012), no. 6, 1709–1714.
[13] M. G. Kendall and P. A. P. Moran, *Geometrical Probability*, Griffin's Statistical Monographs & Courses, No. 10, Hafner Publishing Co., New York, 1963.
[14] E. V. Koroleva and Ya. Yu. Nikitin, *U-max-statistics and limit theorems for perimeters and areas of random polygons*, J. Multivariate Anal. **127** (2014), 98–111. https://doi.org/10.1016/j.jmva.2014.02.006
[15] M. R. Leadbetter, G. Lindgren, and H. Rootzén, *Extremes and related properties of random sequences and processes*, Springer Series in Statistics, Springer-Verlag, New York, 1983.
[16] G. D. Lin, *On characterizations of distributions via moments of record values*, Probab. Theory Related Fields **74** (1987), no. 4, 479–483. https://doi.org/10.1007/BF00363510
[17] L. Marinoff, *A resolution of Bertrand's paradox*, Philos. Sci. **61** (1994), no. 1, 1–24. https://doi.org/10.1086/289777

[18] V. B. Nevzorov, *Records: mathematical theory*, translated from the Russian manuscript by D. M. Chibisov, Translations of Mathematical Monographs, 194, American Mathematical Society, Providence, RI, 2001. https://doi.org/10.1090/mmono/194

[19] R Core Team, *R: A Language and Environment for Statistical Computing*, R Foundation for Statistical Computing, Vienna, Austria, 2015.

[20] S. I. Resnick, *Limit laws for record values*, Stochastic Process. Appl. **1** (1973), 67–82. https://doi.org/10.1016/0304-4149(73)90033-1

[21] D. Rizza, *A study of mathematical determination through Bertrand's paradox*, Philos. Math. (3) **26** (2018), no. 3, 375–395. https://doi.org/10.1093/philmat/nkx035

[22] N. Shackel, *Bertrand's paradox and the principle of indifference*, Philos. Sci. **74** (2007), no. 2, 150–175. https://doi.org/10.1086/519028

[23] A. Soranzo and A. Volčič, *On the Bertrand paradox*, Rend. Circ. Mat. Palermo (2) **47** (1998), no. 3, 503–509. https://doi.org/10.1007/BF02851396

[24] Z. Vidović, *Limit distribution of maximal random chord length*, J. Appl. Stat. Probab. (JASP) **5** (2016), no. 2, 213–220.

[25] D. Zwillinger, *Table of Integrals, Series and Products*, Academic Press, 2007.

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