Bull. Korean Math. Soc. **58** (2021), No. 2, pp. 403–417
https://doi.org/10.4134/BKMS.b200318
pISSN: 1015-8634 / eISSN: 2234-3016

ON THE MIXED RADIAL-ANGULAR INTEGRABILITY OF LITTLEWOOD-PALEY FUNCTIONS

Xiao Zhang

Abstract. This note is devoted to establishing the boundedness for some classes of Littlewood-Paley square operators defined by the kernels without any regularity on the mixed radial-angular spaces. The corresponding vector-valued versions are also presented. As applications, the corresponding results for the Littlewood-Paley g_λ^* function and the Littlewood-Paley function related to the area integrals are also obtained.

1. Introduction

Let $\mathbb{R}^n$ be the Euclidean space of dimension n and S^{n-1} denote the unit sphere in $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma$. The mixed radial-angular spaces $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$, consist of all functions u satisfying $\|u\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} < \infty$, where

$$\|u\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} := \Big(\int_0^\infty \|u(\rho\cdot)\|^p_{L^q(\mathrm{S}^{n-1})} \rho^{n-1} d\rho \Big)^{1/p},$$

$$\|u\|_{L^\infty_{|x|}L^q_\theta(\mathbb{R}^n)} := \sup_{\rho>0} \|u(\rho\cdot)\|_{L^q(\mathrm{S}^{n-1})}.$$

Note that the spaces $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ enjoy the following easy properties.

(i) If $1 \leq p \leq \infty$ and $q = p$, then

$$\|u\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} = \|u\|_{L^p(\mathbb{R}^n)}. \tag{1}$$

(ii) If u is a radial function on $\mathbb{R}^n$ and $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, then

$$\|u\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \simeq \|u\|_{L^p(\mathbb{R}^n)}.$$

(iii) If $1 \leq p \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, then

$$\|u\|_{L^p_{|x|}L^{q_1}_\theta(\mathbb{R}^n)} \leq C_{n,p,q_1,q_2} \|u\|_{L^p_{|x|}L^{q_2}_\theta(\mathbb{R}^n)}.$$

Received April 6, 2020; Accepted July 9, 2020.
2010 *Mathematics Subject Classification.* Primary 42B25, 42B20.
Key words and phrases. Littlewood-Paley function, weighted norm inequality, mixed radial-angular space.

Here the notation $A \simeq B$ means that there are two positive constants C, C' such that $A \le CB$ and $B \le C'A$. Throughout this paper, we use $C_{\alpha,\beta,\ldots}$ to denote positive constants that depend on parameters α, $\beta, \ldots$.

Based on the definition of $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ and (1), one might think that the mixed radial-angular space $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ can be seen as an formal extension of the Lebesgue space $L^p(\mathbb{R}^n)$. Over the last several years it has been successfully used in studying Strichartz estimates and dispersive equations (see [4,18,24] for example). Recently the mixed radial-angular space $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ is also playing active roles in the theory of singular integral operator. The first work in this topic was due to Córdoba [8] who proved that, among other things, the rough singular integral operator

$$T_\Omega(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}dy,$$

is bounded on $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ for all $1 < p < \infty$ and $q = 2$, provided that $\Omega \in \mathcal{C}^1(\mathrm{S}^{n-1})$ with vanishing integral $\int_{\mathrm{S}^{n-1}} \Omega(\theta)d\sigma(\theta) = 0$. By using the same argument in [8, Theorem 2.1], P. D'Ancona and R. Lucà [9] extended the above index $q = 2$ to the range $1 < q < \infty$. The corresponding radial weighted results were established by Cacciafesta and R. Lucà [5] and Duoandikoetxea and Oruetxebarria [11]. Recently, Liu and Fan [15] and Liu et al. [16] improved the above unweighted results to the case $\Omega \in L^q(\mathrm{S}^{n-1})$ or $\Omega \in \mathcal{F}_\beta(\mathrm{S}^{n-1})$ (the Grafakos-Stefanov class) and extended the above results to the singular integral operators along polynomial curves.

On the other hand, the theory of the Littlewood-Paley functions, as everyone knows, has been an important part of harmonic analysis. One can consult Stein's works [21–23] for its origin and significance. Recall that the square function of Littlewood-Paley type is defined in the following way

$$g_\psi(f)(x) = \Big(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\Big)^{1/2},$$

where ψ is a function in $L^1(\mathbb{R}^n)$ with vanishing integral $\int_{\mathbb{R}^n} \psi(x)dx = 0$ and $\psi_t(x) = t^{-n}\psi(t^{-1}x)$ for $t > 0$.

Over the last several years a considerable amount of attention has been given to study the boundedness for the Littlewood-Paley functions on various function spaces. For example, see [1,2,6,10,12,19,20] for the Lebesgue spaces, [14,25] for the Triebel-Lizorkin spaces. A well-known result for Littlewood-Paley functions was given by Benedek, Calderón and Panzone [2] who proved the following.

Theorem A ([2]). *Suppose that ψ satisfies*

$$|\psi(x)| \le C(1+|x|)^{-n-\epsilon} \quad \textit{for some } \epsilon > 0,$$

$$\int_{\mathbb{R}^n} |\psi(x-y) - \psi(x)|dx \le C|y|^\epsilon \quad \textit{for some } \epsilon > 0.$$

Then g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$.

Subsequently, the condition on ψ in Theorem A was relaxed by Fan and Sato [12] who established the following result.

Theorem B ([12])**.** *Suppose that the function* ψ *satisfies the following conditions*:

(i) $\int_{|x|\geq 1} |\psi(x)||x|^\epsilon dx < \infty$ *for some* $\epsilon > 0$.

(ii) $\big(\int_{|x|<1} |\psi(x)|^u dx\big)^{1/u} < \infty$ *for some* $u > 1$.

(iii) $|\psi(x)| \leq h(x)\Omega(x')$ *for all* $x \in \mathbb{R}^n \setminus \{0\}$, *where* $x' = x/|x|$, *for some non-negative function* h *on* $(0,\infty)$ *and* Ω *on* S^{n-1} (*the unit sphere in* $\mathbb{R}^n$) *such that*

(a) $h(r)$ *is non-increasing on* $(0,\infty)$ *and* $h(|x|) \in L^1(\mathbb{R}^n)$,

(b) $\Omega \in L^s(\mathrm{S}^{n-1})$ *for some* $1 < s \leq \infty$.

Then g_ψ *is bounded on* $L^p(\mathbb{R}^n)$ *for all* $1 < p < \infty$.

Recently, Sato [20] obtained the following refined result via a minimum condition on ψ.

Theorem C ([20])**.** *Suppose that* $|\psi(x)| \leq h(x)\Omega(x')$ *for all* $x \in \mathbb{R}^n\setminus\{0\}$, *where* h *is a non-negative, non-increasing function on* $(0,\infty)$ *with supported in* $(0,1]$ *and* Ω *is a non-negative function on* S^{n-1}. *We assume that* $h(|x|) \in L^1(\mathbb{R}^n)$, $\Omega \in L^1(\mathrm{S}^{n-1})$ *and* $\psi \in L^s(\mathbb{R}^n)$ *for some* $1 < s \leq \infty$. *Put* $m_\psi(x) = h(|x|)\Omega(x')$. *Then*

$$\|g_\psi(f)\|_{L^p(\mathbb{R}^n)} \leq C_p(s/(s-1))^{1/2}(\|\psi\|_{L^s(\mathbb{R}^n)} + \|m_\psi\|_{L^1(\mathbb{R}^n)})\|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$, *where the constant* $C_p > 0$ *is independent of* s, ψ, h, Ω.

Based on Theorems B and C and (1), a question that arises naturally is the following.

Question 1.1. *Is the operator* g_ψ *bounded on* $L^p_{|x|}L^q_\theta(\mathbb{R}^n)$ *for* $p \neq q$ *under the same assumptions on* ψ *as in one of Theorems B and C?*

Question 1.1 is the main motivation of this work. In this paper we shall give an affirmative answer to the above question. Our result can be formulated as follows.

Theorem 1.2. *Suppose that* ψ *satisfies the condition of Theorem* B *or* C. *Then for* $1 < q < 2$ *and* $q \leq p < \frac{2q}{2-q}$ *or* $2 \leq q \leq p < \infty$, *the following inequalities hold*:

$$\|g_\psi(f)\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leq C_{p,q}\|f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |g_\psi(f_j)|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leq C_{p,q}\Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^q\Big)^{1/q}\Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)};$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |g_\psi(f_j)|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)} \leq C_{p,q}\Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^n)}.$$

Actually, Theorem 1.2 will be derived from the following more general one.

Theorem 1.3. *Assume that ψ satisfies the following conditions*:

(i) *There exist ϵ, $\delta > 0$ and $C > 0$ such that*

$$\int_{2^k}^{2^{k+1}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \leq C \min\{1, |2^k\xi|^{\epsilon}, |2^k\xi|^{-\delta}\}$$

for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$;

(ii) *There exists a constant $C > 0$ such that*

$$\Big\| \sup_{t>0} \big| |\psi_t| * f(x) \big| \Big\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$.

Then for $1 < q < 2$ and $q \leq p < \frac{2q}{2-q}$ or $2 \leq q \leq p < \infty$, the following inequalities hold:

$$\|g_\psi(f)\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leq C_{p,q} \|f\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}; \tag{2}$$

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |g_\psi(f_j)|^q \Big)^{1/q} \Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)} \leq C_{p,q} \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^q \Big)^{1/q} \Big\|_{L^p_{|x|}L^q_\theta(\mathbb{R}^n)}; \tag{3}$$

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |g_\psi(f_j)|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \leq C_{p,q} \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}. \tag{4}$$

This paper will be organized as follows. Section 2 contains some key propositions, which are the main ingredients of our proofs. The proofs of Theorems 1.2 and 1.3 will be given in Section 3. Finally, as applications of our main results, the mixed radial-angular integrability for the Littlewood-Paley g_λ^* function and the Littlewood-Paley function related to the area integrals will be established in Section 4. We would like to remark that some ideas in the proofs of our main results are taken from [13,16,19,20].

Throughout this paper, for any function f, we denote $\tilde{f}$ by $\tilde{f}(x) = f(-x)$. For any $p \in (1,\infty)$, we let p' denote the dual exponent to p defined as $1/p + 1/p' = 1$. For any nonnegative measurable function ω, we set

$$\|f\|_{L^p(w)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \Big)^{1/p}.$$

Thus $L^p(\omega)$ associated with the function ω is defined by

$$L^p(\mathbb{R}^n, \omega(x)dx) = \{f : \|f\|_{L^p(\omega)} < \infty\}.$$

We also denote by M the usual Hardy-Littlewood maximal function. For $s > 1$, we define the operator M_s by $\mathrm{M}_s(f) = (\mathrm{M}(f^s))^{1/s}$.

2. Some key propositions

This section is devoted to presenting a general criterion on the weighted boundedness of Littlewood-Paley functions, which is a key of our proofs.

Proposition 2.1. *Assume that ψ satisfies the following conditions*:
(a) *There exist ϵ, $\delta > 0$ and $C > 0$ such that*

$$\int_{2^k}^{2^{k+1}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \leq C \min\{1, |2^k\xi|^{\epsilon}, |2^k\xi|^{-\delta}\} \tag{5}$$

for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$.
(b) *There exists a constant $C > 0$ such that*

$$\|\sigma(f)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \tag{6}$$

for all $1 < p < \infty$. Here

$$\sigma(f)(x) := \sup_{t>0} \big||\psi_t| * f(x)\big|.$$

For $s > 0$, we define the operators $\tilde{\sigma}$ and $\tilde{\sigma}_s$ by

$$\tilde{\sigma}(f)(x) := \sup_{t>0} \big||\widetilde{\psi_t}| * f(x)\big|, \quad \tilde{\sigma}_s(f)(x) := (\tilde{\sigma}(|f|^s)(x))^{1/s}.$$

Here $\widetilde{\psi_t}(x) = \psi_t(-x)$. Then for all nonnegative measurable functions u on $\mathbb{R}^n$, the following inequality

$$\|g_\psi(f)\|_{L^p(u)} \leq C_p\|f\|_{L^p(\mathrm{M}_s(\tilde{\sigma}_s(u)))} \tag{7}$$

holds, provided that one of the following conditions holds:
(i) $1 < p < 2$ *and* $s > 2/p$;
(ii) $2 \leq p < \infty$ *and* $s > 1$.

Proof. This proof will be divided into two steps:

Step 1: Proof of (7) for the case (i). Let $\Phi(t) \in \mathcal{C}_c^\infty((1/4, 1))$ such that $0 \leq \Phi \leq 1$ and $\sum_{k\in\mathbb{Z}}(\Phi(2^k|\xi|))^2 = 1$. Define the Fourier multiplier operators $\{S_k\}_{k\in\mathbb{Z}}$ by $S_kf(x) = \Theta_k * f(x)$, where $\widehat{\Theta_k}(\xi) = \Phi(2^k|\xi|)$. By the arguments similar to those used in deriving (5) in [13], one can get

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |S_kf|^2\Big)^{1/2}\Big\|_{L^p(w)} \leq C_{p,w}\|f\|_{L^p(w)} \tag{8}$$

for all $1 < p < \infty$ and $w \in A_p$.

By the changes of variables and Minkowski's inequality, we can write

$$\begin{aligned} g_\psi(f)(x) &= \Big(\sum_{k\in\mathbb{Z}}\int_{2^k}^{2^{k+1}} |\psi_t * f(x)|^2 \frac{dt}{t}\Big)^{1/2} \\ &= \Big(\sum_{k\in\mathbb{Z}}\int_{1}^{2} |\psi_{2^k t} * f(x)|^2 \frac{dt}{t}\Big)^{1/2} \\ &= \Big(\sum_{k\in\mathbb{Z}}\int_{1}^{2} \Big|\sum_{j\in\mathbb{Z}} S_{j+k}^2(\psi_{2^k t} * f)(x)\Big|^2 \frac{dt}{t}\Big)^{1/2} \\ &\le \sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}}\int_{1}^{2} |\psi_{2^k t} * S_{j+k}^2 f(x)|^2 \frac{dt}{t}\Big)^{1/2} \\ &=: \sum_{j\in\mathbb{Z}} T_j f(x). \end{aligned} \tag{9}$$

Hence, by (9), to prove (7), it suffices to show that for any $1 < p < 2$ and $s > 2/p$, there exists a constant $\alpha > 0$ independent of j such that

$$\|T_j f\|_{L^p(u)} \le C_{p,q} 2^{-\alpha|j|}\|f\|_{L^p(\mathrm{M}_s M_{\lambda,s}^{\tilde{\mu}} u)}. \tag{10}$$

Now we shall prove (10). Fix $k \in \mathbb{Z}$, $t \in [1, 2]$ and a nonnegative measurable function u on $\mathbb{R}^n$, it holds that

$$\|\psi_{2^k t} * f\|_{L^\infty(\mathbb{R}^n)} \le \|\psi\|_{L^1(\mathbb{R}^n)}\|f\|_{L^\infty(\mathbb{R}^n)}; \tag{11}$$

$$\|\psi_{2^k t} * f\|_{L^1(u)} \le \|f\|_{L^1(\tilde{\sigma}(u))}. \tag{12}$$

By the interpolation of L^p-spaces with change of measure ([3, Corollary 5.5.4]) between (11) and (12), one get

$$\|\psi_{2^k t} * f\|_{L^p(u)} \le C_p\|f\|_{L^p(\tilde{\sigma}(u))} \tag{13}$$

for all $1 < p < 2$. Here $C_p > 0$ is independent of k, t. We get from (13) that

$$\int_{\mathbb{R}^n}\int_1^2 |\psi_{2^k t} * f_k(x)|^p \frac{dt}{t} u(x)dx \le C_p \int_{\mathbb{R}^n} |f_k(x)|^p \tilde{\sigma}(u)(x)dx \tag{14}$$

for all $1 < p < 2$. Then we get from (14) that

$$\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_1^2 |\psi_{2^k t} * f_k(x)|^p \frac{dt}{t} u(x)dx \le C_p \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}} |f_k(x)|^p \tilde{\sigma}(u)(x)dx \tag{15}$$

for all $1 < p < 2$. By (6) one has

$$\begin{aligned} \int_{\mathbb{R}^n}\Big(\sup_{k\in\mathbb{Z}}\sup_{t\in[1,2]} |\psi_{2^k t} * f_k(x)|\Big)^p dx &\le \int_{\mathbb{R}^n}\Big(\sigma\Big(\sup_{k\in\mathbb{Z}}|f_k(x)|\Big)\Big)^p dx \\ &\le C_p\int_{\mathbb{R}^n}\Big(\sup_{k\in\mathbb{Z}}|f_k(x)|\Big)^p dx \end{aligned} \tag{16}$$

for all $1 < p < 2$. Let $t_1 = 2/p$. The interpolation between (15) and (16) gives that

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\psi_{2^kt}*f_k(x)|^2\frac{dt}{t}\Big)^{p/2}u^{1/t_1}(x)dx\\ &\le C_p\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}|f_k(x)|^2\Big)^{p/2}(\tilde{\sigma}(u))^{1/t_1}(x)dx\end{aligned}\tag{17}$$

for all $1 < p < 2$. By substituting u^{t_1} for u in (17), one has

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\psi_{2^kt}*f_k(x)|^2\frac{dt}{t}\Big)^{p/2}u(x)dx\\ &\le C_p\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}|f_k(x)|^2\Big)^{p/2}\tilde{\sigma}_{t_1}(u)(x)dx\end{aligned}\tag{18}$$

for all $1 < p < 2$. Note that the fact that $\mathrm{M}_s u \in A_1$ (see [7]) and $u \le M_{t_1}u$. Then we have $\mathrm{M}_{t_1}(\tilde{\sigma}_{t_1}(u)) \in A_1$. It follows from (18) that

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\psi_{2^kt}*f_k(x)|^2\frac{dt}{t}\Big)^{p/2}u(x)dx\\ &\le C_p\int_{\mathbb{R}^n}\Big(\sum_{k\in\mathbb{Z}}|f_k(x)|^2\Big)^{p/2}\mathrm{M}_{t_1}(\tilde{\sigma}_{t_1}(u))(x)dx\end{aligned}\tag{19}$$

for all $1 < p < 2$. In light of (8) and (19) we would have

$$\begin{aligned}\|T_jf\|_{L^p(u)} &= \Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2|\psi_{2^kt}*S^2_{j+k}f|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(u)}\\ &\le C_p\Big\|\Big(\sum_{k\in\mathbb{Z}}|S^2_{j+k}f|^2\Big)^{1/2}\Big\|_{L^p(\mathrm{M}_{t_1}(\tilde{\sigma}_{t_1}(u)))}\\ &\le C_p\|f\|_{L^p(\mathrm{M}_{t_1}(\tilde{\sigma}_{t_1}(u)))}\end{aligned}\tag{20}$$

for all $1 < p < 2$.

Next we estimate $\|T_jf\|_{L^2(u)}$. By (5) and Plancherel's theorem, there exists a constant $\beta > 0$ such that

$$\int_{\mathbb{R}^n}\int_1^2|\psi_{2^kt}*S_{j+k}w(x)|^2\frac{dt}{t}dx \le C2^{-\beta|j|}\int_{\mathbb{R}^n}|w(x)|^2dx\tag{21}$$

for arbitrary function w on $\mathbb{R}^n$. One can easily check that

$$|\psi_{2^kt}*S_{j+k}w(x)|^2 \le \|\psi\|_{L^1(\mathbb{R}^n)}\|\Theta_{j+k}\|_{L^1(\mathbb{R}^n)}|\psi_{2^kt}|*|\Theta_{j+k}|*|w(x)|^2,$$

which gives

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_1^2 |\psi_{2^k t} * S_{j+k}w(x)|^2 \frac{dt}{t} u^s(x)dx\\ &\le \|\psi\|_{L^1(\mathbb{R}^n)}\|\Theta_{j+k}\|_{L^1(\mathbb{R}^n)}\\ &\quad\times \int_1^2\int_{\mathbb{R}^n} |\psi_{2^k t}| * |\Theta_{j+k}| * |w|^2(x)u^s(x)dx\frac{dt}{t}\\ &\le C\int_{\mathbb{R}^n} |w(x)|^2 \mathrm{M}(\tilde{\sigma}u^s)(x)dx\end{aligned} \tag{22}$$

for any $s > 1$. Combining (21) with (22) and the interpolation of L^2-spaces with change of measure ([3, Theorem 5.4.1]) implies

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_1^2 |\psi_{2^k t} * S_{j+k}w(x)|^2 \frac{dt}{t} u(x)dx\\ &\le C_s 2^{-\frac{\beta}{s'}|j|}\int_{\mathbb{R}^n} |w(x)|^2 \mathrm{M}_s(\tilde{\sigma}_s(u))(x)dx\end{aligned} \tag{23}$$

for any $s > 1$ and arbitrary function w on $\mathbb{R}^n$. Noting that $\mathrm{M}_s(\tilde{\sigma}_s(u)) \in A_1$. By using (11), (23) with $w = S_{j+k}f$ and the well-known property of the Rademacher's function, we have

$$\begin{aligned}\|T_j f\|^2_{L^2(u)} &= \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_1^2 |\psi_{2^k t} * S^2_{j+k}f(x)|^2 \frac{dt}{t} u(x)dx\\ &\le \sum_{k\in\mathbb{Z}}\int_1^2\int_{\mathbb{R}^n} |\psi_{2^k t} * S^2_{j+k}f(x)|^2 u(x)dx\frac{dt}{t}\\ &\le C_s 2^{-\frac{\beta}{s'}|j|}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}} |S_{j+k}f(x)|^2 \mathrm{M}_s(\tilde{\sigma}_s(u))(x)dx\\ &\le C_s 2^{-\frac{\beta}{s'}|j|}\|f\|^2_{L^2(\mathrm{M}_s(\tilde{\sigma}_s(u)))}.\end{aligned}$$

It follows that

$$\|T_j f\|_{L^2(u)} \le C_p 2^{-\frac{\beta}{2s'}|j|}\|f\|_{L^2(\mathrm{M}_s(\tilde{\sigma}_s(u)))} \tag{24}$$

for any $s > 1$. By an interpolation between (20) and (24) with $s = t_1$, there exist $C_p > 0$ and $\alpha > 0$ such that

$$\|T_j f\|_{L^p(u)} \le C_p 2^{-\alpha|j|}\|f\|_{L^p(\mathrm{M}_{t_1}(\tilde{\sigma}_{t_1}(u)))} \tag{25}$$

for all $1 < p < 2$. By Hölder's inequality, one can check that $\mathrm{M}_{t_1}(\tilde{\sigma}_{t_1}(u)) \le C\mathrm{M}_s(\tilde{\sigma}_s(u))$ for $s > t_1$. This together with (25) yields (10).

Step 2: Proof of (7) for the case (ii). By (9), to prove (7) in this case, it suffices to show that there exist $C_{p,s} > 0$ and $\gamma > 0$ such that

$$\|T_j f\|_{L^p(u)} \le C_{p,s} 2^{-\gamma|j|}\|f\|_{L^p(\mathrm{M}_s(\tilde{\sigma}_s(u)))} \tag{26}$$

for $2 \le p < \infty$ and $s > 1$. We only show that

$$\|T_j f\|_{L^p(u)} \le C_{p,s}\|f\|_{L^p(\mathrm{M}_s(\tilde{\sigma}_s(u)))} \tag{27}$$

for all $2 < p < \infty$ and $s > 1$. Actually, inequality (26) follows from (24), (27) and an interpolation (see [3, Corollary 5.5.4]).

Next we shall prove (27). Fix $2 < p < \infty$. By duality we can find a function $v \in L^{(\frac{p}{2})'}(u)$ with unit norm such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\psi_{2^k t} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(u)}^2 = \int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_1^2 |\psi_{2^k t} * g_k(x)|^2 \frac{dt}{t}\cdot v(x)u(x)dx,$$

which together with the fact that $\|\psi_{2^k t}\|_{L^1(\mathbb{R}^n)} = \|\psi\|_{L^1(\mathbb{R}^n)}$ implies

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\psi_{2^k t} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(u)}^2 \\ &\le \|\psi\|_{L^1(\mathbb{R}^n)}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|^2\int_1^2 \big||\widetilde{\psi_{2^k t}}| * (vu)(x)\big|\frac{dt}{t}dx.\end{aligned} \tag{28}$$

Fix $s > 1$ and let $r = \frac{ps}{2}$. By Hölder's inequality

$$\begin{aligned}\big||\widetilde{\psi_{2^k t}}| * (vu)\big| &\le (|\widetilde{\psi_{2^k t}}| * u^s)^{1/r}(|\widetilde{\psi_{2^k t}}| * (u^{r'/(p/2)'}v^{r'}))^{1/r'} \\ &\le (\tilde{\sigma}(u^s))^{1/r}(\tilde{\sigma}(u^{r'/(p/2)'}v^{r'}))^{1/r'}.\end{aligned} \tag{29}$$

By Hölder's inequality with exponents $\frac{p}{2}$ and $(\frac{p}{2})'$, we get from (28) and (29) that

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\psi_{2^k t} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(u)}^2 \\ &\le C\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}|g_k(x)|^2(\tilde{\sigma}(u^s))^{1/r}(x)(\tilde{\sigma}(u^{r'/(p/2)'}v^{r'}))^{1/r'}(x)dx \\ &\le C_p\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\tilde{\sigma}_s(u))}^2\|\tilde{\sigma}(u^{r'/(p/2)'}v^{r'})\|_{L^{(p/2)'/r'}(\mathbb{R}^n)}^{1/r'}.\end{aligned} \tag{30}$$

Note that $(\frac{p}{2})' > r'$ since $\frac{p}{2} = \frac{r}{s} < r$, by (6) we get

$$\|\tilde{\sigma}(u^{r'/(p/2)'}v^{r'})\|_{L^{(p/2)'/r'}(\mathbb{R}^n)}^{1/r'} \le C_p\|u^{r'/(p/2)'}v^{r'}\|_{L^{(p/2)'/r'}(\mathbb{R}^n)}^{1/r'} \le C_p.$$

This together with (30) yields that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\psi_{2^k t} * g_k|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(u)} \le C_p\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_{L^p(\tilde{\sigma}_s(u))} \tag{31}$$

for all $2 < p < \infty$ and any $s > 1$.

Finally, by the facts $\tilde{\sigma}_s(u) \le \mathrm{M}_s(\tilde{\sigma}_s(u))$ and $\mathrm{M}_s(\tilde{\sigma}_s(u)) \in A_1$, we get from (8) and (31) that

$$\begin{aligned}\|T_j f\|_{L^p(u)} &= \Big\|\Big(\sum_{k\in\mathbb{Z}}\int_1^2 |\psi_{2^k t} * S_{j+k}^2 f|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{L^p(u)} \\ &\le C_p \Big\|\Big(\sum_{k\in\mathbb{Z}} |S_{j+k}^2 f|^2\Big)^{1/2}\Big\|_{L^p(\mathrm{M}_s(\tilde{\sigma}_s(u)))} \\ &\le C_p \|f\|_{L^p(\mathrm{M}_s(\tilde{\sigma}_s(u)))}\end{aligned}$$

for all $2 < p < \infty$ and any $s > 1$. This proves (27) and finishes the proof. $\square$

As an application of Proposition 2.1, we can get the following weighted inequalities for Littlewood-Paley functions, which are of interest in its own right.

Proposition 2.2. *Suppose that ψ satisfies the condition of Theorem* B *or* C. *We define the operator M_Ω by*

$$M_\Omega(f)(x) = \sup_{t>0} t^{-n} \int_{|y|<t} |f(x+y)|\Omega(y/|y|)dy.$$

For $s > 0$, we define the operator $M_{s,\Omega}$ by $M_{s,\Omega}(f) = (M_\Omega(f^s))^{1/s}$. Then the following inequality

$$\|g_\psi(f)\|_{L^p(u)} \le C_p \|f\|_{L^p(\mathrm{M}_s(M_{s,\Omega}(f)))} \tag{32}$$

holds for all nonnegative measurable functions u on $\mathbb{R}^n$, provided that one of the following conditions holds:
(i) $1 < p < 2$ *and* $s > 2/p$;
(ii) $2 \le p < \infty$ *and* $s > 1$.

Proof. By Lemmas 1-3 of [19], we get

$$\int_{2^k}^{2^{k+1}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \le C \min\{1, |2^k\xi|^\epsilon, |2^k\xi|^{-\epsilon}\} \tag{33}$$

for all $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$ and some $\epsilon > 0$, provided that ψ satisfies the condition of Theorem B. It was also shown in [20, Lemma 2] that

$$\int_{2^k}^{2^{k+1}} |\hat{\psi}(t\xi)|^2 \frac{dt}{t} \le C\|\psi\|^2_{L^s(\mathbb{R}^n)} \min\{1, |2^k\xi|^{1/(2s')}, |2^k\xi|^{-1/(2s')}\} \tag{34}$$

for all $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n$ if $\psi \in L^s(\mathbb{R}^n)$ for some $s > 1$.

On the other hand, if ψ satisfies the condition of Theorem B or C, one can use the arguments as in Stein [22, pp. 63–64] to obtain

$$\sup_{t>0} \big||\psi_t| * f(x)\big| \le \|h\|_{L^1(\mathbb{R}^n)} v_n^{-1} A_\Omega(f)(x), \tag{35}$$

where v_n is the volume of the unit ball in $\mathbb{R}^n$ and

$$A_\Omega(f)(x) = \sup_{t>0} t^{-n} \int_{|y|<t} |f(x-y)|\Omega(y/|y|)dy.$$

One can easily check that

$$A_\Omega(f)(x) \le \int_{S^{n-1}} |\Omega(y')| M_{y'}(f)(x) d\sigma(y'),$$

where $y' = y/|y|$ and $M_{y'}$ is the directional Hardy-Littlewood maximal function along y'. Noting that

$$\|M_{y'}(f)\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$, where the constant $C_p > 0$ is independent of y'. Hence, by Minkowski's inequality and (35), it holds that

$$\Big\| \sup_{t>0} \big||\psi_t| * f\big| \Big\|_{L^p(\mathbb{R}^n)} \le C_p \|\Omega\|_{L^1(S^{n-1})} \|f\|_{L^p(\mathbb{R}^n)} \tag{36}$$

for all $1 < p < \infty$.

We also get from (35) that

$$\sup_{t>0} \big|\widetilde{|\psi_t|} * f(x)\big| \le \|h\|_{L^1(\mathbb{R}^n)} v_n^{-1} A_\Omega(\tilde{f})(-x) = \|h\|_{L^1(\mathbb{R}^n)} v_n^{-1} M_\Omega(f)(x), \tag{37}$$

where M_Ω is given as in Proposition 2.2. By (33)-(37) and applying Proposition 2.1, we get (32). This proves Proposition 2.2. □

3. Proofs of Theorems 1.2 and 1.3

This section is devoted to presenting the proofs of Theorems 1.2 and 1.3. To prove our theorems, we also need the following criterion about the boundedness of the operators on the mixed radial-angular spaces.

Proposition 3.1 ([16])**.** *Let $1 < q < \infty$, $\delta \in [1, \infty)$ and $s_0 \in [1, \infty)$. Let T be a sublinear operator such that*

$$\|Tf\|_{L^q(u)} \le C_{q,s,s_0} \|f\|_{L^q(\Theta_s(u))}$$

for all $s \in (s_0, \infty)$ and any nonnegative measurable function u on $\mathbb{R}^n$, where the operator Θ_s satisfies

$$\|\Theta_s(f)\|_{L^r(\mathbb{R}^n)} \le C_r \|f\|_{L^r(\mathbb{R}^n)}$$

for all $r \in (s\delta, \infty)$ and all radial functions f. Then for any fixed $s \in [s_0, \infty)$ and $p \in (q, \frac{q\delta s}{\delta s - 1})$, the following inequalities hold:

$$\|Tf\|_{L^p_{|x|} L^q_\theta(\mathbb{R}^n)} \le C_{p,q} \|f\|_{L^p_{|x|} L^q_\theta(\mathbb{R}^n)};$$

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |Tf_j|^q \Big)^{1/q} \Big\|_{L^p_{|x|} L^q_\theta(\mathbb{R}^n)} \le C_{p,q} \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^q \Big)^{1/q} \Big\|_{L^p_{|x|} L^q_\theta(\mathbb{R}^n)};$$

$$\Big\| \Big(\sum_{j\in\mathbb{Z}} |Tf_j|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} \le C_{p,q} \Big\| \Big(\sum_{j\in\mathbb{Z}} |f_j|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)}.$$

We now turn to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $\tilde{\sigma}_t$ be given as in Proposition 2.1. Fix $s > 1$. By (2) and the L^p bounds for M, we get

$$\|\mathrm{M}_s(\tilde{\sigma}_s(u))\|_{L^r(\mathbb{R}^n)} \leq C_p \|u\|_{L^r(\mathbb{R}^n)} \tag{38}$$

for any $r > s$. By (38), Propositions 2.1 and 3.1, we have that inequalities (2)-(4) hold for the case $1 < q < 2$ and $q < p < \frac{2q}{2-q}$ or $2 \leq q < p < \infty$. These together with (1) and Theorems B and C yield the conclusions of Theorem 1.2. □

Proof of Theorem 1.3. Theorem 1.3 can be proved by Proposition 2.2 and the arguments similar to those used to derive Theorem 1.2. Actually, Theorem 1.3 follows directly from Theorem 1.2 according to inequalities (33)-(36). □

4. Additional results

As applications of our main results, we shall establish the mixed radial-angular integrability for Littlewood-Paley g_λ^* function and Littlewood-Paley function related to the area integral S, which are defined as

$$g_{\psi,\lambda}^*(f)(x) = \Big(\int\int_{\mathbb{R}_+^{n+1}} \Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} |\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\lambda > 0$ and $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0,\infty)$, and

$$g_{\psi,S}(f)(x) := \Big(\iint_{\Gamma(x)} |\psi_t * f(y)|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : \ |x-y| < t\}$.

The main result of this section can be listed as follows:

Theorem 4.1. *Suppose that ψ satisfies the condition of Theorem* B *or* C. *Then for $\lambda > 1$ and $2 \leq p < \infty$, the following inequalities hold*:

$$\|g_{\psi,\lambda}^*(f)\|_{L_{|x|}^p L_\theta^2(\mathbb{R}^n)} \leq C_p \|f\|_{L_{|x|}^p L_\theta^2(\mathbb{R}^n)}; \tag{39}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |g_{\psi,\lambda}^*(f_j)|^2\Big)^{1/2}\Big\|_{L_{|x|}^p L_\theta^2(\mathbb{R}^n)} \leq C_p \Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^2\Big)^{1/2}\Big\|_{L_{|x|}^p L_\theta^2(\mathbb{R}^n)}; \tag{40}$$

$$\Big\|\Big(\sum_{j\in\mathbb{Z}} |g_{\psi,\lambda}^*(f_j)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \leq C_p \Big\|\Big(\sum_{j\in\mathbb{Z}} |f_j|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)}. \tag{41}$$

The same results hold for the operator $g_{\psi,S}$.

In order to prove Theorem 4.1, we need the following lemma, which can be proved by the arguments similar to those used in deriving [17, Lemma7].

Lemma 4.2. *Let $\lambda > 1$. Then there exists a constant $C_{\lambda,n}$ such that for any nonnegative locally integrable function g on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} (g^*_{\psi,\lambda}(f)(x))^2 g(x)dx \le C_{\lambda,n} \int_{\mathbb{R}^n} (g_\psi(f)(x))^2 \mathrm{M}(g)(x)dx.$$

Next we prove Theorem 4.1.

Proof of Theorem 4.1. We shall adopt the method in [15] to prove (39) since (40) and (41) are analogues. Fix $2 \le p < \infty$ and set $q = (p/2)'$. Denote by X the set of all Schwartz functions h defined on $\mathbb{R}$ with $\int_0^\infty h(r)^{\tilde{p}_0} r^{n-1} dr \le 1$. We can write

$$\begin{aligned} \|g^*_{\psi,\lambda}(f)\|^2_{L^p_{|x|}L^2_\theta(\mathbb{R}^n)} &= \Big(\int_0^\infty \Big(\int_{\mathrm{S}^{n-1}} \big(g^*_{\psi,\lambda}(f)(r\theta)\big)^2 d\sigma(\theta)\Big)^{p/2} r^{n-1} dr\Big)^{2/p} \\ &= \sup_{h\in X} \int_0^\infty \int_{\mathrm{S}^{n-1}} \big(g^*_{\psi,\lambda}(f)(r\theta)\big)^2 h(r) r^{n-1} d\sigma(\theta)dr \qquad (42) \\ &= \sup_{h\in X} \int_{\mathbb{R}^n} \big(g^*_{\psi,\lambda}(f)(x)\big)^2 h(|x|)dx. \end{aligned}$$

For $g \in X$, let

$$I(h) := \int_{\mathbb{R}^n} \big(g^*_{\psi,\lambda}(f)(x)\big)^2 h(|x|)dx.$$

Invoking Lemma 4.2 and using Hölder's inequality and Theorem 1.2, we get

$$\begin{aligned} I(h) &\le C_{\lambda,n} \int_{\mathbb{R}^n} \big(g_\psi(f)(x)\big)^2 \mathrm{M}(h)(x)dx \\ &= C_{\lambda,n} \int_0^\infty \int_{\mathrm{S}^{n-1}} \big(g_\psi(f)(r\theta)\big)^2 d\sigma(\theta) \mathrm{M}(h)(r) r^{n-1} dr \\ &\le C_{\lambda,n} \Big(\int_0^\infty \Big(\int_{\mathrm{S}^{n-1}} \big(g_\psi(f)(r\theta)\big)^2 d\sigma(\theta)\Big)^{p/2} r^{n-1} dr\Big)^{2/p} \\ &\quad \times \Big(\int_0^\infty (\mathrm{M}(h)(r))^q r^{n-1} dr\Big)^{1/q} \\ &\le C_{p,\lambda,n} \|g_\psi(f)\|^2_{L^p_{|x|}L^2_\theta(\mathbb{R}^n)} \|\mathrm{M}(h)\|_{L^q(\mathbb{R}^n)} \\ &\le C_{p,\lambda,n} \|f\|^2_{L^p_{|x|}L^2_\theta(\mathbb{R}^n)}. \end{aligned}$$

This together with (42) yields (39).

On the other hand, it is easy to check that

$$g_{\psi,S}(f)(x) \le C_\lambda g^*_{\psi,\lambda}(f)(x).$$

Combining this with (39)-(41) leads to the conclusions for $g_{\psi,S}$. Then theorem 4.1 is proved. □

Acknowledgements. The author wants to express her sincerely thanks to the referee for his or her valuable remarks and suggestions, which made this paper more readable.

References

[1] H. Al-Qassem, L. Cheng, and Y. Pan, *On generalized Littlewood-Paley functions*, Collect. Math. **69** (2018), no. 2, 297–314. https://doi.org/10.1007/s13348-017-0208-4

[2] A. Benedek, A.-P. Calderón, and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 356–365. https://doi.org/10.1073/pnas.48.3.356

[3] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976.

[4] F. Cacciafesta and P. D'Ancona, *Endpoint estimates and global existence for the nonlinear Dirac equation with potential*, J. Differential Equations **254** (2013), no. 5, 2233–2260. https://doi.org/10.1016/j.jde.2012.12.002

[5] F. Cacciafesta and R. Lucà, *Singular integrals with angular integrability*, Proc. Amer. Math. Soc. **144** (2016), no. 8, 3413–3418. https://doi.org/10.1090/proc/13123

[6] L. C. Cheng, *On Littlewood-Paley functions*, Proc. Amer. Math. Soc. **135** (2007), no. 10, 3241–3247. https://doi.org/10.1090/S0002-9939-07-08917-4

[7] R. R. Coifman and R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc. **79** (1980), no. 2, 249–254. https://doi.org/10.2307/2043245

[8] A. Córdoba, *Singular integrals, maximal functions and Fourier restriction to spheres: the disk multiplier revisited*, Adv. Math. **290** (2016), 208–235.

[9] P. D'Ancona and R. Lucà, *On the regularity set and angular integrability for the Navier-Stokes equation*, Arch. Ration. Mech. Anal. **221** (2016), no. 3, 1255–1284. https://doi.org/10.1007/s00205-016-0982-2

[10] J. Duoandikoetxea, *Sharp L^p boundedness for a class of square functions*, Rev. Mat. Complut. **26** (2013), no. 2, 535–548. https://doi.org/10.1007/s13163-012-0106-y

[11] J. Duoandikoetxea and O. Oruetxebarria, *Weighted mixed-norm inequalities through extrapolation*, Math. Nachr. **292** (2019), no. 7, 1482–1489. https://doi.org/10.1002/mana.201800311

[12] D. Fan and S. Sato, *Remarks on Littlewood-Paley functions and singular integrals*, J. Math. Soc. Japan **54** (2002), no. 3, 565–585. https://doi.org/10.2969/jmsj/1191593909

[13] S. Hofmann, *Weighted norm inequalities and vector valued inequalities for certain rough operators*, Indiana Univ. Math. J. **42** (1993), no. 1, 1–14. https://doi.org/10.1512/iumj.1993.42.42001

[14] F. Liu, *A note of Littlewood-Paley functions on Triebel-Lizorkin spaces*, Bull. Korean Math. Soc. **55** (2018), no. 2, 659–672. https://doi.org/10.4134/BKMS.b170212

[15] F. Liu and D. Fan, *Weighted estimates for rough singular integrals with applications to angular integrability*, Pacific J. Math. **301** (2019), no. 1, 267–295. https://doi.org/10.2140/pjm.2019.301.267

[16] F. Liu, R. Liu, and H. Wu, *Weighted estimates for rough singular integrals with applications to angular integrability, II*, Math. Inequal. Appl. **23** (2020), no. 1, 393–418. https://doi.org/10.7153/mia-2020-23-31

[17] F. Liu, H. Wu, and D. Zhang, *L^p bounds for parametric Marcinkiewicz integrals with mixed homogeneity*, Math. Inequal. Appl. **18** (2015), no. 2, 453–469. https://doi.org/10.7153/mia-18-34

[18] S. Machihara, M. Nakamura, K. Nakanishi, and T. Ozawa, *Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation*, J. Funct. Anal. **219** (2005), no. 1, 1–20. https://doi.org/10.1016/j.jfa.2004.07.005

[19] S. Sato, *Remarks on square functions in the Littlewood-Paley theory*, Bull. Austral. Math. Soc. **58** (1998), no. 2, 199–211. https://doi.org/10.1017/S0004972700032172

[20] ———, *Estimates for Littlewood-Paley functions and extrapolation*, Integral Equations Operator Theory **62** (2008), no. 3, 429–440. https://doi.org/10.1007/s00020-008-1631-4

[21] E. M. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. **88** (1958), 430–466. https://doi.org/10.2307/1993226

[22] ______, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[23] ______, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, 1993.

[24] T. Tao, *Spherically averaged endpoint Strichartz estimates for the two-dimensional Schrödinger equation*, Comm. Partial Differential Equations **25** (2000), no. 7-8, 1471–1485. https://doi.org/10.1080/03605300008821556

[25] C. Zhang and J. Chen, *Boundedness of g-functions on Triebel-Lizorkin spaces*, Taiwanese J. Math. **13** (2009), no. 3, 973–981. https://doi.org/10.11650/twjm/1500405452

Xiao Zhang
College of Mathematics and Systems Science
Shandong University of Science and Technology
Qingdao 266590, P. R. China
Email address: skd994405@sdust.edu.cn