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DUALITIES OF VARIABLE ANISOTROPIC HARDY SPACES AND BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS

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Abstract. Let A be an expansive dilation on $\mathbb{R}^n$, and $p(\cdot):\mathbb{R}^n\to(0,\,\infty)$ be a variable exponent function satisfying the globally log-Hölder continuous condition. Let $H_A^{p(\cdot)}(\mathbb{R}^n)$ be the variable anisotropic Hardy space defined via the non-tangential grand maximal function. In this paper, the author obtains the boundedness of anisotropic convolutional δ-type Calderón-Zygmund operators from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ or from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to itself. In addition, the author also obtains the duality between $H_A^{p(\cdot)}(\mathbb{R}^n)$ and the anisotropic Campanato spaces with variable exponents.

1. Introduction

As a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ when $p\in(0,1]$, Hardy space $H^p(\mathbb{R}^n)$ plays an important role in various fields of analysis and partial differential equations; see, for examples, [10, 17–19, 21, 23]. On the other hand, variable exponent function spaces have their applications in fluid dynamics [1], image processing [3], partial differential equations and variational calculus [9, 20, 21].

Let $p(\cdot):\mathbb{R}^n\to(0,\,\infty)$ be a variable exponent function satisfying the globally log-Hölder continuous condition (see Section 2 below for its definition). Recently, Nakai and Sawano [15] introduced the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, via the radial grand maximal function, and then obtained some real-variable characterizations of the space, such as the characterizations in terms of the atomic and the molecular decompositions. Moreover, they obtained the boundedness of δ-type Calderón-Zygmund operators from $H^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ or from $H^{p(\cdot)}(\mathbb{R}^n)$ to itself. Then Sawano [16], Yang et al. [22] and Zhuo et al. [24] further contributed to the theory.

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Very recently, Liu et al. [12] introduced the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ associated with a general expansive matrix A, via the non-tangential grand maximal function, and then established its various real-variable characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$, respectively, in terms of the atomic characterization and the Littlewood-Paley characterization.

To complete the theory of the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$, in this article, as applications of the atomic characterization, we obtain the boundedness of anisotropic convolutional δ-type Calderón-Zygmund operators from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ and from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to itself. In addition, we also obtain the dual space of $H_A^{p(\cdot)}(\mathbb{R}^n)$ is the anisotropic Campanato space with variable exponents.

The rest of this paper is organized as follows.

In Section 2, we first recall some notation and definitions concerning expansive dilations, variable exponent, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$, via the non-tangential grand maximal function.

Section 3 is devoted to getting the boundedness of anisotropic convolutional δ-type Calderón-Zygmund operators from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ and from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to itself, by using the atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$ established in [12, Theorem 4.8] (see also Lemma 3.3 below). It is worth pointing out that some of the proof methods of the boundedness of Calderón-Zygmund operators T on $H_A^p(\mathbb{R}^n) = H_A^{p,\,p}(\mathbb{R}^n)$ ([13, Theorem 3.9]) and $H^{p(\cdot)}(\mathbb{R}^n)$ ([15, Theorem 5.2]) don't work anymore in the present setting. For example, we search out some estimates related to $L^{p(\cdot)}(\mathbb{R}^n)$ norms for some series of functions which can be reduced into dealing with the $L^q(\mathbb{R}^n)$ norms of the corresponding functions (see Lemma 3.6 below).

In Section 4, we prove that the dual space of $H_A^{p(\cdot)}(\mathbb{R}^n)$ is the anisotropic Campanato space with variable exponents (see Theorem 4.4 below). For this purpose, we first introduce a new kind of anisotropic Campanato spaces with variable exponents $\mathcal{L}_A^{p(\cdot),\,q,\,s}(\mathbb{R}^n)$ in Definition 4.1 below, which includes the anisotropic Campanato space of Bownik (see [2, p. 50, Definition 8.1]) and the space $BMO(\mathbb{R}^n)$ of John and Nirenberg [11].

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. For any $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n := (\mathbb{Z}_+)^n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and

$$\partial^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. For any $q \in [1, \infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$. For any $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the *maximal integer* not larger than a. The *symbol* $D \lesssim F$ means that $D \le CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its *characteristic function*. If there are no special instructions, any space $\mathcal{X}(\mathbb{R}^n)$ is denoted simply by $\mathcal{X}$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2. Denote by $\mathcal{S}$ the *space of all Schwartz functions* and $\mathcal{S}'$ its *dual space* (namely, the *space of all tempered distributions*).

2. Variable anisotropic Hardy space $H_A^{p(\cdot)}$

In this section, we first recall the notion of variable anisotropic Hardy space $H_A^{p(\cdot)}$, via the non-tangential grand maximal function $M_N(f)$, and then given its molecular characterization.

We begin with recalling the notion of expansive dilations on $\mathbb{R}^n$; see [2, p. 5]. A real $n \times n$ matrix A is called an *expansive dilation*, shortly a *dilation*, if $\min_{\lambda\in\sigma(A)} |\lambda| > 1$, where $\sigma(A)$ denotes the set of all *eigenvalues* of A. Let λ_- and λ_+ be two *positive numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \le \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

In the case when A is diagonalizable over $\mathbb{C}$, we can even take $\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\}$ and $\lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}$. Otherwise, we need to choose them sufficiently close to these equalities according to what we need in our arguments.

It was proved in [2, p. 5, Lemma 2.2] that, for a given dilation A, there exist a number $r \in (1, \infty)$ and a set $\Delta := \{x \in \mathbb{R}^n : |Px| < 1\}$, where P is some non-degenerate $n \times n$ matrix, such that $\Delta \subset r\Delta \subset A\Delta$, and one can additionally assume that $|\Delta| = 1$, where $|\Delta|$ denotes the n-dimensional Lebesgue measure of the set Δ. Let $B_k := A^k\Delta$ for $k \in \mathbb{Z}$. Then B_k is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$, here and hereafter, $b := |\det A|$. An ellipsoid $x + B_k$ for some $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ is called a *dilated ball*. Denote by $\mathfrak{B}$ the set of all such dilated balls, namely,

$$\mathfrak{B} := \{x + B_k : x \in \mathbb{R}^n,\ k \in \mathbb{Z}\}. \tag{2.1}$$

Throughout the whole paper, let σ be the *smallest integer* such that $2B_0 \subset A^\sigma B_0$ and, for any subset E of $\mathbb{R}^n$, let $E^{\complement} := \mathbb{R}^n \setminus E$. Then, for all $k, j \in \mathbb{Z}$ with $k \le j$, it holds true that

$$B_k + B_j \subset B_{j+\sigma}, \tag{2.2}$$

$$B_k + (B_{k+\sigma})^{\complement} \subset (B_k)^{\complement}, \tag{2.3}$$

where $E+F$ denotes the *algebraic sum* $\{x+y : x \in E,\ y \in F\}$ of sets $E, F \subset \mathbb{R}^n$.

Definition 2.1. A *quasi-norm*, associated with dilation A, is a Borel measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$, for simplicity, denoted by ρ, satisfying

(i) $\rho(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, here and hereafter, $\vec{0}_n$ denotes the origin of $\mathbb{R}^n$;
(ii) $\rho(Ax) = b\rho(x)$ for all $x \in \mathbb{R}^n$, where, as above, $b := |\det A|$;
(iii) $\rho(x + y) \le H[\rho(x) + \rho(y)]$ for all $x, y \in \mathbb{R}^n$, where $H \in [1, \infty)$ is a constant independent of x and y.

In the standard dyadic case $A := 2\mathrm{I}_{n\times n}$, $\rho(x) := |x|^n$ for all $x \in \mathbb{R}^n$ is an example of homogeneous quasi-norms associated with A, here and hereafter, $\mathrm{I}_{n\times n}$ denotes the $n \times n$ *unit matrix*, $|\cdot|$ always denotes the *Euclidean norm* in $\mathbb{R}^n$.

It was proved, in [2, p. 6, Lemma 2.4], that all homogeneous quasi-norms associated with a given dilation A are equivalent. Therefore, for a given dilation A, in what follows, for simplicity, we always use the *step homogeneous quasi-norm* ρ defined by setting, for all $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{k\in\mathbb{Z}} b^k \chi_{B_{k+1}\setminus B_k}(x) \text{ if } x \neq \vec{0}_n, \quad \text{or else} \quad \rho(\vec{0}_n) := 0.$$

By (2.2), we know that, for all $x,\, y \in \mathbb{R}^n$,

$$\rho(x+y) \le b^\sigma \left(\max\{\rho(x),\, \rho(y)\}\right) \le b^\sigma[\rho(x)+\rho(y)];$$

see [2, p. 8]. Moreover, $(\mathbb{R}^n,\, \rho,\, dx)$ is a space of homogeneous type in the sense of Coifman and Weiss [4, 5], where dx denotes the *n-dimensional Lebesgue measure.*

Now we recall that a measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ is called a *variable exponent.* For any variable exponent $p(\cdot)$, let

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x). \tag{2.4}$$

Denote by $\mathcal{P}$ the set of all variable exponents $p(\cdot)$ satisfying $0 < p_- \le p_+ < \infty$.

Let f be a measurable function on $\mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}$. Then the *modular function* (or, for simplicity, the *modular*) $\varrho_{p(\cdot)}$, associated with $p(\cdot)$, is defined by setting

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

and the *Luxemburg* (also called *Luxemburg-Nakano*) *quasi-norm* $\|f\|_{L^{p(\cdot)}}$ by

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda \in (0, \infty) : \varrho_{p(\cdot)}(f/\lambda) \le 1\right\}.$$

Moreover, the *variable Lebesgue space* $L^{p(\cdot)}$ is defined to be the set of all measurable functions f satisfying that $\varrho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}}$.

The following remark comes from [14, Remark 2.3].

Remark 2.2. Let $p(\cdot) \in \mathcal{P}$.

(i) Obviously, for any $r \in (0, \infty)$ and $f \in L^{p(\cdot)}$,

$$\||f|^r\|_{L^{p(\cdot)}} = \|f\|^r_{L^{rp(\cdot)}}.$$

Moreover, for any $\mu \in \mathbb{C}$ and $f, g \in L^{p(\cdot)}$, $\|\mu f\|_{L^{p(\cdot)}} = |\mu|\, \|f\|_{L^{p(\cdot)}}$ and

$$\|f+g\|^{\underline{p}}_{L^{p(\cdot)}} \le \|f\|^{\underline{p}}_{L^{p(\cdot)}} + \|g\|^{\underline{p}}_{L^{p(\cdot)}},$$

here and hereafter,

$$\underline{p} := \min\{p_-,\, 1\} \tag{2.5}$$

with p_- as in (2.4). In particular, when $p_- \in [1, \infty]$, $L^{p(\cdot)}$ is a Banach space (see [8, Theorem 3.2.7]).

(ii) It was proved in [6, Proposition 2.21] that, for any function $f \in L^{p(\cdot)}$ with $\|f\|_{L^{p(\cdot)}} > 0$, $\varrho_{p(\cdot)}(f/\|f\|_{L^{p(\cdot)}}) = 1$ and, in [6, Corollary 2.22] that, if $\|f\|_{L^{p(\cdot)}} \le 1$, then $\varrho_{p(\cdot)}(f) \le \|f\|_{L^{p(\cdot)}}$.

A function $p(\cdot) \in \mathcal{P}$ is said to satisfy the *globally log-Hölder continuous condition*, denoted by $p(\cdot) \in C^{\log}$, if there exist two positive constants $C_{\log}(p)$ and C_∞, and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e + 1/\rho(x-y))}$$

and

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + \rho(x))}.$$

A C^∞ function φ is said to belong to the Schwartz class $\mathcal{S}$ if, for every integer $\ell \in \mathbb{Z}_+$ and multi-index α, $\|\varphi\|_{\alpha,\ell} := \sup_{x\in\mathbb{R}^n} [\rho(x)]^\ell |\partial^\alpha \varphi(x)| < \infty$. The dual space of $\mathcal{S}$, namely, the space of all tempered distributions on $\mathbb{R}^n$ equipped with the weak-$*$ topology, is denoted by $\mathcal{S}'$. For any $N \in \mathbb{Z}_+$, let

$$\mathcal{S}_N := \{\varphi \in \mathcal{S} : \|\varphi\|_{\alpha,\ell} \le 1,\ |\alpha| \le N,\ \ell \le N\}.$$

In what follows, for $\varphi \in \mathcal{S}$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\varphi_k(x) := b^{-k}\varphi\left(A^{-k}x\right)$.

Definition 2.3. Let $\varphi \in \mathcal{S}$ and $f \in \mathcal{S}'$. The *non-tangential maximal function* $M_\varphi(f)$ with respect to φ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\varphi(f)(x) := \sup_{y\in x+B_k, k\in\mathbb{Z}} |f * \varphi_k(y)|.$$

Moreover, for any given $N \in \mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ of $f \in \mathcal{S}'$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N} M_\varphi(f)(x).$$

The following variable anisotropic Hardy space $H_A^{p(\cdot)}$ was introduced in [12, Definition 2.4].

Definition 2.4. Let $p(\cdot) \in C^{\log}$, A be a dilation and $N \in [\lfloor(1/\underline{p}-1)/\ln\lambda_-\rfloor + 2, \infty)$, where $\underline{p}$ is as in (2.5). The *variable anisotropic Hardy space* $H_A^{p(\cdot)}$ is defined as

$$H_A^{p(\cdot)} := \left\{f \in \mathcal{S}' : M_N(f) \in L^{p(\cdot)}\right\}$$

and, for any $f \in H_A^{p(\cdot)}$, let $\|f\|_{H_A^{p(\cdot)}} := \|M_N(f)\|_{L^{p(\cdot)}}$.

Remark 2.5. Let $p(\cdot) \in C^{\log}$.

(i) The quasi-norm of $H_A^{p(\cdot)}$ in Definition 2.4 depends on N, however, by [12, Theorem 3.10], we know that the $H_A^{p(\cdot)}$ is independent of the choice of N, as long as $N \in [\lfloor (1/\underline{p}-1)/\ln\lambda_- \rfloor + 2, \infty)$.

(ii) When $p(\cdot) := p$, where $p \in (0, \infty)$, the space $H_A^{p(\cdot)}$ is reduced to the anisotropic Hardy H_A^p studied in [2, Definition 3.11].

(iii) When $A := 2\mathrm{I}_{n\times n}$, the space $H_A^{p(\cdot)}$ is reduced to the variable Hardy space $H^{p(\cdot)}$ studied in [15, p. 3674].

We begin with the following notion of anisotropic $(p(\cdot), q, s)$-atoms introduced in [14, Definition 4.1].

Definition 2.6. Let $p(\cdot) \in \mathcal{P}$, $q \in (1, \infty]$ and $s \in [\lfloor (1/p_- - 1)\ln b/\ln\lambda_- \rfloor, \infty) \cap \mathbb{Z}_+$ with p_- as in (2.4). An *anisotropic* $(p(\cdot), q, s)$*-atom* is a measurable function a on $\mathbb{R}^n$ satisfying

(i) $\operatorname{supp} a \subset B$, where $B \in \mathfrak{B}$ and $\mathfrak{B}$ is as in (2.1);

(ii) $\|a\|_{L^q} \le \frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le s$.

Throughout this article, we call an anisotropic $(p(\cdot), q, s)$-atom simply by a $(p(\cdot), q, s)$-atom. The following variable anisotropic atomic Hardy space was introduced in [12, Definition 4.2].

Definition 2.7. Let $p(\cdot) \in C^{\log}$, $q \in (1, \infty]$, $s \in [\lfloor (1/p_- - 1)\ln b/\ln\lambda_- \rfloor, \infty) \cap \mathbb{Z}_+$ with p_- as in (2.4), and A be a dilation. The *variable anisotropic atomic Hardy space* $H_A^{p(\cdot), q, s}$ is defined to be the set of all distributions $f \in \mathcal{S}'$ satisfying that there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot), q, s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, on $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in} \quad \mathcal{S}'.$$

Moreover, for any $f \in H_A^{p(\cdot), q, s}$, let

$$\|f\|_{H_A^{p(\cdot), q, s}} := \inf \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}},$$

where the infimum is taken over all the decompositions of f as above.

3. Calderón-Zygmund operators on $H_A^{p(\cdot)}$

In this section, we obtain the boundedness of anisotropic convolutional δ-type Calderón-Zygmund operators from $H_A^{p(\cdot)}$ to $L^{p(\cdot)}$ or from $H_A^{p(\cdot)}$ to itself. Let us begin with the notion of anisotropic Calderón-Zygmund operators associated with dilation A.

Let $\delta \in (0, \frac{\ln\lambda_+}{\ln b})$. We call a linear operator T is an anisotropic convolutional δ-type Calderón-Zygmund operator, if T is bounded on L^2 with kernel $k \in \mathcal{S}'$

coinciding with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$, and satisfying that there exists a positive constant C such that, for any $x, y \in \mathbb{R}^n$ with $\rho(x) > b^{2\sigma}\rho(y)$,

$$|k(x-y) - k(x)| \le C\frac{[\rho(y)]^\delta}{[\rho(x)]^{1+\delta}}.$$

For any $f \in L^2$, $T(f) := \text{p.v.}k * f(x)$.

Theorem 3.1. *Let $p(\cdot) \in C^{\log}$ and $\delta \in (0, \frac{\ln \lambda_+}{\ln b})$. Assume T is an anisotropic convolutional δ-type Calderón-Zygmund operator. If $p_- \in (\frac{1}{1+\delta}, 1)$ with p_- as in* (2.4)*, then T can be extended to a bounded linear operator from $H_A^{p(\cdot)}$ to $L^{p(\cdot)}$ and from $H_A^{p(\cdot)}$ to $H_A^{p(\cdot)}$. Moreover, there exists a positive constant C such that, for any $H_A^{p(\cdot)}$,*

(i) $\|T(f)\|_{L^{p(\cdot)}} \le C\|f\|_{H_A^{p(\cdot)}}$;
(ii) $\|T(f)\|_{H_A^{p(\cdot)}} \le C\|f\|_{H_A^{p(\cdot)}}$.

Remark 3.2. When $A := 2\mathrm{I}_{n\times n}$ and T is a Calderon-Zygmund operator of convolution type, Theorem 3.1 coincides with [15, Proposition 5.3, Theorem 5.5] of Nakai and Sawano, respectively.

To prove Theorem 3.1, we need some technical lemmas. The following lemma reveals the atomic characterization of the variable anisotropic Hardy space (see [12, Theorem 4.8]).

Lemma 3.3. *Let $p(\cdot) \in C^{\log}$, $q \in (\max\{p_+, 1\}, \infty]$ with p_+ as in* (2.4)*, $s \in [\lfloor(1/p_- - 1)\ln b/\ln\lambda_-\rfloor, \infty) \cap \mathbb{Z}_+$ with p_- as in* (2.4) *and $N \in \mathbb{N} \cap [\lfloor(1/\underline{p} - 1)\ln b/\ln\lambda_-\rfloor + 2, \infty)$. Then*

$$H_A^{p(\cdot)} = H_A^{p(\cdot), q, s}$$

with equivalent quasi-norms.

By the proof of [12, Theorem 4.8], we obtain the following conclusion, which plays an important role in the section.

Lemma 3.4. *Let $p(\cdot) \in C^{\log}$, $r \in (1, \infty]$ and $s \in [\lfloor(1/p_- - 1)\ln b/\ln\lambda_-\rfloor, \infty) \cap \mathbb{Z}_+$ with p_- as in* (2.4)*. Then, for any $f \in H_A^{p(\cdot)} \cap L^r$, there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$, dilated balls $\{x_i + B_{\ell_i}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ and $(p(\cdot), \infty, s)$-atoms $\{a_i\}_{i\in\mathbb{N}}$ such that*

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \text{ in } L^r \text{ and } H_A^{p(\cdot)},$$

where the series also converges almost everywhere.

Proof. Let $f \in H_A^{p(\cdot)} \cap L^r$. For any $k \in \mathbb{Z}$, by the proof of [12, Theorem 4.8], we know that there exist $\{x_i^k\}_{i\in\mathbb{N}} \subset \Omega_k = \{x \in \mathbb{R}^n : M_N f(x) > 2^k\}$,

$\{\ell_i^k\}_{i\in\mathbb{N}} \subset \mathbb{Z}$, a sequence of $(p(\cdot),\infty,s)$-atoms, $\{a_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}}$, supported on $\{x_i^k + B_{\ell_i^k+4\sigma}\}_{k\in\mathbb{Z},i\in\mathbb{N}}$, respectively, and $\{\lambda_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}} \subset \mathbb{C}$, such that

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k =: \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} h_i^k \quad \text{in} \quad \mathcal{S}', \tag{3.1}$$

and for any $k\in\mathbb{Z}$ and $i\in\mathbb{N}$, $\operatorname{supp} h_i^k \subset x_i^k + B_{\ell_i^k+4\sigma} \subset \Omega_k$,

$$\|h_i^k\|_{L^\infty} \lesssim 2^k \quad \text{and} \quad \sharp\{j\in\mathbb{N} : (x_i^k + B_{\ell_i^k+4\sigma}) \cap (x_j^k + B_{\ell_j^k+4\sigma}) \neq \emptyset\} \le R, \tag{3.2}$$

where R is as in [12, Lemma 4.5]. Moreover, by $f \in H_A^{p(\cdot)} \cap L^r$, we have, for almost every $x \in \Omega_k$, there exists a $k(x)\in\mathbb{Z}$ such that $2^{k(x)} < M_N f(x) \le 2^{k(x)+1}$. From this, $\operatorname{supp} h_i^k \subset \Omega_k$ and (3.2), we deduce that, for a.e. $x\in\mathbb{R}^n$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} |h_i^k(x)| &\sim \sum_{k\in(-\infty,k(x)]\cap\mathbb{Z}} \sum_{i\in\mathbb{N}} |h_i^k(x)| \\ &\lesssim \sum_{k\in(-\infty,k(x)]\cap\mathbb{Z}} \sum_{i\in\mathbb{N}} 2^k \chi_{x_i^k + B_{\ell_i^k+4\sigma}}(x) \\ &\sim \sum_{k\in(-\infty,k(x)]\cap\mathbb{Z}} 2^k \sim 2^{k(x)} \sim M_N f(x).\end{aligned} \tag{3.3}$$

This implies that there exists a subsequence of the series

$$\left\{\sum_{k=-K}^{K} \sum_{i\in\mathbb{Z}} h_i^k\right\}_{K\in\mathbb{N}},$$

denoted still by itself without loss of generality, which converges to some measurable function $\widetilde{f}$ almost everywhere in $\mathbb{R}^n$.

On the other hand, from (3.3), it follows that, for any $K\in\mathbb{N}$ and a.e. $x\in\mathbb{R}^n$,

$$\begin{aligned}\left|\widetilde{f}(x) - \sum_{k=-K}^{K} h_i^k(x)\right| &\lesssim |\widetilde{f}(x)| + \sum_{k\in(-\infty,k(x)]\cap\mathbb{Z}} \sum_{i\in\mathbb{N}} |h_i^k(x)| \\ &\lesssim |\widetilde{f}(x)| + M_N f(x) \lesssim M_N f(x).\end{aligned}$$

From this, the fact that $M_N(f) \in L^r$ with $1 < r \le \infty$, and the Lebesgue dominated convergence theorem, we further deduce that $\widetilde{f} = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} h_i^k$ in L^r. By this and (3.3), we know $f = \widetilde{f} \in L^r$ and hence

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} h_i^k \quad \text{in} \quad L^r \text{ and } H_A^{p(\cdot)},$$

and also almost everywhere. □

We recall the definition of anisotropic Hardy-Littlewood maximal function $M_{HL}(f)$. For any $f \in L^1_{\rm loc}$ and $x \in \mathbb{R}^n$,

$$M_{HL}(f)(x) := \sup_{x\in B\in\mathfrak{B}} \frac{1}{|B|}\int_B |f(z)|\,dz, \tag{3.4}$$

where $\mathfrak{B}$ is as in (2.1).

The following lemma is just [14, Lemma 4.3].

Lemma 3.5. *Let $q \in (1, \infty]$. Assume that $p(\cdot) \in C^{\log}$ satisfies $1 < p_- \le p_+ < \infty$, where p_- and p_+ are as in* (2.4). *Then there exists a positive constant C such that, for any sequence $\{f_k\}_{k\in\mathbb{N}}$ of measurable functions,*

$$\left\|\left\{\sum_{k\in\mathbb{N}}[M_{HL}(f_k)]^q\right\}^{1/q}\right\|_{L^{p(\cdot)}} \le C\left\|\left(\sum_{k\in\mathbb{N}}|f_k|^q\right)^{1/q}\right\|_{L^{p(\cdot)}}$$

with the usual modification made when $q = \infty$, where M_{HL} denotes the Hardy-Littlewood maximal operator as in (3.4).

Let us recall some auxiliary estimates together with the completeness of function spaces. The following Lemma 3.6, Lemma 3.7 and Lemma 3.8, respectively, come from [12, Lemma 4.6, Lemma 4.7] and [6, Lemma 2.71].

Lemma 3.6. *Let $p(\cdot) \in C^{\log}$ and $q \in (1, \infty] \cap (p_+, \infty]$ with p_+ as in* (2.4). *Assume that $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$, $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ and $\{a_i\}_{i\in\mathbb{N}} \in L^q$ satisfy, for any $i \in \mathbb{N}$, $\operatorname{supp} a_i \subset B^{(i)}$,*

$$\|a\|_{L^q} \le \frac{|B^{(i)}|^{1/q}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}}}$$

and

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}} < \infty.$$

Then

$$\left\|\left[\sum_{i\in\mathbb{N}}|\lambda_i a_i|^{\underline{p}}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}} \le C\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}},$$

where $\underline{p}$ is as in (2.5) *and C is a positive constant independent of $\{\lambda_i\}_{i\in\mathbb{N}}$, $\{B^{(i)}\}_{i\in\mathbb{N}}$ and $\{a_i\}_{i\in\mathbb{N}}$.*

Lemma 3.7. *Let $p(\cdot) \in C^{\log}$ and $r \in (1, \infty] \cap (p_+, \infty]$ with p_+ as in* (2.4). *Then $H_A^{p(\cdot)} \cap L^r$ is dense in $H_A^{p(\cdot)}$.*

Lemma 3.8. *Given $p(\cdot) \in \mathcal{P}$, $L^{p(\cdot)}$ is complete: every Cauchy sequence in $L^{p(\cdot)}$ converges in norm.*

Lemma 3.9. *Let $p(\cdot) \in C^{\log}$. Then $H_A^{p(\cdot)}$ is complete: every Cauchy sequence in $H_A^{p(\cdot)}$ converges in norm.*

Proof. We show this lemma by borrowing some ideas from the proof of [7, Proposition 4.1]. To prove that $H_A^{p(\cdot)}$ is complete, it is sufficient to prove that if $\{f_k\}_{k=1}^\infty$ is a sequence in $H_A^{p(\cdot)}$ such that

$$\sum_{k\in\mathbb{N}} \|f_k\|_{H_A^{p(\cdot)}}^{\underline{p}} < \infty,$$

where $\underline{p}$ is as in (2.5), then the series $\sum_{k\in\mathbb{N}} f_k$ in $H_A^{p(\cdot)}$ converges in norm. For any $j \in \mathbb{N}$, let $F_j := \sum_{k=1}^j f_k$. From Remark 2.2(i) and the fact that $\underline{p}$ is as in (2.5), we conclude that, for any $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}
\|F_m - F_n\|_{H_A^{p(\cdot)}}^{\underline{p}} &= \left\|\sum_{k=n+1}^m f_k\right\|_{H_A^{p(\cdot)}}^{\underline{p}} \leq \left\|\sum_{k=n+1}^m M_N(f_k)\right\|_{L^{p(\cdot)}}^{\underline{p}} \\
&= \left\|\left[\sum_{k=n+1}^m M_N(f_k)\right]^{\underline{p}}\right\|_{L^{p(\cdot)/\underline{p}}} \leq \left\|\sum_{k=n+1}^m [M_N(f_k)]^{\underline{p}}\right\|_{L^{p(\cdot)/\underline{p}}} \\
&\leq \sum_{k=n+1}^m \|[M_N(f_k)]^{\underline{p}}\|_{L^{p(\cdot)/\underline{p}}} = \sum_{k=n+1}^m \|M_N(f_k)\|_{L^{p(\cdot)}}^{\underline{p}} \\
&= \sum_{k=n+1}^m \|f_k\|_{H_A^{p(\cdot)}}^{\underline{p}}.
\end{aligned}$$

By this, we know that $\{F_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $H_A^{p(\cdot)}$. Since $H_A^{p(\cdot)}$ is continuously contained in $\mathcal{S}'$ (see [12, Lemma 4.3]), thus $\{F_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}'$. Therefore we know that there exists a tempered distribution $f \in \mathcal{S}'$ such that $F_j \to f$ in $\mathcal{S}'$ as $j \to \infty$.

Next we prove $f \in H_A^{p(\cdot)}$. Since

$$M_N(f) \leq \lim_{j\to\infty} \sum_{k=1}^j M_N(f_k),$$

by Remark 2.2(i), we obtain

$$\begin{aligned}
\|M_N(f)\|_{L^{p(\cdot)}}^{\underline{p}} &\leq \left\|\lim_{j\to\infty} \sum_{k=1}^j M_N(f_k)\right\|_{L^{p(\cdot)}}^{\underline{p}} = \lim_{j\to\infty} \left\|\sum_{k=1}^j M_N(f_k)\right\|_{L^{p(\cdot)}}^{\underline{p}} \\
&\leq \lim_{j\to\infty} \left\|\sum_{k=1}^j [M_N(f_k)]^{\underline{p}}\right\|_{L^{p(\cdot)/\underline{p}}} = \sum_{k=1}^\infty \|[M_N(f_k)]^{\underline{p}}\|_{L^{p(\cdot)/\underline{p}}} \\
&= \sum_{k=1}^\infty \|f_k\|_{H_A^{p(\cdot)}}^{\underline{p}} < \infty,
\end{aligned}$$

which implies that $f \in H_A^{p(\cdot)}$.

Finally, by Remark 2.2(i), we find that

$$\begin{aligned}\left\|f-\sum_{k=1}^{j} f_k\right\|_{H_A^{p(\cdot)}}^{\underline{p}} &= \left\|\lim_{s\to\infty}\sum_{k=j+1}^{s} f_k\right\|_{H_A^{p(\cdot)}}^{\underline{p}} \le \lim_{s\to\infty}\left\|\sum_{k=j+1}^{s} f_k\right\|_{H_A^{p(\cdot)}}^{\underline{p}}\\ &\le \sum_{k=j+1}^{\infty}\|f_k\|_{H_A^{p(\cdot)}}^{\underline{p}} \to 0 \quad\text{as } j\to\infty,\end{aligned}$$

which implies that $\{F_j\}_{j\in\mathbb{N}}$ in $H_A^{p(\cdot)}$ converges to $f=\sum_{k\in\mathbb{N}} f_k$ in norm. This finishes the proof of Lemma 3.9. $\square$

Proof of Theorem 3.1. We only prove (i), by using Lemma 3.9, (ii) can be proved in the same way.

First, we show that (i) holds true for any $f\in H_A^{p(\cdot)}\cap L^r$ with $r\in(1,\infty]\cap(p_+,\infty]$. For any $f\in H_A^{p(\cdot)}\cap L^r$, from Lemma 3.4, we know that there exist numbers $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$ and a sequence of $(p(\cdot),\infty,s)$-atom, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, on $\{x_i+B_{\ell_i}\}_{i\in\mathbb{N}}\subset\mathfrak{B}$ such that

$$f=\sum_{i\in\mathbb{N}}\lambda_i a_i \quad\text{in } L^r$$

and

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{x_i+B_{\ell_i}}}{\left\|\chi_{x_i+B_{\ell_i}}\right\|_{L^{p(\cdot)}}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}} \lesssim \|f\|_{H_A^{p(\cdot)}}.$$

By the assumption that the operator T is bounded on L^r, we further have

$$T(f)=\sum_{i\in\mathbb{N}}\lambda_i(Ta_i) \quad\text{in } L^r,$$

and hence in $\mathcal{S}'$. Then, by Remark 2.2(i), we find that, for any $x\in\mathbb{R}^n$,

$$\begin{aligned}\|T(f)\|_{L^{p(\cdot)}}^{\underline{p}} &= \left\|T\left(\sum_{i\in\mathbb{N}}\lambda_i a_i\right)\right\|_{L^{p(\cdot)}}^{\underline{p}}\\ &= \left\|\sum_{i\in\mathbb{N}}\lambda_i T(a_i)\right\|_{L^{p(\cdot)}}^{\underline{p}}\\ &\le \left\|\sum_{i\in\mathbb{N}}|\lambda_i|T(a_i)\chi_{x_i+A^\sigma B_{\ell_i}}\right\|_{L^{p(\cdot)}}^{\underline{p}} + \left\|\sum_{i\in\mathbb{N}}|\lambda_i|T(a_i)\chi_{(x_i+A^\sigma B_{\ell_i})^\complement}\right\|_{L^{p(\cdot)}}^{\underline{p}}\\ &\lesssim \left\|\left\{\sum_{i\in\mathbb{N}}\left[|\lambda_i|T(a_i)\chi_{x_i+A^\sigma B_{\ell_i}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}}^{\underline{p}}\end{aligned}$$

$$+\left\|\sum_{i\in\mathbb{N}}|\lambda_i|T(a_i)\chi_{(x_i+A^\sigma B_{\ell_i})^\complement}\right\|_{L^{p(\cdot)}}^{\underline{p}} =: \mathrm{L}_1+\mathrm{L}_2.$$

For the term L_1, by the boundedness of T on L^r with $r\in(\max\{p_+,1\},\infty)$, and Lemma 3.6, we conclude that

$$\mathrm{L}_1 \lesssim \left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{x_i+B_{\ell_i}}}{\left\|\chi_{x_i+B_{\ell_i}}\right\|_{L^{p(\cdot)}}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}}^{\underline{p}} \lesssim \|f\|_{H_A^{p(\cdot)}}^{\underline{p}}.$$

For the term L_2, assume $a_i(x)$ is a $(p(\cdot),\infty,s)$-atom supported on $x_i+B_{\ell_i}$. From the size condition of $a_i(x)$, we conclude that, for any $x\in(x_i+A^\sigma B_{\ell_i})^\complement$,

$$\begin{aligned}
Ta_i(x) &= k*a_i(x)\\
&\le \int_{x_i+B_{\ell_i+\sigma}}|k(x-y)-k(x-x_i)|\,|a_i(y)|\,dy\\
&\lesssim \int_{x_i+B_{\ell_i+\sigma}}\frac{\rho(y-x_i)^\delta}{\rho(x-x_i)^{1+\delta}}|a_i(y)|\,dy\\
&\lesssim \frac{|x_i+B_{\ell_i}|^\delta}{\rho(x-x_i)^{1+\delta}}\|a_i\|_{L^r}|x_i+B_{\ell_i}|^{1/r'}\\
&\lesssim \frac{|x_i+B_{\ell_i}|^\delta}{\rho(x-x_i)^{1+\delta}}\frac{1}{\|\chi_{B_{\ell_i}}\|_{L^{p(\cdot)}}} \lesssim \left[M_{HL}(\chi_{x_i+B_{\ell_i}})(x)\right]^{1+\delta}\frac{1}{\|\chi_{B_{\ell_i}}\|_{L^{p(\cdot)}}}.
\end{aligned}$$

By this, Remark 2.2(i), Lemma 3.5 and the fact that $\beta:=1+\delta>1/\underline{p}$, we obtain

$$\begin{aligned}
\mathrm{L}_2 &\lesssim \left\|\sum_{i\in\mathbb{N}}\frac{|\lambda_i|}{\|\chi_{x_i+B_{\ell_i}}\|_{L^{p(\cdot)}}}\left[M_{HL}(\chi_{x_i+B_{l_i}})\right]^\beta\right\|_{L^{p(\cdot)}}^{\underline{p}}\\
&\sim \left\|\left\{\sum_{i\in\mathbb{N}}\frac{|\lambda_i|}{\|\chi_{x_i+B_{\ell_i}}\|_{L^{p(\cdot)}}}\left[M_{HL}(\chi_{x_i+B_{\ell_i}})\right]^\beta\right\}^{1/\beta}\right\|_{L^{\beta p(\cdot)}}^{\beta\underline{p}}\\
&\lesssim \left\|\left\{\sum_{i\in\mathbb{N}}\frac{|\lambda_i|\chi_{x_i+B_{\ell_i}}}{\|\chi_{x_i+B_{\ell_i}}\|_{L^{p(\cdot)}}}\right\}^{1/\beta}\right\|_{L^{\beta p(\cdot)}}^{\beta\underline{p}} \sim \left\|\sum_{i\in\mathbb{N}}\frac{|\lambda_i|\chi_{x_i+B_{\ell_i}}}{\|\chi_{x_i+B_{\ell_i}}\|_{L^{p(\cdot)}}}\right\|_{L^{p(\cdot)}}^{\underline{p}}\\
&\lesssim \left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{x_i+B_{\ell_i}}}{\|\chi_{x_i+B_{\ell_i}}\|_{L^{p(\cdot)}}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}}^{\underline{p}} \sim \|f\|_{H_A^{p(\cdot),q,s}}^{\underline{p}}.
\end{aligned}$$

Combining the estimates of L_1 and L_2, we further conclude that, for any $f\in H_A^{p(\cdot)}\cap L^r$,

$$\|T(f)\|_{L^{p(\cdot)}} \lesssim \|f\|_{H_A^{p(\cdot)}}.$$

Next, we prove that (i) also holds true for any $f \in H_A^{p(\cdot)}$. Let $f \in H_A^{p(\cdot)}$, by Lemma 3.7, we know that there exists a sequence $\{f_j\}_{j\in\mathbb{Z}_+} \subset H_A^{p(\cdot)} \cap L^r$ with $r \in (1, \infty] \cap (p_+, \infty]$ such that $f_j \to f$ as $j \to \infty$ in $H_A^{p(\cdot)}$. Therefore, $\{f_j\}_{j\in\mathbb{Z}_+}$ is a Cauchy sequence in $H_A^{p(\cdot)}$. By this, we see that, for any $j,\, k \in \mathbb{Z}_+$,

$$\|T(f_j) - T(f_k)\|_{L^{p(\cdot)}} = \|T(f_j - f_k)\|_{L^{p(\cdot)}} \lesssim \|f_j - f_k\|_{H_A^{p(\cdot)}}.$$

Notice that $\{T(f_j)\}_{j\in\mathbb{Z}_+}$ is also a Cauchy sequence in $L^{p(\cdot)}$. Applying Lemma 3.8, we conclude that there exist a $g \in L^{p(\cdot)}$ such that $T(f_j) \to g$ as $j \to \infty$ in $L^{p(\cdot)}$. Let $T(f) := g$. We claim that $T(f)$ is well defined. Indeed, for any other sequence $\{h_j\}_{j\in\mathbb{Z}_+} \subset H_A^{p(\cdot)} \cap L^r$ satisfying $h_j \to f$ as $j \to \infty$ in $H_A^{p(\cdot)}$, by Remark 2.2(i), we have

$$\begin{aligned}\|T(h_j) - T(f)\|_{L^{p(\cdot)}}^{\underline{p}} &\le \|T(h_j) - T(f_j)\|_{L^{p(\cdot)}}^{\underline{p}} + \|T(f_j) - g\|_{L^{p(\cdot)}}^{\underline{p}}.\\ &\lesssim \|h_j - f_j\|_{H_A^{p(\cdot)}}^{\underline{p}} + \|T(f_j) - g\|_{L^{p(\cdot)}}^{\underline{p}}\\ &\lesssim \|h_j - f\|_{H_A^{p(\cdot)}}^{\underline{p}} + \|f - f_j\|_{H_A^{p(\cdot)}}^{\underline{p}} + \|T(f_j) - g\|_{L^{p(\cdot)}}^{\underline{p}} \to 0,\\ &\quad \text{as } j \to 0,\end{aligned}$$

which is wished.

From this, we see that, for any $f \in H_A^{p(\cdot)}$,

$$\|T(f)\|_{L^{p(\cdot)}} = \|g\|_{L^{p(\cdot)}} = \lim_{j\to\infty} \|T(f_j)\|_{L^{p(\cdot)}} \lesssim \lim_{j\to\infty} \|f_j\|_{H_A^{p(\cdot)}} \sim \|f\|_{H_A^{p(\cdot)}},$$

which implies that (i) also holds true for any $f \in H_A^{p(\cdot)}$ and hence completes the proof of Theorem 3.1. □

4. Dual spaces of $H_A^{p(\cdot)}$

In this section, we give the dual space of $H_A^{p(\cdot)}$. More precisely, as an application of the atomic characterizations of $H_A^{p(\cdot)}$ obtained in Lemma 3.3, we prove that the dual space of $H_A^{p(\cdot)}$ is the variable anisotropic Campanato space $\mathcal{L}_A^{p(\cdot),\, q,\, s}$.

Now, we introduce the notion of the anisotropic Campanato space with variable exponents $\mathcal{L}_A^{p(\cdot),\, q,\, s}$.

Definition 4.1. Let A be a given dilation, $p(\cdot) \in \mathcal{P}$, s be a nonnegative integer and $r \in [1, \infty)$. Then *the anisotropic Campanato space with variable exponent* $\mathcal{L}_A^{p(\cdot),\, q,\, s}$ is defined to be the set of all $f \in L^q_{\text{loc}}$ such that

$$\|f\|_{\mathcal{L}_A^{p(\cdot),\, q,\, s}} := \sup_{B\in\mathfrak{B}} \inf_{P\in P_s} \frac{|B|}{\|\chi_B\|_{L^{p(\cdot)}}} \left[\frac{1}{|B|}\int_B |f(x) - P(x)|^q\, dx\right]^{1/q} < \infty$$

and

$$\|f\|_{\mathcal{L}_A^{p(\cdot),\, \infty,\, s}} := \sup_{B\in\mathfrak{B}} \inf_{P\in P_s} \frac{|B|}{\|\chi_B\|_{L^{p(\cdot)}}} \|f(x) - P(x)\|_{L^\infty} < \infty,$$

where $\mathfrak{B}$ is as in (2.1).

Lemma 4.2. *Let $p(\cdot) \in \mathcal{P}$. Then, for any $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$,*

$$\sum_{i\in\mathbb{N}} |\lambda_i| \le \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}},$$

where $\underline{p}$ is as in (2.5)*.*

Proof. Since $p(\cdot) \in \mathcal{P}$ and the well-known inequality that, $\|\cdot\|_{\ell^1} \le \|\cdot\|_{\ell^p}$ with $p \in (0, 1]$, then

$$\begin{aligned}
\sum_{i\in\mathbb{N}} |\lambda_i| &= \sum_{i\in\mathbb{N}} |\lambda_i| \left\| \frac{\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}}} \right\|_{L^{p(\cdot)}} \\
&\le \left\| \sum_{i\in\mathbb{N}} \frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}}} \right\|_{L^{p(\cdot)}} \\
&\le \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}}.
\end{aligned}$$

This finishes the proof of Lemma 4.2. □

Lemma 4.3. *Let A be a given dilation, $p(\cdot) \in C^{\log}$, s be a nonnegative integer and $q \in [1, \infty)$. Then, for any continuous linear functional ℓ on $H_A^{p(\cdot)} = H_A^{p(\cdot),\, q,\, s}$,*

$$\begin{aligned}
\|\ell\|_{(H_A^{p(\cdot),\, q,\, s})^*} :=& \sup\left\{ |\ell(f)| : \|f\|_{H_A^{p(\cdot),\, q,\, s}} \le 1 \right\} \\
=& \sup\{ |\ell(a)| : a \text{ is } (p(\cdot),\, q,\, s)\text{-}atom \},
\end{aligned}$$

here and hereafter, $(H_A^{p(\cdot),\, q,\, s})^*$ denotes the dual space of $H_A^{p(\cdot),\, q,\, s}$.

Proof. For any $(p(\cdot),\, q,\, s)$-atom a, it is easy to show that $\|f\|_{H_A^{p(\cdot),\, q,\, s}} \le 1$. Therefore,

$$\sup\{ |\ell(a)| : a \text{ is } (p(\cdot),\, q,\, s)\text{-atom} \} \le \sup\left\{ |\ell(f)| : \|f\|_{H_A^{p(\cdot),\, q,\, s}} \le 1 \right\}.$$

On the other hand, let $f \in H_A^{p(\cdot),\, q,\, s}$ and $\|f\|_{H_A^{p(\cdot),\, q,\, s}} \le 1$. Then, for any $\varepsilon > 0$, we know that there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot),\, \infty,\, s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, on $\{B_i\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in } \mathcal{S}' \text{ and almost everywhere}$$

and

$$\left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i| \chi_{B_i}}{\|\chi_{B_i}\|_{L^{p(\cdot)}}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}} \le 1 + \varepsilon.$$

From this, the continuity of ℓ and Lemma 4.3, we further conclude that

$$|\ell(f)| \le \sum_{i\in\mathbb{N}} |\lambda_i||\ell(a_i)| \le \sum_{i\in\mathbb{N}} |\lambda_i| \sup\{|\ell(a)| : a \text{ is } (p(\cdot),\, q,\, s)\text{-atom}\}$$
$$\le (1+\varepsilon)\sup\{|\ell(a)| : a \text{ is } (p(\cdot),\, q,\, s)\text{-atom}\}.$$

Combined with the arbitrariness of ε and hence finishes the proof of Lemma 4.3. □

Let $q \in [1, \infty]$ and $s \in \mathbb{Z}_+$. Denote by $L^q_{\rm comp}$ the set of all function $f \in L^\infty$ with compact and

$$L^{q,\,s}_{\rm comp} := \{f \in L^q_{\rm comp} : \int_{\mathbb{R}^n} f(x)x^\alpha\,dx = 0,\ |\alpha| \le s\}.$$

In this paper, for any $r \in \mathbb{Z}_+$, we use P_r to denote the set of polynomials on $\mathbb{R}^n$ with order not more than r.

The main result of this section is the following a theorem.

Theorem 4.4. *Let A be a given dilation, $p(\cdot) \in C^{\log}$, $p_+ \in (0, 1]$, $q \in (p_+, \infty)$ and $s \in [\lfloor(1/p_- - 1)\ln b/\ln\lambda_-\rfloor, \infty) \cap \mathbb{Z}_+$ with p_- as in (2.4). Then the dual space of $H^{p(\cdot)}_A = H^{p(\cdot),\,q,\,s}_A$, denoted by $(H^{p(\cdot),\,q,\,s}_A)^*$, is the variable anisotropic Campanato space $\mathcal{L}^{p(\cdot),\,q',\,s}_A$ in the following sense: for any $b \in \mathcal{L}^{p(\cdot),\,q',\,s}_A$, the linear functional*

$$\ell_b(g) := \int_{\mathbb{R}^n} b(x)g(x)\,dx, \tag{4.1}$$

initial defined for all $g \in L^{q,\,s}_{\rm comp}$, has a bounded extension to $H^{p(\cdot),\,q,\,s}_A = H^{p(\cdot)}_A$.

Conversely, if ℓ is a bounded linear functional on $H^{p(\cdot),\,q,\,s}_A = H^{p(\cdot)}_A$, then ℓ has the form as in (4.1) with a unique $b \in \mathcal{L}^{p(\cdot),\,q',\,s}_A$.

Moreover,

$$\|b\|_{\mathcal{L}^{p(\cdot),\,q',\,s}_A} \sim \|\ell_b\|_{(H^{p(\cdot),\,q,\,s}_A)^*},$$

where the implicit positive constants are independent of b.

Remark 4.5. (i) When $A := 2\mathrm{I}_{n\times n}$, Theorem 4.4 goes back to [15, Theorem 7.5].

(ii) When $p(\cdot) := p$, where $p \in (0, 1]$, Theorem 4.4 is reduced to [2, Theorem 8.3].

Proof of Theorem 4.4. By Lemma 3.3, we only need to show

$$\mathcal{L}^{p(\cdot),\,q',\,s}_A \subset (H^{p(\cdot),\,q,\,s}_A)^*.$$

Let $b \in \mathcal{L}^{p(\cdot),\,q',\,s}_A$ and a be a $(p(\cdot),\, q,\, s)$-atom supported on $B \in \mathfrak{B}$. Then, by the vanishing moment condition of a, Hölder's inequality and the size condition of a, we obtain

$$\left|\int_{\mathbb{R}^n} b(x)a(x)\,dx\right| = \inf_{P\in P_s}\left|\int_B (b(x) - P(x))a(x)\,dx\right| \tag{4.2}$$

$$
\begin{aligned}
&\le \|a\|_{L^q} \inf_{P\in P_s}\left[\int_B |b(x)-P(x)|^{q'}\,dx\right]^{1/q'}\\
&\le \frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}} \inf_{P\in P_s}\left[\int_B |b(x)-P(x)|^{q'}\,dx\right]^{1/q'}\\
&\le \|b\|_{\mathcal{L}_A^{p(\cdot),\,q',\,s}}.
\end{aligned}
$$

Therefore, for $\{\lambda_i\}_{i\in\mathbb{N}}\subset\mathbb{C}$ and a sequence $\{a_i\}_{i\in\mathbb{N}}$ of $(p(\cdot),\,q,\,s)$-atoms supported, respectively, on $\{B^{(i)}\}_{i\in\mathbb{N}}\subset\mathfrak{B}$ and

$$
g=\sum_{i\in\mathbb{N}}\lambda_i a_i\in H_A^{p(\cdot),\,q,\,s},
$$

by Lemma 4.2 and (4.2), we have

$$
\begin{aligned}
|\ell_b(g)| &= \left|\int_{\mathbb{R}^n} b(x)g(x)\,dx\right| \le \sum_{i\in\mathbb{N}}|\lambda_i|\left|\int_B |b(x)-P(x)|\,|a_i(x)|\,dx\right|\\
&\le \sum_{i\in\mathbb{N}}|\lambda_i|\|b\|_{\mathcal{L}_A^{p(\cdot),\,q',\,s}} \le \|g\|_{H_A^{p(\cdot),\,q,\,s}}\|b\|_{\mathcal{L}_A^{p(\cdot),\,q',\,s}}.
\end{aligned}
$$

This implies that $\mathcal{L}_A^{p(\cdot),\,q',\,s}\subset(H_A^{p(\cdot),\,q,\,s})^*$ holds true.

Now we prove $(H_A^{p(\cdot),\,q,\,s})^*\subset\mathcal{L}_A^{p(\cdot),\,q',\,s}$. For any $B\in\mathfrak{B}$, let

$$
A_B: L^1(B)\to P_s
$$

be the natural projection satisfying, for any $g\in L^1$ and $q\in P_s$,

$$
\int_B A_B(g)(x)q(x)\,dx=\int_B g(x)q(x)\,dx.
$$

From the similar proof of [2, (8.9)], we know that there exists a positive constant C_s such that, for any $B\in\mathfrak{B}$ and $g\in L^1(B)$,

$$
\sup_{x\in B}|A_B(g)(x)|\le C_s\frac{\int_B |g(z)|\,dz}{|B|}.
$$

For any $q\in(1,\,\infty]$ and $B\in\mathfrak{B}$, let

$$
L^q(B):=\{f\in L^q:\operatorname{supp}f\subset B\}.
$$

Set

$$
L_0^q(B):=\{g\in L^q(B): A_B(g)(x)=0 \text{ and } g \text{ is not zero almost everywhere}\}.
$$

For any $g\in L_0^q(B)$, set

$$
a(x):=\begin{cases}\frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}}\|g\|_{L^q(B)}^{-1}g(x), & x\in B;\\ 0, & x\notin B.\end{cases} \tag{4.3}
$$

Then $a(x)$ is a $(p(\cdot), q, s)$-atom. By this, we obtain, for any $\ell \in (H_A^{p(\cdot), q, s})^*$ and $g \in L_0^q(B)$,

$$|\ell(g)| \le \frac{\|\chi_B\|_{L^{p(\cdot)}}}{|B|^{1/q}} \|\ell\|_{(H_A^{p(\cdot), q, s})^*} \|g\|_{L^q(B)}. \tag{4.4}$$

This show that ℓ is a bounded linear function on $L_0^q(B)$. By the Hahn-Banach theorem, it can be extended to $L^q(B)$ without increasing its norm.

When $q \in (1, \infty)$, noticing the fact that the duality of $L^q(B)$ is $L^{q'}(B)$, we know that there exists a $\xi \in L^{q'}(B)$ such that, for any $f \in L_0^q(B)$,

$$\ell(f) = \int_B f(x)\xi(x)\,dx.$$

When $q = \infty$, it's easy to see that there exists a $\xi \in L^{q'}(B)$ such that, for any $f \in L_0^\infty(B)$,

$$\ell(f) = \int_B f(x)\xi(x)\,dx.$$

Therefore, for any $q \in (1, \infty]$, it's easy to see that there exists a $\xi \in L^{q'}(B)$ such that, for any $f \in L_0^q(B)$,

$$\ell(f) = \int_B f(x)\xi(x)\,dx.$$

Let $q \in (1, \infty]$. Now we prove that, if there exists another function $\xi' \in L^{q'}(B)$ such that, for any $f \in L_0^q(B)$ and

$$\ell(f) = \int_B f(x)\xi(x)\,dx,$$

then $\xi' - \xi \in P_s(B)$. To prove this, we only need to prove that, if $\xi, \xi' \in L^1(B)$ such that, for any $f \in L_0^\infty(B)$, $\int_B f(x)\xi'(x)\,dx = \int_B f(x)\xi(x)\,dx$, then $\xi - \xi' \in P_s(B)$. In fact, for any $f \in L_0^\infty(B)$, we obtain

$$\begin{aligned}
0 &= \int_B [f(x) - A_B(f)(x)][\xi'(x) - \xi(x)]\,dx \\
&= \int_B f(x)[\xi'(x) - \xi(x)]\,dx - \int_B f(x)A_B[\xi'(x) - \xi(x)]\,dx \\
&= \int_B f(x)[\xi'(x) - \xi(x) - A_B(\xi' - \xi)(x)]\,dx.
\end{aligned}$$

Therefore, for a.e. $x \in B \subset \mathfrak{B}$, we have

$$\xi'(x) - \xi(x) = A_B(\xi' - \xi)(x).$$

Hence $\xi' - \xi \in P_s(B)$. From this, we see that, for any $q \in (1, \infty]$ and $f \in L_0^q(B)$, there exists a unique $\xi \in L^{q'}(B)/P_s(B)$ such that

$$\ell(f) = \int_B f(x)\xi(x)\,dx.$$

If $q \in (1, \infty]$, for any $j \in \mathbb{N}$ and $g \in L_0^q(B_j)$ with $q \in (1, \infty]$, let $f_j \in L^{q'}(B)/P_s(B)$ be a unique function such that

$$\ell(g) = \int_{B_j} f_j(x)g(x)\,dx.$$

Then, we see that, for any $i, j \in \mathbb{N}$ with $i < j$, $f_j|_{B_i} = f_i$. From this and the fact that, for any $g \in (H_A^{p(\cdot),\,q,\,s})^*$, there exists a $j_0 \in \mathbb{N}$ such that $g \in L_0^q(B_{j_0})$. Thus, for any $g \in (H_A^{p(\cdot),\,q,\,s})^*$, we have

$$\ell(g) = \int_B b(x)g(x)\,dx, \tag{4.5}$$

where $b(x) := f_j(x)$ with $x \in B_j$.

Next we need to show that $b \in \mathcal{L}_A^{p(\cdot),\,q',\,s}$. From (4.5) and (4.4), we conclude that, for any $q \in (1, \infty]$, $B \in \mathfrak{B}$,

$$\|b\|_{(L_0^q(B))^*} \le \frac{\|\chi_B\|_{L^{p(\cdot)}}}{|B|^{1/q}}\|\ell\|_{(H_A^{p(\cdot),\,q,\,s})^*}. \tag{4.6}$$

Moreover, by [2, (8.12)], we know that

$$\|b\|_{(L_0^q(B))^*} = \inf_{P\in P_s}\|b - P\|_{(L^{q'}(B))}.$$

From this and (4.6), we conclude that, for any $q \in (1, \infty]$,

$$\begin{aligned}\|b\|_{\mathcal{L}_A^{p(\cdot),\,q',\,s}} &= \sup_{B\in\mathfrak{B}} \frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}} \inf_{P\in P_s}\|b - P\|_{(L^{q'}(B))}\\ &= \sup_{B\in\mathfrak{B}} \frac{|B|^{1/q}}{\|\chi_B\|_{L^{p(\cdot)}}}\|b\|_{(L_0^q(B))^*}\\ &\le \|\ell\|_{(H_A^{p(\cdot),\,q,\,s})^*} < \infty.\end{aligned}$$

This finishes the proof of Theorem 4.4. □

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