Bull. Korean Math. Soc. **58** (2021), No. 2, pp. 349–364
https://doi.org/10.4134/BKMS.b200296
pISSN: 1015-8634 / eISSN: 2234-3016

OPTIMAL SURRENDER TIME FOR A VARIABLE ANNUITY WITH A FIXED INSURANCE FEE

Junkee Jeon and Kyunghyun Park

Abstract. This paper studies the optimal surrender policies for a variable annuity (VA) contract with a surrender option and a fixed insurance fee for guaranteed minimum maturity benefits (GMMB). In our proposed model, a policyholder pays the fixed insurance fee. Based on the integral transform techniques, we derive the analytic integral equations for the optimal surrender boundary and the value function of the VA contract that can be solved numerically by recursive integration method. We provide numerical values for the value function, the optimal surrender boundary, and the expected optimal surrender time.

1. Introduction

In this paper, we propose a new type of a variable annuity contract where the policyholder can choose an option to surrender the contract any time before maturity. In contrast to the traditional annuity contracts that offer fixed amount at a predetermined date, variable annuity (VA) contracts provide a random payoff to the policyholder. The policyholder pays a single premium to her account, and this can be invested in the risky assets or the market index by the insurer. Depending on the financial market environments, the policyholder may receive a high annuity income, meanwhile he/she is also exposed to the downside risk. To protect the policyholder from the risk, the VA contract guarantees the minimum payoff at the maturity and the insurer usually charges the insurance fee on the contract to collect the funding of the guarantees. VA contracts usually permit that the policyholder can choose the option to surrender the contract. In the VA contract with the surrender option, it is important to investigate when the policyholder surrenders the contract.

Received April 2, 2020; Revised August 21, 2020; Accepted December 23, 2020.
2010 *Mathematics Subject Classification.* Primary 37H10, 35Q93, 91-08.
Key words and phrases. Variable annuity, surrender option, guaranteed minimum maturity benefits, fixed insurance fee, integral equation.
Junkee Jeon received support from the Young Research Program of the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (NRF-2020R1C1C1A01007313). Kyunghyun Park received support from the NRF Global Ph.D Fellowship (2016H1A2A1908911).

There have been previous studies on VA contracts with many types of guarantees and options (e.g., [2,4,8,10]). In particular, [2] studied the VA contract with the surrender option when the insurance company withdraws a proportional rate of the policyholder's account. In contrast to [2], we consider the VA contract with a surrender option that the insurance company does not withdraw any insurance fee from the underlying fund or the policyholder's account. Instead, the policyholder continuously pays the fixed insurance fee to the insurer until the contract is terminated. This contract is thought of as a VA version of an installment option (see [3]). Thus, the main feature of our paper is that regardless of the fund value, the policyholder pays a fixed insurance fee.

Since the policyholder has a right to stop paying the insurance fee at any time, thereby surrendering the VA contract, this problem can be considered as an optimal stopping problem. We utilize a standard approach which formulates the optimal stopping problem as a *variational inequality* of the contract value (see [9]). The *variational inequality* yields a *free boundary* which corresponds to the optimal surrender boundary of the contract (see [6]). When the underlying fund rises enough to reach the optimal surrender boundary, the policyholder exercises the surrender option. Based on the Mellin transform techniques, we obtain the integral equation representations for the VA contract and the optimal surrender boundary. Then, we numerically solve the integral equations using the *recursive integration method* provided by [7].

The rest of this paper is organized as follows: In Section 2, we introduce and formulate the model of our VA contract with a fixed insurance fee. In Section 3, we derive the *variational inequality* arising from the optimal stopping problem which is satisfied by the value of the VA contract and obtain the integral equation representation of the optimal surrender boundary by the Mellin transform. We present the VA value through a table on several parameters and present the sensitivity analysis of the optimal surrender boundary and the expected optimal surrender time in Section 4. In Section 5, we conclude.

2. Model formulation

For a given maturity $T > 0$, we consider a VA contract with a guaranteed minimum maturity benefit $G > 0$. We assume that the contract is initiated at time $t = 0$ when the policyholder pays a single premium $F_0 > 0$. The insurer invests the single premium in the underlying fund or market index.

Let us denote the accumulated fund value of the VA at time t as $(F_t)_{t=0}^T$. In the case of [2], the insurer continuously withdraws a proportional rate of the accumulated fund value $(F_t)_{t=0}^T$ as an insurance fee. In contrast to [2], we consider a VA contract with a fixed insurance fee. More precisely, it is not that the insurer withdraws any fee from the policyholder's account value but that the policyholder continuously pays a fixed rate of the initial premium to the insurer at the cost of the insurance company managing the underlying account. We adopt the usual Black-Scholes framework and thus assume that underlying fund $(S_t)_{t=0}^T$ follows a geometric Brownian motion. Under the risk-neutral measure $\mathbb{Q}$, the dynamics of underlying fund $(S)_{t=0}^T$ are given by

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{2.1}$$

where r is the risk-free interest rate and $\sigma > 0$ is the constant volatility and W_t is a standard Brownian motion in the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with the filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ generated by the Brownian motion $(W_t)_{0\leq t\leq T}$.

Since the insurer does not withdraw any fee from the policyholder's account during the contract, we have the following relationship between the policyholder's account and underlying fund:

$$\frac{F_t}{F_0} = \frac{S_t}{S_0},$$

and the dynamics of policyholder's account value at time t is given by

$$dF_t = rF_t dt + \sigma F_t dW_t. \tag{2.2}$$

Since we consider the VA contract with the guaranteed minimum maturity benefit, we assume that the policyholder's benefit at the maturity T is given by

$$\max(F_T, G).$$

The policyholder has a right to exercise the surrender option at anytime before the maturity T. Then, the surrender benefit at time t is given by $(1-\kappa_t)F_t$, where κ_t is the penalty percentage charged on the accumulated fund when surrendering the contract at time t. As in [2], we assume that κ_t is exponentially decreasing in time t and defined by $1 - e^{-\kappa(T-t)}$, so that the surrender benefit is given by

$$e^{-\kappa(T-t)} F_t, \tag{2.3}$$

where $\kappa > 0$ is a fixed rate of the surrender penalty.

The main feature of our model is that the structure of the insurance fee is similar to that of the installment option (see [3]). This means that the insurance fee in our VA contract is paid continuously until the policyholder chooses the option to surrender the VA contract. Thus, the present value of the cumulative insurance fee of our VA contract at time t is given by

$$\int_t^\theta e^{-r(s-t)} c \, ds, \tag{2.4}$$

where $c > 0$ is the fixed insurance fee and $\theta \in [t, T)$ is the time when the policyholder surrenders the VA contract.

Then, the concern is on finding the optimal time for the policyholder to surrender the VA contract. Here, we present the *optimal stopping problem*:

Problem 2.1. *In the absence of arbitrage opportunities, the value $V(t, F_t)$ of the VA contract at time t is expressed by*

$$V(t, F_t) = \sup_{\theta_t \in \mathcal{S}(t,T)} \mathbb{E}_t^{\mathbb{Q}} \left[e^{-r(\theta_t - t)} \left(e^{-\kappa(T-\theta_t)} F_{\theta_t} 1_{\{\theta_t < T\}} + \max(F_T, G) 1_{\{\theta_t = T\}} \right) \right.$$

$$-\int_t^{\theta_t} e^{-r(s-t)} c\, ds\Bigg], \tag{2.5}$$

where $\mathcal{S}(t,T)$ is the set of all stopping times of the filtration $(\mathcal{F}_t)_{t=0}^T$ taking values in $[t,T]$, and the conditional expectation $\mathbb{E}_t[\cdot \mid \mathcal{F}_t]$ is calculated under the risk-neutral measure $\mathbb{Q}$.

By a standard approach for the optimal stopping problem (see [9]), the value $V(t,f)$ satisfies the following *variational inequality* (VI):

Variational Inequality 1.

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{L}V - c \le 0, & \text{if} \quad V(t,f) = e^{-\kappa(T-t)}f, \\ \dfrac{\partial V}{\partial t} + \mathcal{L}V - c = 0, & \text{if} \quad V(t,f) > e^{-\kappa(T-t)}f, \\ V(T,f) = \max(f,G), \end{cases} \tag{2.6}$$

on domain of state variable $\mathcal{D} = \{(t,f) \mid 0 \le t < T,\ 0 < f < \infty\}$ and the operator $\mathcal{L}$ is given by

$$\mathcal{L} \equiv \frac{\sigma^2}{2} f^2 \frac{\partial^2}{\partial f} + rf\frac{\partial}{\partial f} - r. \tag{2.7}$$

From the standard theory of the *variational inequality* (see [6]), we can define a *free boundary* of the *variational inequality.* In this model, the optimal surrender boundary can be defined as

$$\mathcal{B}(t) \equiv \sup\left\{ f > 0 \mid V(t,f) > e^{-\kappa(T-t)}f \right\}. \tag{2.8}$$

In terms of the optimal surrender boundary, the domain $\mathcal{D}$ can be divided into two regions: one is the *continuation-region* (**CR**) and the other is the *surrender-region* (**SR**), i.e.,

$$\begin{aligned} \mathbf{CR} &\equiv \{(t,f) \in \mathcal{D} \mid V(t,f) > e^{-\kappa(T-t)}f\} = \{(t,f) \in \mathcal{D} \mid 0 < f < \mathcal{B}(t)\}, \\ \mathbf{SR} &\equiv \{(t,f) \in \mathcal{D} \mid V(t,f) = e^{-\kappa(T-t)}f\} = \{(t,f) \in \mathcal{D} \mid f \ge \mathcal{B}(t)\}. \end{aligned} \tag{2.9}$$

Moreover, the value function $V(t,f)$ satisfies the following *smooth-pasting* conditions at the optimal surrender boundary:

$$V(t,\mathcal{B}(t)) = e^{-\kappa(T-t)}\mathcal{B}(t), \quad \left.\frac{\partial V}{\partial f}\right|_{f=\mathcal{B}(t)} = e^{-\kappa(T-t)}. \tag{2.10}$$

Since

$$\frac{\partial V}{\partial t} + \mathcal{L}V = c \quad \text{in} \ \ (t,f) \in \mathbf{CR},$$

and

$$V(t,f) = e^{-\kappa(T-t)}f \quad \text{in} \ \ (t,f) \in \mathbf{SR},$$

it is easily confirmed that the value $V(t,f)$ satisfies the following non-homogeneous partial differential equation (PDE):

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{L}V = c \cdot 1_{\{f<\mathcal{B}(t)\}} + \kappa e^{-\kappa(T-t)} f \cdot 1_{\{f \geq \mathcal{B}(t)\}}, \ (t,f) \in \mathcal{D} \\ V(t,f) = \max(f,G), \end{cases} \tag{2.11}$$

with the *smooth pasting* condition (2.10).

Before proceeding to our main theorem, let us simply derive the value of the VA without the surrender option. The value of the VA contract without the surrender option means that the policyholder cannot choose the option to surrender. Then, the insurance fee c should be continuously paid to the insurer until the maturity T. Here, we call the value without surrender option as *European part* of the value function $V(t,f)$ and denote the value as $V_E(t,f)$. Then, the value of European part $V_E(t,f)$ at time t is given by

$$\begin{aligned} V_E(t,F_t) &\equiv \mathbb{E}_t^{\mathbb{Q}} \left[e^{-r(T-t)} \max(F_T,G) - \int_t^T e^{-r(s-t)} c \, ds \right] \\ &= \mathbb{E}_t^{\mathbb{Q}} \left[e^{-r(T-t)} \max(F_T,G) \right] - \frac{1-e^{-r(T-t)}}{r} c, \end{aligned} \tag{2.12}$$

where $\frac{1-e^{-r(T-t)}}{r}c$ is the present value of the cumulative insurance fee imposed on the VA contract without the option from the present time t to the maturity T.

From the definition of the fair insurance fee c^* in [2], we can deduce that the fair insurance fee c^* in our VA contract satisfies the following relationship:

$$F_0 = V_E^{c^*}(0,F_0) = \mathbb{E}^{\mathbb{Q}} \left[e^{-rT} \max(F_T,G) \right] - \frac{1-e^{-rT}}{r} c^*, \tag{2.13}$$

where F_0 is the single premium initially paid by the policyholder[1].

Since the closed-form of the expectation $\mathbb{E}_t^{\mathbb{Q}} \left[e^{-r(T-t)} \max(F_T,G) \right]$ is well known, we just present the result. By using the closed-form of $V_E(t,f)$, we can easily compute the fair insurance fee c^* in the following proposition.

Proposition 2.1 (European part for the VA value and fair insurance fee)**.**

(a) *The value of the European part $V_E(t,F_t)$ at time t is given by*

$$\begin{aligned} V_E(t,F_t) = &\ F_t \mathcal{N}\left(\frac{\log(\frac{F_t}{G}) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \right) \\ &+ G e^{-r(T-t)} \mathcal{N}\left(-\frac{\log(\frac{F_t}{G}) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \right) - \frac{1-e^{-r(T-t)}}{r} c, \end{aligned} \tag{2.14}$$

where $\mathcal{N}(\cdot)$ is a standard cumulative normal distribution function.

[1]In the VA contract without surrender option, it is fair that the initial fund premium F_0 should be equal to the value $V_E^c(0,F_0)$.

(b) *The fair insurance fee* c^* *of the VA contract is given by*

$$c^* = \left(\frac{1-e^{-rT}}{r}\right)^{-1}\left[Ge^{-rT}\cdot\mathcal{N}\left(-\frac{\log(\frac{F_0}{G})+(r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right.$$
$$\left.-F_0\mathcal{N}\left(-\frac{\log(\frac{F_0}{G})+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)\right]. \tag{2.15}$$

3. Integral equation representation

In this section, we will derive the integral equation solutions for the value $V(t,f)$ of VA contract and the optimal surrender boundary $\mathcal{B}(t)$. To solve the PDE problem (2.11) with boundary conditions (2.10), we use the Mellin transform, which is one of the transform techniques to derive the analytic solution of the Black-Scholes PDE with boundary conditions (see [1], [5], and [11]). In Appendix A, we briefly review the definition of the Mellin transform and some basic properties.

The next theorem provides the integral equation solutions for $V(t,f)$ and the corresponding optimal surrender boundary $\mathcal{B}(t)$ by applying the Mellin transform techniques.

Theorem 3.1 (Main theorem)**.**

(a) *The value* $V(t,F_t)$ *of the VA contract can be decomposed into the European part* $V_E(t,F_t)$ *and the early surrender premium part* $V_P(t,F_t)$*,*

$$V(t,F_t) = V_E(t,F_t) + V_P(t,F_t), \tag{3.1}$$

where the European part is given in (2.14) *and the early surrender premium* $V_P(t,F_t)$ *is represented by*

$$V_P(t,F_t) = -\kappa e^{-\kappa(T-t)}F_t\int_t^T e^{\kappa(\xi-t)}\mathcal{N}\left(\frac{\log(\frac{F_t}{\mathcal{B}(\xi)})+(r+\frac{\sigma^2}{2})(\xi-t)}{\sigma\sqrt{\xi-t}}\right)d\xi$$
$$+c\int_t^T e^{-r(\xi-t)}\mathcal{N}\left(\frac{\log(\frac{F_t}{\mathcal{B}(\xi)})+(r-\frac{\sigma^2}{2})(\xi-t)}{\sigma\sqrt{\xi-t}}\right)d\xi. \tag{3.2}$$

(b) *The optimal surrender boundary is the solution of the following integral equation*:

$$V(t,\mathcal{B}(t)) = e^{-\kappa(T-t)}\mathcal{B}(t). \tag{3.3}$$

Proof. See Appendix B. ☐

4. Implications

Based on the integral equations in Theorem 3.1, we evaluate the accuracy of values derived from the integral equations by comparing them with benchmark values derived from the *binomial tree method* (BTM). To obtain the numerical values for the integral equations of $V(t,f)$ and $\mathcal{B}(t)$, we utilize the *recursive integration method* (RIM) proposed by [7].

TABLE 1. Comparison of the VA values (BTM and RIM) under the $F_0 = 100$, $r = 0.03$, $\sigma = 0.2$ and $T = 10$.

c^*	κ	G	BTM (10^4)	BTM (10^5)	RIM (20)	RIM (200)	RIM (2000)	RIM (4-pt)
0.0126	0	100	102.2088	102.2096	102.3212	102.1946	102.1808	102.1750
0.0126	$c^*/6$	100	101.1762	101.1764	101.2528	101.1442	101.1326	101.1511
0.0126	$c^*/4$	100	100.7403	100.7405	100.7855	100.6859	100.6754	100.6904
0.0212	0	120	102.9602	102.9602	103.0065	103.0219	103.0226	102.9466
0.0212	$c^*/6$	120	101.4007	101.4010	101.4208	101.4310	101.4114	101.3873
0.0212	$c^*/4$	120	100.7875	100.7871	100.7570	100.7648	100.7651	100.7928
RMSE			·	·	1.546×10^{-1}	9.720×10^{-2}	1.070×10^{-1}	6.824×10^{-2}
CPU time (sec)			5.910	7.127×10^{2}	1.771×10^{-1}	1.302×10^{1}	1.201×10^{2}	5.067×10^{-2}

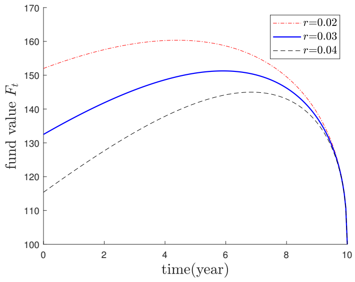
Table 1 provides the VA contract values derived by the BTM and the RIM for varying κ and G, respectively. We set the single premium F_0 equal to 100. The values of the fair insurance fee c^* in the first column are determined from the formula in Proposition 2.1(b). The contract values via the BTM are presented for each time-steps $n = 10000$ and 100000, respectively, and the values via the RIM are presented for each time-steps $n = 20$, 200, and 2000, respectively. Moreover, we provide other values of the RIM where a four-point Richardson extrapolation scheme is used to accelerate the convergence of the RIM. We measure the accuracy of the integral equations by calculating the root-mean-square errors (RMSEs) between the RIM values and the BTM values with time-steps $n = 100000$, and provide computing times (CPU time) for using each method. The RMSE values indicate that the RIM with time step $n = 200$, having 9.7199×10^{-2} RMSE, is more accurate than the RIM with other time-steps $n = 20$ and 2000. Meanwhile, the RIM with a four-point Richardson extrapolation scheme shows the best results in terms of both accuracy and efficiency, having 6.8235×10^{-2} RMSE and taking 5.0667×10^{-2} seconds. This implies that the RIM under this extrapolation scheme does shed light on the advantage of our integral representation results.

Note that in our parameter sets, all the VA values are greater than 100. Because of the early surrender premium $V_P^{c^*}(0, F_0)$, the VA value $V^{c^*}(0, F_0)$ is higher than the single premium F_0 paid initially, i.e.,

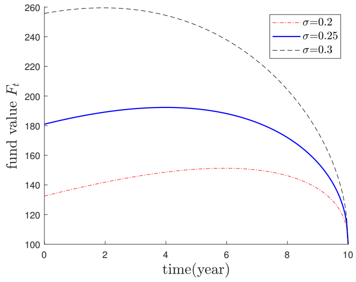
$$V^{c^*}(0, F_0) = V_E^{c^*}(0, F_0) + V_P^{c^*}(0, F_0) = F_0 + V_P^{c^*}(0, F_0) > F_0.$$

In Figure 1, we illustrate the sensitivity of the optimal surrender boundary $\mathcal{B}(t)$ for sevaral parameters. The baseline parameters are given as follows:

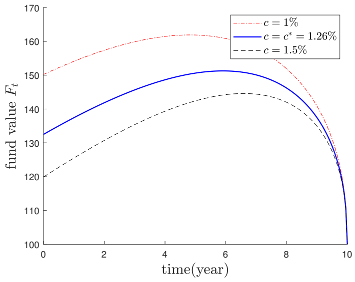
$$T = 10,\ r = 0.03,\ c^* = 1.26,\ \kappa = 0,\ \sigma = 0.2 \text{ and } G = 100. \tag{4.1}$$



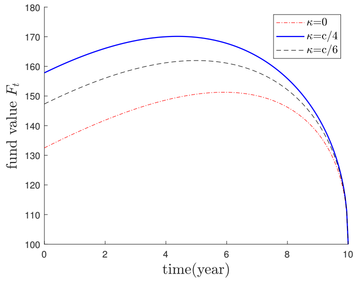
(a) Sensitivity to r



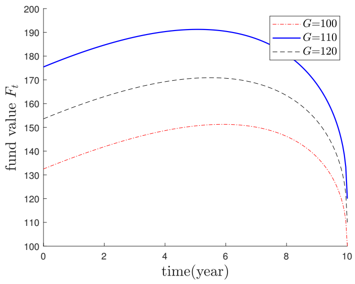
(b) Sensitivity to σ



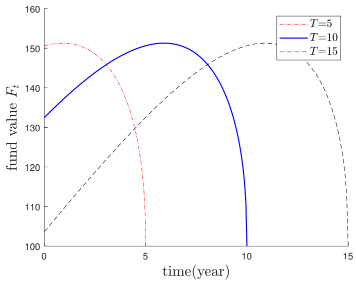
(c) Sensitivity to c



(d) Sensitivity to κ



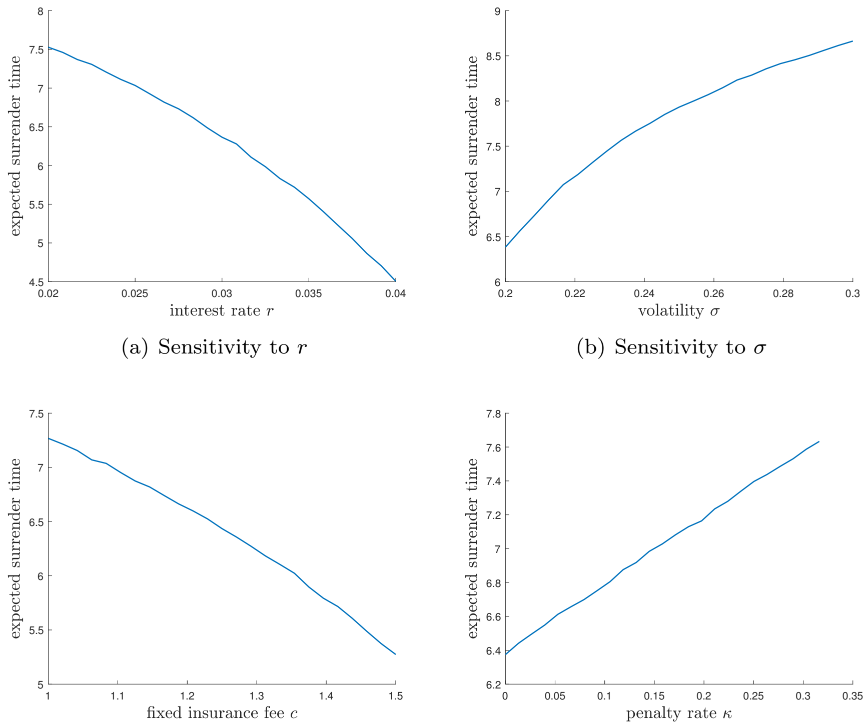
(e) Sensitivity to G



(f) Sensitivity to T

FIGURE 1. Sensitivity analysis of the optimal surrender boundary.

Figure 1(a) shows that the higher the risk-free interest rate is, the lower the surrender boundary is. This implies that in the policyholder's view, the higher the interest rate of the fund is, the more advantageous it is to own the underlying fund by surrendering the VA contract. That is, the policyholder tends to surrender the contract at the lower surrender boundary. On the other hand, Figure 1(b) shows that the surrender boundary is high, when the volatility is large. This indicates that if the volatility σ is large, the possession of the fund is disadvantageous. Figure 1(c) shows that the lower the fair insurance fee is, the higher the surrender boundary is. This implies that the lower the insurance fee c is, the lower the tendency is to terminate the contract by surrendering the VA contract. On the other hand, Figure 1(d) shows that if the surrender penalty constant κ is lower, the policyholder tends to surrender the VA contract at relatively lower surrender boundary. Figure 1(e) shows that the larger the guaranteed minimum benefit G is, the higher the surrender boundary is. In other words, the VA value increases as the guaranteed minimum benefit G increases. Since the optimal surrender boundary is a function of the time-to-maturity $(T-t)$, it is natural to appear like Figure 1(f) where the optimal surrender boundaries under different maturities are drawn over time t.



(a) Sensitivity to r (b) Sensitivity to σ

(c) Sensitivity to c (d) Sensitivity to κ

FIGURE 2. Sensitivity analysis of the expected optimal surrender time.

Now, we investigate the expected optimal surrender time when the policyholder is expected to surrender the contract based on the optimal surrender boundary in (3.3). Let us denote θ^* as the optimal surrender time. Note that since θ^* is a random time depending on the scenario of the underlying fund value, we estimate the expected optimal surrender time under the risk-neutral measure $\mathbb{Q}$. Then, we define the expected optimal surrender time by, respectively,

$$\mathbb{E}^{\mathbb{Q}}[\theta^*] \quad \text{where } \theta^* \equiv \inf\left\{t \in [0,T] \mid F_t \geq \mathcal{B}(t)\right\}. \tag{4.2}$$

By utilizing the Monte-Carlo simulation method with 100 time-steps and 100,000 paths, we estimate values of the expected optimal surrender time values under several parameters and illustrate the estimated values in Figure 2. The baseline parameters are the same as the values in (4.1). In Figures 2(a) and 2(b), we observe the negative relationship between the interest rate and the expected surrender time, and the positive relationship between the volatility and the expected surrender time, respectively. These relationships are consistent with the tendency of the optimal exercise time of general American-type derivative in that the optimal exercise time of the American-type derivative shortens as the sharpe ratio of the underlying asset increases. In Figures 2(c) and 2(d), we observe the negative relationship between the fixed insurance fee and the expected optimal surrender time, and the positive relationship between the penalty rate and the expected surrender time, respectively. These relationships are intuitive in that an increase in the insurance fee is a decrease in the contract value, and an increase in the penalty rate is an increase in the value. In terms of the policyholder, thus, it is beneficial to surrender early if the insurance fee increases, meanwhile it is beneficial to surrender late if the penalty rate increases.

5. Concluding remarks

In this paper, we have proposed a new model of a VA contract with a surrender option. In our model, the insurer does not withdraw any insurance fee from the policyholder's account, but the policyholder continuously pays the fixed insurance fee until the policyholder exercises the surrender option. Our model's problem can be categorized as an optimal stopping problem. Based on the Mellin transform, we have derived the integral equation solutions for the VA value and the optimal surrender boundary. We have solved numerically the integral equations and provide the values of the contract value function, the optimal surrender boundary, and the expected optimal surrender time under several model parameters.

For the analytic tractability of our model, we have adopted the Black-Scholes market framework. Consideration of stochastic volatility models or other general market frameworks, however, will not only make some technical contributions, but also shed more insight on understanding the effects of the surrender option and the fixed insurance fee.

Appendix A. Review of Mellin transform

Definition (Definition of the Mellin transform and inverse Mellin transform)**.** Let $g(\cdot)$ be a complex-valued and locally integrable function on $(0,\infty)$. Then, the Mellin transform $\mathcal{M}(g(\cdot);w)$ is defined as

$$\mathcal{M}(g(f);w)=\widehat{g}(w)\equiv\int_0^\infty g(f)f^{w-1}df,\ w\in\mathbb{C}. \tag{A.1}$$

If this integral converges for $a<Re(w)<b$, then the inverse of the Mellin transform is

$$g(f)=\mathcal{M}^{-1}(\widehat{g}(w);f)\equiv\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\widehat{g}(w)f^{-w}dw. \tag{A.2}$$

Remark A.1 (Basic properties of the Mellin transform).

(a) (Convolution property of Mellin transform). Let $g(\cdot)$ and $h(\cdot)$ be locally integrable functions on $(0,\infty)$. For given $a<w<b$, let us denote the Mellin transform of $g(f)$ and $h(f)$ as $\widehat{g}(w)$ and $\widehat{h}(w)$, respectively. Then, we define the Mellin convolution which is given by the inverse Mellin transform of $\widehat{g}(w)\widehat{h}(w)$:

$$\begin{aligned}g(f)\vee h(f)&\equiv\mathcal{M}^{-1}(\widehat{g}(w)\widehat{h}(w);f)\\ &=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\widehat{g}(w)\widehat{h}(w)f^{-w}dw=\int_0^\infty g\left(\frac{f}{u}\right)h(u)\frac{du}{u}.\end{aligned} \tag{A.3}$$

(b) (Inverse Mellin transform of exponential function). For given c_1, c_2 with $Re(c_1)>0$ and Mellin transform $\widehat{g}(w)=e^{c_1(w+c_2)^2}$ of $g(\cdot)$, the inverse Mellin transform of $\widehat{g}(w)$ is given

$$g(f)=\mathcal{M}^{-1}(\widehat{g}(w);f)=\frac{1}{2}(\pi c_1)^{-\frac{1}{2}}f^{c_2}e^{-\frac{1}{4c_1}(\log(f))^2}. \tag{A.4}$$

Appendix B. Proof of Theorem 3.1

Proof of (a). The non-homogeneous PDE in (2.11) can be rewritten as

$$\frac{\partial V}{\partial t}+\mathcal{L}V=\psi(t,f)\text{ and }V(T,f)=\phi(f), \tag{B.1}$$

where

$$\psi(t,f)=c1_{\{f<\mathcal{B}(t)\}}+\kappa e^{-\kappa(T-t)}f1_{\{f\geq\mathcal{B}(t)\}}\quad\text{and}\quad\phi(f)=\max(f,G).$$

By applying the Mellin transform to the PDE (B.1), the non-homogeneous PDE problem (B.1) can be transformed to the following non-homogeneous ordinary differential equation (ODE):

$$\frac{d\widehat{V}}{dt}(t,w)+\frac{1}{2}\sigma^2\mathcal{A}(w)\widehat{V}(t,w)=\widehat{\psi}(t,w), \tag{B.2}$$

where $\widehat{V}$ and $\widehat{\psi}$ are the Mellin transforms of the value function V and ψ, respectively, and $\mathcal{A}(w)$ is a quadratic function given $\mathcal{A}(w)=w^2+(1-\frac{2r}{\sigma^2})w-\frac{2r}{\sigma^2}$.

Then, we can easily deduce that the solution of non-homogeneous ODE (B.2) is given by

$$\widehat{V}(t,w) = e^{\frac{1}{2}\sigma^2\mathcal{A}(w)(T-t)}\widehat{\phi}(w) - \int_t^T e^{\frac{1}{2}\sigma^2\mathcal{A}(w)(\xi-t)}\widehat{\psi}(\xi,w)d\xi, \tag{B.3}$$

where $\widehat{\phi}$ is the Mellin transform of the function ϕ.

From the definition of the inverse Mellin transform in Appendix A, we can derive the following equation:

$$\begin{aligned} V(t,f) = {} & \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2\mathcal{A}(w)(T-t)}\widehat{\phi}(w)dw \\ & - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T e^{\frac{1}{2}\sigma^2\mathcal{A}(w)(\xi-t)}\widehat{\psi}(\xi,w)d\xi dw. \end{aligned} \tag{B.4}$$

Here, we will derive the analytic one-dimensional integral solution by using the properties of the Mellin transform. Let us define the kernel function $\mathcal{K}(t,f)$ as follows:

$$\mathcal{K}(s,f) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}\sigma^2\mathcal{A}(w)s} f^{-w}dw.$$

Then, $\mathcal{K}(t,f)$ can be expressed in the following form:

$$\mathcal{K}(t,f) = \exp\left(-\alpha(\tfrac{1+2r/\sigma^2}{2})^2\right)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha(w+\beta)^2} f^{-w}dw, \tag{B.5}$$

where $\alpha = \frac{\sigma^2 t}{2}$ and $\beta = \frac{1-2r/\sigma^2}{2}$.

We can easily deduce that the right-hand side of the equation (B.5) is the inverse Mellin transform of the exponential function $e^{\alpha(w+\beta)^2}$.

By using the property in A, we can get the following equation:

$$\mathcal{K}(s,f) = \exp\left(-\tfrac{\sigma^2 s}{2}\left(\tfrac{1+2r/\sigma^2}{2}\right)^2\right)\frac{f^{\frac{1-2r/\sigma^2}{2}}}{\sigma\sqrt{2\pi s}}e^{-\frac{1}{2}(\frac{\log(f)}{\sigma\sqrt{s}})^2}. \tag{B.6}$$

Then, the value function (B.4) can be expressed in terms of the Mellin transform of the kernel function $\mathcal{K}(t,f)$:

$$\begin{aligned} V(t,f) = {} & \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\widehat{\mathcal{K}}(T-t,w)\widehat{\phi}(w)dw \\ & - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_t^T \widehat{\mathcal{K}}(\xi-t,w)\widehat{\psi}(\xi,w)d\xi dw. \end{aligned} \tag{B.7}$$

By using the convolution property of the Mellin transform in Appendix A, the value $V(t,f)$ can be given as the following integral equation:

$$\begin{aligned} V(t,f) = {} & \int_0^\infty \phi(u)\mathcal{K}\left(T-t,\tfrac{f}{u}\right)\frac{1}{u}du \\ & - \int_t^T\int_0^\infty \psi(\xi,u)\mathcal{K}\left(\xi-t,\tfrac{f}{u}\right)\frac{1}{u}dud\xi. \end{aligned} \tag{B.8}$$

Let us calculate the first integral term of the right-hand side in (B.8). If the function ϕ is directly assigned to the first integral term, the integral equations are given as follows:

$$
\begin{aligned}
\int_0^\infty \phi(u)\mathcal{K}\left(T-t, \tfrac{f}{u}\right)\frac{1}{u}du &= \int_0^\infty \max(u,G)\mathcal{K}\left(T-t, \tfrac{f}{u}\right)\frac{1}{u}du \\
&= \int_0^G G\cdot\mathcal{K}\left(T-t, \tfrac{f}{u}\right)\frac{1}{u}du \\
&\quad + \int_G^\infty u\cdot\mathcal{K}\left(T-t, \tfrac{f}{u}\right)\frac{1}{u}du.
\end{aligned} \tag{B.9}
$$

By assigning the kernel function $\mathcal{K}$ in (B.6) to the first term of the above integral equation (B.9), we can derive the following equations:

(B.10)

$$
\begin{aligned}
&\int_0^G G\cdot\mathcal{K}\left(T-t, \tfrac{f}{u}\right)\frac{1}{u}du \\
&= \int_0^G Ge^{-\frac{\sigma^2}{2}(T-t)\left(\frac{1+2r/\sigma^2}{2}\right)^2}\frac{(f/u)^{\frac{1-2r/\sigma^2}{2}}}{\sigma\sqrt{2\pi(T-t)}}e^{-\frac{1}{2}(\frac{\log(f/u)}{\sigma\sqrt{T-t}})^2}\frac{1}{u}du \\
&= Ge^{-\frac{\sigma^2}{2}(T-t)\left(\frac{1+2r/\sigma^2}{2}\right)^2}\int_{\log(f/G)}^\infty \frac{e^{w(\frac{1-2r/\sigma^2}{2})}}{\sigma\sqrt{2\pi(T-t)}}e^{-\frac{1}{2}(\frac{w}{\sigma\sqrt{T-t}})^2}dw,\ (w=\log(f/u)) \\
&= Ge^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{\log(f/G)}^\infty \exp\left\{-\frac{1}{2}\left(\frac{w-\frac{1-2r/\sigma^2}{2}(\sigma^2(T-t))}{\sigma\sqrt{T-t}}\right)^2\right\}dw \\
&= Ge^{-r(T-t)}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{\log(f/G)+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{z^2}{2}}dz,\ \left(z=\frac{w-\frac{1-2r/\sigma^2}{2}(\sigma^2(T-t))}{\sigma\sqrt{T-t}}\right) \\
&= Ge^{-r(T-t)}\mathcal{N}\left(-\frac{\log(f/G)+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

where $\mathcal{N}(\cdot)$ is a standard cumulative normal distribution function.

Similar to the second term of the integral equation (B.9), we can get the following equations:

(B.11)

$$
\begin{aligned}
&\int_G^\infty u\cdot\mathcal{K}\left(T-t, \tfrac{f}{u}\right)\frac{1}{u}du \\
&= \int_G^\infty u\cdot e^{-\frac{\sigma^2}{2}(T-t)\left(\frac{1+2r/\sigma^2}{2}\right)^2}\frac{(f/u)^{\frac{1-2r/\sigma^2}{2}}}{\sigma\sqrt{2\pi(T-t)}}e^{-\frac{1}{2}(\frac{\log(f/u)}{\sigma\sqrt{T-t}})^2}\frac{1}{u}du \\
&= fe^{-\frac{\sigma^2}{2}(T-t)\left(\frac{1+2r/\sigma^2}{2}\right)^2}\int_{-\infty}^{\log(f/G)} \frac{e^{-w(\frac{1+2r/\sigma^2}{2})}}{\sigma\sqrt{2\pi(T-t)}}e^{-\frac{1}{2}(\frac{w}{\sigma\sqrt{T-t}})^2}dw,\ (w=\log(f/u))
\end{aligned}
$$

$$= f\frac{1}{\sigma\sqrt{2\pi(T-t)}}\int_{-\infty}^{\log(f/G)}\exp\left\{-\frac{1}{2}\left(\frac{w+(\frac{1+2r/\sigma^2}{2})\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)^2\right\}dw$$

$$= f\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{\log(f/G)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}} e^{-\frac{z^2}{2}}dz,\ \left(z=\frac{w+\frac{1+2r/\sigma^2}{2}(\sigma^2(T-t))}{\sigma\sqrt{T-t}}\right)$$

$$= f\cdot\mathcal{N}\left(\frac{\log(f/G)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right).$$

Since the computation of the second integral term in (B.8) is almost same as that of the first term, we omit the detailed computation here. If the function ψ is directly assigned to the second term, the integral equation can be rewritten by

(B.12)

$$\begin{aligned}
&\int_t^T\int_0^\infty \psi(\xi,u)\mathcal{K}\left(\xi-t,\tfrac{f}{u}\right)\frac{1}{u}dud\xi\\
&=\int_t^T\int_0^\infty\left(c\cdot 1_{\{u<\mathcal{B}(\xi)\}}+\kappa e^{-\kappa(T-\xi)}u\cdot 1_{\{u\geq\mathcal{B}(\xi)\}}\right)\mathcal{K}\left(\xi-t,\tfrac{f}{u}\right)\frac{1}{u}dud\xi\\
&=\int_t^T\int_0^{\mathcal{B}(\xi)} c\cdot\mathcal{K}\left(\xi-t,\tfrac{f}{u}\right)\frac{1}{u}dud\xi\\
&\quad+\int_t^T\int_{\mathcal{B}(\xi)}^\infty \kappa e^{-\kappa(T-\xi)}u\cdot\mathcal{K}\left(\xi-t,\tfrac{f}{u}\right)\frac{1}{u}dud\xi.
\end{aligned}$$

The computation in the first term of the last equation (B.12) is the same as that of the equation (B.10). Thus, we can easily deduce the following equation:

$$\begin{aligned}
&\int_t^T\int_0^{\mathcal{B}(\xi)} c\cdot\mathcal{K}\left(\xi-t,\tfrac{f}{u}\right)\frac{1}{u}dud\xi\\
&=\int_t^T e^{-r(\xi-t)}c\cdot\mathcal{N}\left(-\frac{\log(\frac{f}{\mathcal{B}(\xi)})+(r-\frac{\sigma^2}{2})(\xi-t)}{\sigma\sqrt{\xi-t}}\right)d\xi\\
&=\int_t^T e^{-r(\xi-t)}c\cdot\left\{1-\mathcal{N}\left(\frac{\log(\frac{f}{\mathcal{B}(\xi)})+(r-\frac{\sigma^2}{2})(\xi-t)}{\sigma\sqrt{\xi-t}}\right)\right\}d\xi\\
&=\frac{1-e^{-r(T-t)}}{r}c-c\cdot\int_t^T e^{-r(\xi-t)}\cdot\mathcal{N}\left(\frac{\log(\frac{f}{\mathcal{B}(\xi)})+(r-\frac{\sigma^2}{2})(\xi-t)}{\sigma\sqrt{\xi-t}}\right)d\xi.
\end{aligned}\tag{B.13}$$

Similarly, the second term of the last equation (B.12) is almost similar to the computation of the equation (B.11). Thus, we can derive the following equation:

$$\begin{aligned}&\int_t^T \int_{\mathcal{B}(\xi)}^{\infty} \kappa e^{-\kappa(T-\xi)} u \cdot \mathcal{K}\left(\xi - t, \tfrac{f}{u}\right) \frac{1}{u} du d\xi \\ &= \kappa e^{-\kappa(T-t)} f \cdot \int_t^T e^{\kappa(\xi-t)} \mathcal{N}\left(\frac{\log(\frac{f}{\mathcal{B}(\xi)}) + (r + \frac{\sigma^2}{2})(\xi - t)}{\sigma\sqrt{\xi - t}}\right) d\xi.\end{aligned} \tag{B.14}$$

Combining all the integral equations of (B.10), (B.11), (B.13) and (B.14), we can derive the following analytic integral equation solution:

$$\begin{aligned}&V(t,f) \\ &= f\mathcal{N}\left(\frac{\log(\frac{f}{G})+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) + Ge^{-r(T-t)}\mathcal{N}\left(-\frac{\log(\frac{f}{G})+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad - \frac{1-e^{-r(T-t)}}{r}c - \kappa e^{-\kappa(T-t)} f \cdot \int_t^T e^{\kappa(\xi-t)}\mathcal{N}\left(\frac{\log(\frac{f}{\mathcal{B}(\xi)})+(r+\frac{\sigma^2}{2})(\xi-t)}{\sigma\sqrt{\xi-t}}\right) d\xi \\ &\quad + c \cdot \int_t^T e^{-r(\xi-t)}\mathcal{N}\left(\frac{\log(\frac{f}{\mathcal{B}(\xi)})+(r-\frac{\sigma^2}{2})(\xi-t)}{\sigma\sqrt{\xi-t}}\right) d\xi \\ &= V_E(t,f) + V_P(t,f).\end{aligned}$$

Proof of (b). By using the *smooth-pasting* conditions at the optimal surrender boundary in (2.10) and the analytic integral equation solution in (3.2), the optimal surrender boundary can be given as the solution of the following integral equation:

$$V(t, \mathcal{B}(t)) = e^{-\kappa(T-t)}\mathcal{B}(t).$$

References

[1] J. Bertrand, P. Bertrand, and J. P. Ovarlez, *The Mellin Transform, The Transforms and Applications*, Handbook (A.D. Poularkas, ed.). CRC Press, Boca Raton, FL, 1996.

[2] C. Bernard, A. MacKay, and M. Muehlbeyer, *Optimal surrender policy for variable annuity guarantees*, Insurance Math. Econom. **55** (2014), 116–128. https://doi.org/10.1016/j.insmatheco.2014.01.006

[3] P. Ciurlia, *Valuation of European continuous-installment options*, Comput. Math. Appl. **62** (2011), no. 6, 2518–2534. https://doi.org/10.1016/j.camwa.2011.04.073

[4] T. F. Coleman, Y. Li, and M.-C. Patron, *Hedging guarantees in variable annuities under both equity and interest rate risks*, Insurance Math. Econom. **38** (2006), no. 2, 215–228. https://doi.org/10.1016/j.insmatheco.2005.06.002

[5] A. Erdlyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, vol. 1-2, New York: McGraw-Hill, 1954.

[6] A. Friedman, *Parabolic variational inequalities in one space dimension and smoothness of the free boundary*, J. Functional Analysis **18** (1975), 151–176. https://doi.org/10.1016/0022-1236(75)90022-1

[7] J. Huang, M. Subrahmanyam, and G. Yu, *Pricing and hedging American options: a recursive integration method*, Rev. Financ. Stud. **9** (1996), 277–300.

[8] H. Lee and B. Ko, *Valuing equity-indexed annuities with icicled barrier options*, J. Korean Statist. Soc. **47** (2018), no. 3, 330–346. https://doi.org/10.1016/j.jkss.2018.04.001

[9] G. Peskir and A. Shiryaev, *Optimal Stopping and Free-Boundary Problems*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2006.

[10] Y. Shen, M. Sherris, and J. Ziveyi, *Valuation of guaranteed minimum maturity benefits in variable annuities with surrender options*, Insurance Math. Econom. **69** (2016), 127–137. https://doi.org/10.1016/j.insmatheco.2016.04.006

[11] I. N. Sneddon, *The Use of Integral Transforms*, New York: Mcgraw-Hill, 1972.

Junkee Jeon
Department of Applied Mathematics & Institute of Natural Science
Kyung Hee University
Yongin 17104, Korea
Email address: junkeejeon@khu.ac.kr

Kyunghyun Park
Department of Mathematical Sciences
Seoul National University
Seoul 08826, Korea
Email address: luminary9699@snu.ac.kr