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UNIFIED APOSTOL-KOROBOV TYPE POLYNOMIALS AND RELATED POLYNOMIALS

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Abstract. Korobov type polynomials are introduced and extensively investigated many mathematicians ([1, 8–10, 12–14]). In this work, we define unified Apostol Korobov type polynomials and give some recurrences relations for these polynomials. Further, we consider the q-poly Korobov polynomials and the q-poly-Korobov type Changhee polynomials. We give some explicit relations and identities above mentioned functions.

1. Introduction

As usual, throughout this paper, $\mathbb{N}$ denotes the set natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{R}$ denotes the set of real numbers.

The generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ of order α and the generalized Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α are defined by the following generating functions (see, for detail [16–18])

$$\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}=\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{xt}, \tag{1}$$

$$(|t|<2\pi \text{ when } \lambda=1,\ |t|<|\log\lambda| \text{ when } \lambda\neq 1)$$

and

$$\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}=\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}e^{xt}, \tag{2}$$

$$(|t|<\pi \text{ when } \lambda=1,\ |t|<|\log(-\lambda)| \text{ when } \lambda\neq 1)\,.$$

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For $\lambda = 1$, we get the Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x)$ of order α and the Euler polynomials $\mathcal{E}_n^{(\alpha)}(x)$ of order α.

The Stirling numbers of the second kind $S_2(n,k)$ are defined ([7], [11]) and weight Stirling numbers of the second kind $S_2(n,m,x)$ [4] are defined the following generating functions, respectively

$$\sum_{n=0}^{\infty} S_2(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!} \tag{3}$$

and

$$\sum_{n=0}^{\infty} S_2(n,m,x)\frac{t^n}{n!} = \frac{e^{xt}(e^t-1)^m}{m!}. \tag{4}$$

The polylogarithm function $Li_k(z)$ ([2], [4], [6,7,11]) is defined by

$$Li_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},\ k \in \mathbb{Z},\ k \geq 1. \tag{5}$$

This function is convergent for $|z| < 1$, when $k = 1$

$$Li_1(z) = -\log(1-z). \tag{6}$$

The multi-logarithm ([7], [11]) is defined by

$$Li_{k_1,\ldots,k_n}(z) = \sum_{0<m_1<\cdots<m_n} \frac{z^{m_n}}{m_1^{k_1}\cdots m_n^{k_n}},\ k_i \geq 1,\ |z| < 1. \tag{7}$$

From (6), the following equation can be obtain

$$Li_{\underbrace{1,\ldots,1}_{n\text{-times}}}(z) = \frac{1}{n!}\left(-\log(1-z)\right)^n. \tag{8}$$

Kim et al. [8] defined the poly-Bernoulli polynomials

$$\sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!} = \frac{Li_k(1-e^{-t})}{e^t-1}e^{xt}, \tag{9}$$

when $k = 1$, $B_n^{(1)}(x) = B_n(x)$.

Hamahata in [5] defined by the poly-Euler polynomials

$$\sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{t(e^t+1)}e^{xt}, \tag{10}$$

when $k = 1$, $E_n^{(1)}(x) = E_n(x)$.

Kim et al. ([9], [10]) defined the Changhee polynomials and the first kind Korobov polynomials the following generating functions, respectively,

$$\sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!} = \frac{2}{t+2}(1+t)^x \tag{11}$$

and

$$\sum_{n=0}^{\infty} K_n(x \mid \lambda)\frac{t^n}{n!} = \frac{\lambda t}{(t+1)^{\lambda}-1}(1+t)^x \tag{12}$$

when $x=0$, $Ch_n(0)=Ch_n$ are called the Changhee numbers and $K_n(0 \mid \lambda) = K_n(\lambda)$ are called the Korobov numbers with $\lambda \in \mathbb{R}$.

The Korobov-type Changhee polynomials [9] are defined the following generating function with $\lambda \in \mathbb{R}$

$$\sum_{n=0}^{\infty} Ch_n(x \mid \lambda)\frac{t^n}{n!} = \frac{2}{(t+1)^{\lambda}+1}(1+t)^x. \tag{13}$$

Note that

$$\lim_{\lambda\to 1} Ch_n(x \mid \lambda) = Ch_n(x) \text{ and } \lim_{\lambda\to 0} Ch_n(x \mid \lambda) = (x)_n,$$

where

$$(x)_n = x(x-1)\cdots(x-(n-1)),$$

$\lambda \in \mathbb{R}$, Carlitz [3] introduced the degenerate Bernoulli polynomials by means of the following generating functions

$$\sum_{n=0}^{\infty} \mathcal{B}_n(x \mid \lambda)\frac{t^n}{n!} = \frac{t}{(\lambda t+1)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda} \tag{14}$$

so that

$$\mathcal{B}_n(x \mid \lambda) = \sum_{m=0}^{n}\binom{n}{m}\mathcal{B}_n(\lambda)\left(\frac{x}{\lambda}\right)_{n-m}\lambda^{n-m}.$$

From (14), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\mathcal{B}_n(x \mid \lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{t}{(\lambda t+1)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda}\\ &= \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},\end{aligned}$$

where $B_n(x)$ are Bernoulli polynomials.

Ozarslan [1] defined unified form of the Apostol-Bernoulli, Euler and Genocchi polynomials

$$\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x:k,a,b)\frac{t^n}{n!} = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt}, \tag{15}$$

$$k \in \mathbb{Z},\ a,\ b \in \mathbb{R}\setminus\{0\},\ \alpha,\ \beta \in \mathbb{C}.$$

2. Unified Apostol-Korobov type polynomials

In this section, we define unified Apostol-Korobov type polynomials. We investigate these polynomials and give some explicit identities and relations for these polynomials.

Firstly, we define the Apostol-Korobov polynomials $K_{n,\alpha}(x:\lambda)$ and the Apostol-Korobov type Changhee polynomials are defined the following generating functions, respectively

$$\sum_{n=0}^{\infty} K_{n,\alpha}(x:\lambda)\frac{t^n}{n!} = \frac{\lambda t}{\alpha(1+t)^{\lambda}-1}(1+t)^x \tag{16}$$

and

$$\sum_{n=0}^{\infty} Ch_{n,\alpha}(x:\lambda)\frac{t^n}{n!} = \frac{2}{\alpha(1+t)^{\lambda}+1}(1+t)^x, \tag{17}$$

where $\alpha \in \mathbb{N}_0$.

We consider the following unified form of the Apostol-Korobov polynomials

$$\sum_{n=0}^{\infty} \mathfrak{R}_{n,\beta}(x:k,a,b,\lambda)\frac{t^n}{n!} = \frac{2^{1-k}(\lambda t)^k}{\beta^b(1+t)^{\lambda}-a^b}(1+t)^x. \tag{18}$$

Remark 2.1. Setting $k=a=b=1$ and $\alpha=\beta$ in (18), we get

$$\mathfrak{R}_{n,\alpha}(x:k,1,1,\lambda) = K_{n,\alpha}(x:\lambda).$$

Remark 2.2. Setting $k=0$, $a=-1$, $b=1$ and $\alpha=\beta$ in (18), we get

$$\mathfrak{R}_{n,\alpha}(x:0,-1,1,\lambda) = Ch_{n,\alpha}(x:\lambda).$$

From (18), we get the following relations

$$\mathfrak{R}_{n,\beta}(x:k,a,b,\lambda) = \sum_{m=0}^{n}\binom{n}{m}\mathfrak{R}_{m,\beta}(0:k,a,b,\lambda)(x)_{n-m}, \tag{i}$$

$$\mathfrak{R}_{n,\beta}(x+y:k,a,b,\lambda) = \sum_{m=0}^{n}\binom{n}{m}\mathfrak{R}_{m,\beta}(x:k,a,b,\lambda)(y)_{n-m} \tag{ii}$$

and

$$\mathfrak{R}_{n,\beta}(x+y:k,a,b,\lambda) = \sum_{m=0}^{n}\binom{n}{m}\mathfrak{R}_{m,\beta}(0:k,a,b,\lambda)(x+y)_{n-m}. \tag{iii}$$

Theorem 2.3. *The following relation holds true*:

$$\beta^b\sum_{m=0}^{l}\frac{1}{m!(l-m)!}\mathfrak{R}_{m,\beta}(x:k,a,b,\lambda)(\lambda)_{l-m} - \frac{a^b}{l!}\mathfrak{R}_{l,\beta}(x:k,a,b,\lambda) = 2^{1-k}\lambda^k\frac{(x)_{l-k}}{(l-k)!}. \tag{19}$$

Proof. By using (18), we write by

$$\sum_{n=0}^{\infty} \mathfrak{R}_{n,\beta}(x:k,a,b,\lambda)\frac{t^n}{n!}\left(\beta^b(1+t)^{\lambda}-a^b\right)=2^{1-k}(\lambda t)^k(1+t)^x$$

and

$$\sum_{l=0}^{\infty}\left(\beta^b\sum_{m=0}^{l}\binom{l}{m}\mathfrak{R}_{m,\beta}(x:k,a,b,\lambda)(\lambda)_{l-m}-a^b\mathfrak{R}_{l,\beta}(x:k,a,b,\lambda)\right)\frac{t^l}{l!}$$
$$=2^{1-k}\lambda^k\sum_{l=0}^{\infty}(x)_l\frac{t^{l+k}}{l!}=2^{1-k}\lambda^k\sum_{l=k}^{\infty}(x)_{l-k}\frac{1}{(l-k)!}t^l.$$

Comparing the coefficients of t^n on both sides, we have (19). $\square$

Theorem 2.4. *There is the following relation between unified Apostol-Korobov polynomials and the weighted Stirling numbers of second kind $S(n,m,x)$ and unified Apostol-Bernoulli, Euler and Genocchi polynomials $\mathfrak{R}_{n,\beta}(x:k,a,b,\lambda)$ as*

$$\sum_{m=0}^{n-k}\frac{\mathfrak{R}_{m,\beta}(x:k,a,b,\lambda)S_2(n-k,m)}{(n-k)!}$$
$$=\sum_{i=0}^{n}\frac{1}{n!(n-i)!}\mathfrak{R}_{n-i,\beta}(0:k,a,b)\lambda^m k!S_2(i,k,x). \tag{20}$$

Proof. By replacing t by e^t-1 in (18), we get

$$\sum_{m=0}^{\infty}\mathfrak{R}_{m,\beta}(x:k,a,b,\lambda)\frac{(e^t-1)^m}{m!}=\frac{2^{1-k}\lambda^k(e^t-1)^k e^{xt}}{\beta^b e^{t\lambda}-a^b},$$

$$\sum_{m=0}^{\infty}\mathfrak{R}_{m,\beta}(x:k,a,b,\lambda)\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}=\frac{2^{1-k}(\lambda t)^k t^{-k}}{\beta^b e^{t\lambda}-a^b}k!\frac{(e^t-1)e^{xt}}{k!}.$$

By using (4) and (15) in the last equations, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\mathfrak{R}_{m,\beta}(x:k,a,b,\lambda)S_2(n,m)\frac{t^n}{n!}$$
$$=t^{-k}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\mathfrak{R}_{n-i,\beta}(0:k,a,b)\lambda^m k!S_2(i,k,x)\frac{t^n}{n!}.$$

From here, comparing the coefficients of t^n, we have (20). $\square$

From $\lim_{\mu\to 0}(1+\mu t)^{1/\mu}=e^t$, we consider the degenerate function of t which are given by

$$t=\lim_{\mu\to 0}\log(1+\mu t)^{1/\mu}.$$

$\frac{\log(1+\mu t)}{\mu}$ is called the degenerate function of t. Now, we consider the degenerate unified Apostol-Korobov type polynomials by

$$\sum_{n=0}^{\infty}\mathfrak{R}_{n,\beta,\mu}\left(x:k,a,b,\lambda\right)\frac{t^n}{n!}$$
$$= \frac{2^{1-k}\lambda^k\left(\log\left(1+\mu t\right)^{1/\mu}\right)^k}{\beta^b\left[1+\log\left(1+\mu t\right)^{1/\mu}\right]^\lambda - a^b}\left[1+\log\left(1+\mu t\right)^{1/\mu}\right]^x, \tag{21}$$

where $\mu \in \mathbb{R}^+$.

Theorem 2.5. *There is the following relation between the degenerate unified Apostol-Korobov type polynomials and weighted Stirling numbers of second kind* $S(n,m,x)$

$$\sum_{p=0}^{j-k}\mu^p l^p S_2\left(j-k,p\right)\sum_{m=0}^{\infty}\mathfrak{R}_{m,\beta,\mu}\left(x:k,a,b,\lambda\right)\frac{\lambda^{-m}}{m!\left(j-k\right)!}\sum_{l=0}^{m}\binom{m}{l}\left(-1\right)^{m-l} \tag{22}$$
$$= \sum_{i=0}^{j}\binom{j}{i}\mathfrak{R}_{j-i,\beta}\left(0:k,a,b\right)\lambda^{j-i}S_2\left(i,k,x\right).$$

Proof. By using t by $\frac{e^{\mu(e^t-1)}-1}{\mu}$ in (21), we get

$$\sum_{m=0}^{\infty}\mathfrak{R}_{m,\beta,\mu}\left(x:k,a,b,\lambda\right)\mu^{-m}\frac{\left(e^{\mu\left(e^t-1\right)}-1\right)^m}{m!} = \frac{2^{1-k}\lambda^k\left(e^t-1\right)^k e^{tx}}{\beta^b e^{t\lambda}-a^b}. \tag{23}$$

The left side of the equation (23),

(24)
$$\sum_{m=0}^{\infty}\mathfrak{R}_{m,\beta,\mu}\left(x:k,a,b,\lambda\right)\frac{\mu^{-m}}{m!}\sum_{l=0}^{m}\binom{m}{l}\left(-1\right)^{m-l}e^{\mu l\left(e^t-1\right)}$$
$$= \sum_{m=0}^{\infty}\mathfrak{R}_{m,\beta,\mu}\left(x:k,a,b,\lambda\right)\frac{\mu^{-m}}{m!}\sum_{l=0}^{m}\binom{m}{l}\left(-1\right)^{m-l}\sum_{p=0}^{\infty}\mu^p l^p\frac{\left(e^t-1\right)^p}{p!}$$
$$= \sum_{m=0}^{\infty}\mathfrak{R}_{m,\beta,\mu}\left(x:k,a,b,\lambda\right)\frac{\mu^{-m}}{m!}\sum_{l=0}^{m}\binom{m}{l}\left(-1\right)^{m-l}\sum_{p=0}^{\infty}\mu^p l^p\sum_{j=p}^{\infty}S_2\left(j,p\right)\frac{t^j}{j!}$$
$$= \sum_{j=0}^{\infty}\left(\sum_{p=0}^{j}\mu^p l^p S_2\left(j,p\right)\sum_{m=0}^{\infty}\mathfrak{R}_{m,\beta,\mu}\left(x:k,a,b,\lambda\right)\frac{\mu^{-m}}{m!}\sum_{l=0}^{m}\binom{m}{l}\left(-1\right)^{m-l}\right)\frac{t^j}{j!}.$$

By using (4) and (15) in right side of the equation (23)

$$= t^{-k}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\mathfrak{R}_{j-i,\beta}\left(0:k,a,b\right)\lambda^{n-i}S_2\left(i,k,x\right)\frac{t^n}{n!}. \tag{25}$$

From (24) and (25), comparing the coefficients of t^n, we have (22). $\square$

3. On the q-poly-Korobov polynomials and the q-poly-Korobov type Changhee polynomials

In this section, we introduce and investigate some properties and explicit relations between the q-poly-Korobov polynomials and the q-poly-Korobov type Changhee polynomials.

We define the q-poly-Korobov polynomials and the q-poly-Korobov type Changhee polynomials the following generating functions, respectively;

$$\sum_{n=0}^{\infty} K_{n,q}^{(k)}(x \mid \lambda)\frac{t^n}{n!} = \frac{\lambda Li_{[k,q]}\left(1-e^{-t}\right)}{(t+1)^{\lambda}-1}(1+t)^x \tag{26}$$

and

$$\sum_{n=0}^{\infty} Ch_{n,q}^{(k)}(x \mid \lambda)\frac{t^n}{n!} = \frac{2Li_{[k,q]}\left(1-e^{-t}\right)}{t\left((t+1)^{\lambda}+1\right)}(1+t)^x, \tag{27}$$

where $q \in \mathbb{R}$, $0 < q \leq 1$ and the poly-logarithm function is defined as

$$Li_{[k,q]}(t) = \sum_{n=1}^{\infty} \frac{t^n}{[n]_q^k}. \tag{28}$$

The polynomials $K_{n,q}^{(k)}(\lambda) := K_{n,q}^{(k)}(0 \mid \lambda)$ are called the q-poly-Korobov numbers and the polynomial $Ch_{n,q}^{(k)}(\lambda) = Ch_{n,q}^{(k)}(0 \mid \lambda)$ are called the q-poly-Korobov type Changhee numbers.

The q-numbers and q-factorial are defined by

$$[n]_q = \frac{1-q^n}{1-q},\ q \neq 1,\ [n]_q! = [n]_q[n-1]_q \cdots [1]_q,$$

$n \in \mathbb{N}$ and $q \in \mathbb{C}$, respectively where $[0]_q! = 1$.

The first values of the q-polylogarithm function for $k \leq 0$,

$$Li_{[0,q]}(t) = \frac{t}{1-t},\ Li_{[-1,q]}(t) = \frac{t}{(1-t)(1-qt)}.$$

The q-polylagarithm function for $k \leq 0$ is a rational functions. For k is a nonnegative integer

$$Li_{[-k,q]}(t) = \frac{1}{(1-q)^k}\sum_{l=0}^{k}\binom{k}{l}(-1)^l\frac{q^l t}{1-q^l t}.$$

For $n = 2$ in (8), we get $Li_{[1,1]}\left(1-e^{-t}\right) = \frac{t^2}{2!}$.

From (26) and (27), for $k = q = 1$, we have

$$K_{l,1}^{(1)}(x \mid \lambda) = \frac{1}{2}l\ K_{l-1}^{(k)}(x \mid \lambda) \text{ and } Ch_{l,1}^{(1)}(x \mid \lambda) = \frac{1}{2}l\ Ch_{l-1}^{(k)}(x \mid \lambda),\ l \geq 1.$$

Theorem 3.1. *The following relations holds true*:

$$K_{n,q}^{(k)}(x \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} K_{m,q}^{(k)}(\lambda)\, (x)_{n-m}\,,$$

$$\text{(i)} \qquad Ch_{n,q}^{(k)}(x \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} Ch_{m,q}^{(k)}(\lambda)\, (x)_{n-m}\,,$$

$$K_{n,q}^{(k)}(x+y \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} K_{m,q}^{(k)}(x \mid \lambda)\, (y)_{n-m}$$

and

$$\text{(ii)} \qquad Ch_{n,q}^{(k)}(x+y \mid \lambda) = \sum_{m=0}^{n} \binom{n}{m} Ch_{m,q}^{(k)}(x \mid \lambda)\, (y)_{n-m}\,.$$

Theorem 3.2. *There are the following relationships for the q-poly-Korobov polynomials and the q-poly-Korobov type Changhee polynomials as*

$$\begin{aligned} &K_{m,q}^{(k)}(x+\lambda \mid \lambda) - K_{m,q}^{(k)}(x \mid \lambda) \\ &= \lambda \sum_{j=0}^{m} \binom{m}{j} \sum_{n=0}^{j-1} \frac{(-1)^{n+1+j}\,(n+1)!}{[n+1]_q^k} S_2\,(j, n+1)\,(x)_{m-j} \end{aligned} \tag{29}$$

and

$$\begin{aligned} &m\left(Ch_{m-1,q}^{(k)}(x+\lambda \mid \lambda) + Ch_{m-1,q}^{(k)}(x \mid \lambda)\right) \\ &= 2 \sum_{j=0}^{m} \binom{m}{j} \sum_{n=0}^{j-1} \frac{(-1)^{n+1+j}\,(n+1)!}{[n+1]_q^k} S_2\,(j, n+1)\,(x)_{m-j}\,. \end{aligned} \tag{30}$$

Proof. By using (3), (26) and (28), we write as

$$\begin{aligned} &\sum_{m=0}^{\infty} \left(K_{m,q}^{(k)}(x+\lambda \mid \lambda) - K_{m,q}^{(k)}(x \mid \lambda)\right) \frac{t^n}{n!} \\ &= \lambda Li_{k,q}\left(1-e^{-t}\right)(1+t)^x \\ &= \lambda \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,(n+1)!}{[n+1]_q^k} \frac{\left(e^{-t}-1\right)^{n+1}}{(n+1)!}\,(1+t)^x \\ &= \lambda \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,(n+1)!}{[n+1]_q^k} \sum_{j=n+1}^{\infty} S_2\,(j, n+1)\,\frac{(-t)^j}{j!} \sum_{l=0}^{\infty} (x)_l\, \frac{t^l}{l!} \\ &= \lambda \sum_{m=0}^{\infty} \left(\sum_{j=0}^{m} \binom{m}{j} \sum_{n=0}^{j-1} \frac{(-1)^{n+1+j}\,(n+1)!}{[n+1]_q^k} S_2\,(j, n+1)\,(x)_{m-j} \right) \frac{t^m}{m!}. \end{aligned}$$

Comparing the coefficients of both sides, we have (29).

Since the proof of (30) is similar to (29), we omit it. □

From (29) and (30), we have the following Corollary 3.3.

Corollary 3.3. *There is the following relation between the q-poly-Korobov polynomials and the q-poly-Korobov type Changhee polynomials*

$$\begin{aligned} &2\left[K_{m,q}^{(k)}(x+\lambda \mid \lambda)-K_{m,q}^{(k)}(x \mid \lambda)\right] \\ &=\lambda m\left[Ch_{m-1,q}^{(k)}(x+\lambda \mid \lambda)+Ch_{m-1,q}^{(k)}(x \mid \lambda)\right]. \end{aligned}$$

Theorem 3.4. *The following relation for the q-poly-Korobov polynomial holds true*:

$$\begin{aligned} &\left(\sum_{m=0}^{n}\binom{n}{m}K_{m,q}^{(k)}(x \mid \lambda)\,(x)_{n-m}-K_{n,q}^{(k)}(x \mid \lambda)\right) \\ &=\lambda\sum_{j=0}^{m}\binom{m}{j}\sum_{n=0}^{j-1}\frac{(-1)^{n+1+j}\,(n+1)!}{[n+1]_q^k}S_2\,(j,n+1)\,(x)_{m-j}\,. \end{aligned} \tag{31}$$

Proof. By using (3) and (26), we write as

$$\sum_{n=0}^{\infty}K_{n,q}^{(k)}(x \mid \lambda)\frac{t^n}{n!}\left((1+t)^{\lambda}-1\right)=\lambda Li_{k,q}\left(1-e^{-t}\right)(1+t)^x$$

and

$$\begin{aligned} &\sum_{m=0}^{\infty}K_{m,q}^{(k)}(x \mid \lambda)\frac{t^m}{m!}\sum_{l=0}^{\infty}(x)_l\,\frac{t^l}{l!}-\sum_{n=0}^{\infty}K_{n,q}^{(k)}(x \mid \lambda)\frac{t^n}{n!} \\ &=\lambda\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}\sum_{n=0}^{j-1}\frac{(-1)^{n+1+j}\,(n+1)!}{[n+1]_q^k}S_2\,(j,n+1)\,(x)_{m-j}\right)\frac{t^m}{m!}. \end{aligned}$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{n!}$, we have (31). □

From (29) and (31), we have the following Corollary 3.5.

Corollary 3.5. *There is another relation for the q-poly-Korobov polynomials as*

$$\begin{aligned} &\left(K_{m,q}^{(k)}(x+\lambda \mid \lambda)-K_{m,q}^{(k)}(x \mid \lambda)\right) \\ &=\sum_{n=0}^{m}\binom{m}{n}K_{n,q}^{(k)}(x \mid \lambda)\,(\lambda)_{m-n}-K_{m,q}^{(k)}(x \mid \lambda). \end{aligned}$$

Theorem 3.6. *The following relation holds true*:

$$\left(\sum_{n=0}^{m}n\binom{m}{n}Ch_{n-1,q}^{(k)}(x \mid \lambda)\,(x)_{m-n}+mCh_{m-1,q}^{(k)}(x \mid \lambda)\right)$$

$$= 2\sum_{j=0}^{m}\binom{m}{j}\sum_{n=0}^{j-1}\frac{(-1)^{n+1+j}(n+1)!}{[n+1]_q^k}S_2(j,n+1)(x)_{m-j}\,. \tag{32}$$

From (30) and (32), we have the following Corollary.

Corollary 3.7. *There is the another relation for the q-poly-Korobov type Changhee polynomial*:

$$\begin{aligned} &m\left(Ch_{m-1,q}^{(k)}(x+\lambda\mid\lambda)+Ch_{m-1,q}^{(k)}(x\mid\lambda)\right)\\ =&\sum_{n=0}^{m}n\binom{m}{n}Ch_{n-1,q}^{(k)}(x\mid\lambda)(x)_{m-n}+mCh_{m-1,q}^{(k)}(x\mid\lambda). \end{aligned}$$

Theorem 3.8. *There is the following relation between the q-poly-Korobov polynomials and the Bernoulli polynomials*

$$\sum_{n=0}^{\infty}K_{n,q}^{(k)}(x\mid\lambda)S_2(m,n)(-1)^m=\sum_{l=0}^{m}\binom{m}{l}\frac{\lambda^{m-l}(-1)^l\,l!}{[l+1]_q^k}B_{m-l}\left(\frac{x}{\lambda}\right). \tag{33}$$

Proof. Replacing t by $e^{-t}-1$ in (26), we get

$$\sum_{n=0}^{\infty}K_{n,q}^{(k)}(x\mid\lambda)\frac{(e^{-t}-1)^n}{n!}=\frac{\lambda e^{-tx}}{e^{-t\lambda}-1}Li_{k,q}(-t)$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty}K_{n,q}^{(k)}(x\mid\lambda)\sum_{m=n}^{\infty}S_2(m,n)(-1)^m\frac{t^m}{m!}\\ =&-\sum_{m=0}^{\infty}B_m\left(\frac{x}{\lambda}\right)(-\lambda)^m\frac{t^m}{m!}\sum_{l=0}^{\infty}\frac{(-1)^{l+1}\,l!}{[l+1]_q^k}\frac{t^l}{l!}. \end{aligned}$$

By using Cauchy product and comparing the coefficients, we have (33). □

Theorem 3.9. *There is the following relation between the q-poly-Korobov type Changhee polynomials and the Euler polynomials*:

$$\begin{aligned} &\sum_{n=0}^{m}nCh_{n-1,q}^{(k)}(x\mid\lambda)S_2(m,n)(-1)^m\\ =&-m\sum_{l=0}^{m-1}\binom{m-1}{l}\frac{(-\lambda)^{m-l-1}\,l!\,(-1)^l}{[l+1]_q^k}E_{m-1-l}\left(\frac{x}{\lambda}\right). \end{aligned} \tag{34}$$

Proof. Using t by $e^{-t}-1$ in (27), we get

$$\sum_{n=0}^{\infty}Ch_{n,q}^{(k)}(x\mid\lambda)\frac{(e^{-t}-1)^{n+1}}{n!}=\frac{2e^{-tx}}{e^{-t\lambda}+1}Li_{k,q}(-t)\,,$$

$$\sum_{n=0}^{\infty}nCh_{n-1,q}^{(k)}(x\mid\lambda)\sum_{m=n}^{\infty}S_2(m,n)(-1)^m\frac{t^m}{m!}$$

$$= -t\sum_{r=0}^{\infty} E_r\left(\frac{x}{\lambda}\right)(-\lambda)^r \frac{t^r}{r!}\sum_{l=0}^{\infty}\frac{l!\,(-1)^l}{[l+1]_q^k}\frac{t^l}{l!}$$

and

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m} nCh_{n-1,q}^{(k)}(x\mid\lambda)S_2(m,n)(-1)^m\frac{t^m}{m!}$$
$$= -\sum_{m=0}^{\infty} m\sum_{l=0}^{m-1}\binom{m-1}{l}E_{m-1-l}\left(\frac{x}{\lambda}\right)(-\lambda)^{m-l-1}\frac{l!\,(-1)^l}{[l+1]_q^k}\frac{t^m}{m!}.$$

Comparing the coefficients, we have (34). □

4. Conclusion

In recent years, many mathematicians studied Korobov polynomials and Korobov type Changhee polynomials. The classical Bernoulli polynomials and Euler polynomials are studied many mathematicians ([1–18]). Srivastava [16], Srivastava et al. in ([17, 18]) introduced and investigated basic properties of these polynomials. They proved some theorems and explicit relations for these polynomials. Carlitz ([3], [4]) introduced degenerate Bernoulli polynomials and weighted degenerate Bernoulli polynomials. Bayad et al. [2] and Hamahata [4], Imatomi et al. [7], Kim et al. ([8–10, 12]) considered and investigated poly-Bernoulli and poly-Euler polynomials. Kruchinin [14] introduced Korobov polynomials. Kim et al. [12] introduced Korobov type polynomials and Korobov-type Changhee polynomials.

In this work, we define the unified form of the Apostol-Korobov, Apostol-Korobov type Changhee polynomials. We give explicit relations for these polynomials. Further, we define the q-poly-Korobov polynomials and the q-poly-Korobov type Changhee polynomials. We prove some relations between these polynomials and the Bernoulli polynomials and the Euler polynomials.

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