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STABILITY OF PARTIALLY PEXIDERIZED EXPONENTIAL-RADICAL FUNCTIONAL EQUATION

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Abstract. Let $\mathbb{R}$ be the set of real numbers, $f, g : \mathbb{R} \to \mathbb{R}$ and $\epsilon \geq 0$. In this paper, we consider the stability of partially pexiderized exponential-radical functional equation

$$f\left(\sqrt[N]{x^N + y^N}\right) = f(x)g(y)$$

for all $x, y \in \mathbb{R}$, i.e., we investigate the functional inequality

$$\left|f\left(\sqrt[N]{x^N + y^N}\right) - f(x)g(y)\right| \leq \epsilon$$

for all $x, y \in \mathbb{R}$.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}$, X, and Y the set of real numbers, a real normed space, and a real Banach space, respectively, and $\epsilon \geq 0$ will be fixed constants. A mapping $f : X \to Y$ is called *a monomial of degree* N if it satisfies the functional equation

$$\Delta_y^N f(x) - N! f(y) = 0$$

for all $x, y \in X$, where the difference operator Δ_y is defined by $\Delta_y f(x) = f(x+y) - f(x)$ for all $x, y \in X$ and Δ_y^N $(N = 2, 3, 4, \ldots)$ are defined by the iteration $\Delta_y^{N+1} f = \Delta_y \left(\Delta_y^N f\right)$ for all $N = 1, 2, 3, \ldots$. Using iterations we can see that

$$\Delta_y^N f(x) = \sum_{k=0}^{N} \binom{N}{k} (-1)^k f\left(x + (N-k)y\right)$$

for all $x, y \in X$. A mapping $f : X \to Y$ is said to be *exponential* if f satisfies the functional equation

$$f(x+y) - f(x)f(y) = 0$$

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for all $x, y \in X$. A mapping $f : X \to Y$ is said to be *radical* if f satisfies the functional equation

$$f\left(\sqrt[N]{x^N + y^N}\right) = f(x)f(y)$$

for all $x, y \in X$.

The Ulam problem for functional equations goes back to 1940 when S. M. Ulam proposed the following [9]:

Let f be a mapping from a group G_1 to a metric group G_2 with metric $d(\cdot, \cdot)$ such that

$$d(f(xy),\, f(x)f(y)) \le \epsilon.$$

Then does there exist a group homomorphism h and $\theta_\epsilon > 0$ such that

$$d(f(x),\, h(x)) \le \theta_\epsilon$$

for all $x \in G_1$?

This problem was solved affirmatively by D. H. Hyers under the assumption that G_2 is a Banach space (see Hyers [6], Hyers-Isac-Rassias [7]).

As a result of the Ulam problem for exponential functional equation, it is proved that if $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x + y) - f(x)f(y)| \le \epsilon$$

for all $x, y \in \mathbb{R}$, then f is either a bounded function satisfying

$$|f(x)| \le \frac{1}{2}\left(1 + \sqrt{1 + 4\epsilon}\right)$$

for all $x \in \mathbb{R}$, or an unbounded function satisfying

$$f(x + y) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$ (see Baker [1] or Baker-Lawrence-Zorzitto [2]). Due to Székelyhidi [8], the above result was generalized to the case when the difference $f(x + y) - f(x)f(y)$ is bounded for each fixed y (or equivalently, for each fixed x). In particular, Chung, Choi and Lee [5] investigated the functional inequality

$$|f(x + y) - f(x)g(y)| \le \epsilon$$

for all $x, y \in G$, where $f, g : G \to \mathbb{R}$ and G be a commutative group which is 2-divisible. Also, Choi in [4] showed that stability of functional equation

$$f\left(\sqrt[N]{x^N + y^N}\right) = f(x)f(y) \tag{1.1}$$

for all $x, y \in \mathbb{R}$, where $f : \mathbb{R} \to \mathbb{R}$, i.e., we consider the functional inequalities

$$\left|f\left(\sqrt[N]{x^N + y^N}\right) - f(x)f(y)\right| \le \phi(x),$$
$$\left|f\left(\sqrt[N]{x^N + y^N}\right) - f(x)f(y)\right| \le \psi(x, y)$$

for all $x, y \in \mathbb{R}$, where $\phi : \mathbb{R} \to \mathbb{R}^+$ is an arbitrary function and $\psi : \mathbb{R}^2 \to \mathbb{R}^+$ satisfies a certain condition.

In this paper, we investigate the stability of partially pexiderized exponential-radical functional equation

$$f\left(\sqrt[N]{x^N+y^N}\right)=f(x)g(y)$$

for all $x,y\in\mathbb{R}$, where $f,g:\mathbb{R}\to\mathbb{R}$, i.e., we investigate the functional inequality

$$\left|f\left(\sqrt[N]{x^N+y^N}\right)-f(x)g(y)\right|\le\epsilon$$

for all $x,y\in\mathbb{R}$, where $f,g:\mathbb{R}\to\mathbb{R}$.

2. Solutions of partially pexiderized exponential-radical functional equation

In this section, we consider the solutions of the functional equation

$$f\left(\sqrt[N]{x^N+y^N}\right)=f(x)g(y)$$

for all $x,y\in\mathbb{R}$, where $f,g:\mathbb{R}\to\mathbb{R}$. We exclude the trivial case when $f(x)=0$ or $g(x)=0$ for all $x\in\mathbb{R}$. We need the following lemma.

Lemma 2.1 ([3, Corollary 2.2]). *A function $f:\mathbb{R}\to\mathbb{R}$ satisfies functional equation*

$$f\left(\sqrt[N]{x^N+y^N}\right)=f(x)f(y)\tag{2.1}$$

if and only if there exists a solution $h:P\to\mathbb{R}$ to the functional equation

$$h(x+y)=h(x)h(y)\tag{2.2}$$

such that $f(x)=h(x^N)$ for all $x\in\mathbb{R}$, where $P:=\{x^N\,|\,x\in\mathbb{R}\}$.

Theorem 2.2. *A function $f:\mathbb{R}\to\mathbb{R}$ satisfies functional equation*

$$f\left(\sqrt[N]{x^N+y^N}\right)=f(x)g(y)\tag{2.3}$$

if and only if there exists a solution $h:P\to\mathbb{R}$ to the functional equation (2.2) *such that*

$$\begin{cases}f(x)=\alpha h(x^N),\\ g(x)=h(x^N)\end{cases}\tag{2.4}$$

for all $x\in\mathbb{R}$, where $P:=\{x^N\,|\,x\in\mathbb{R}\}$ and $\alpha\in\mathbb{R}\setminus\{0\}$.

Proof. Replacing x by y and y by x in (2.3) we have

$$f\left(\sqrt[N]{x^N+y^N}\right)=f(y)g(x)\tag{2.5}$$

for all $x,y\in\mathbb{R}$. Subtracting (2.5) from (2.3) we have

$$g(x)f(y)=f(x)g(y)\tag{2.6}$$

for all $x,y\in\mathbb{R}$. We fix $y=y_0$ with $f(y_0)\neq0$ and obtain

$$g(x)=cf(x)\tag{2.7}$$

for all $x \in \mathbb{R}$, where $c \in \mathbb{R} \setminus \{0\}$ with $c = \frac{g(y_0)}{f(y_0)}$. Hence, (2.7) in (2.3) we get

$$f\left(\sqrt[N]{x^N + y^N}\right) = cf(x)f(y) \tag{2.8}$$

for all $x, y \in \mathbb{R}$, where $c \in \mathbb{R} \setminus \{0\}$. We define $r : \mathbb{R} \to \mathbb{R}$ given by

$$r(x) = cf(x) \tag{2.9}$$

for all $x \in \mathbb{R}$. From (2.8) and (2.9) we get

$$r\left(\sqrt[N]{x^N + y^N}\right) = r(x)r(y) \tag{2.10}$$

for all $x, y \in \mathbb{R}$. Thus, r satisfies Lemma 2.1, i.e., there exists a solution $h : P \to \mathbb{R}$ to the functional equation (2.2) such that $r(x) = h(x^N)$ for all $x \in \mathbb{R}$, where $P := \{x^N \mid x \in \mathbb{R}\}$. Hence, from (2.7), (2.9) we get (2.4). Now, the proof is complete. □

3. Stability of partially pexiderized exponential-radical functional equation

In this section, we consider the stability of functional equation

$$f\left(\sqrt[N]{x^N + y^N}\right) = f(x)g(y)$$

for all $x, y \in \mathbb{R}$, where $f, g : \mathbb{R} \to \mathbb{R}$, i.e., we deal with the functional inequality

$$\left|f\left(\sqrt[N]{x^N + y^N}\right) - f(x)g(y)\right| \leq \epsilon$$

for all $x, y \in \mathbb{R}$, where $f, g : \mathbb{R} \to \mathbb{R}$.

Theorem 3.1. *Let $\epsilon \geq 0$ and $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the functional inequality*

$$\left|f\left(\sqrt[N]{x^N + y^N}\right) - f(x)g(y)\right| \leq \epsilon \tag{3.1}$$

for all $x, y \in \mathbb{R}$. Then either f is a bounded function and

$$\left|g\left(\sqrt[N]{x^N + y^N}\right) - g(x)g(y)\right| \leq \lambda\epsilon, \tag{3.2}$$

$$|f(x) - f(0)g(x)| \leq \epsilon \tag{3.3}$$

for all $x, y \in \mathbb{R}$, where

$$\lambda = \frac{2 + M_g}{M_f}, \quad M_f = \sup_{y \in G} |f(y)|, \quad M_g = \sup_{y \in G} |g(y)|,$$

or f, g satisfy the functional equation

$$f\left(\sqrt[N]{x^N + y^N}\right) = f(x)g(y) \tag{3.4}$$

for all $x, y \in \mathbb{R}$, where $f, g : \mathbb{R} \to \mathbb{R}$.

Proof. First, we assume that g is bounded. From inequality (3.1) we have

$$\begin{aligned}
(3.5)\quad & |f(z)|\left|g\left(\sqrt[N]{x^N+y^N}\right)-g(x)g(y)\right| \\
&= \Big|f(z)g\left(\sqrt[N]{x^N+y^N}\right)-f\left(\sqrt[N]{x^N+y^N+z^N}\right)+f\left(\sqrt[N]{x^N+y^N+z^N}\right) \\
&\quad -f\left(\sqrt[N]{z^N+x^N}\right)g(y)+f\left(\sqrt[N]{z^N+x^N}\right)g(y)-f(z)g(x)g(y)\Big| \\
&\le \left|f(z)g\left(\sqrt[N]{x^N+y^N}\right)-f\left(\sqrt[N]{x^N+y^N+z^N}\right)\right| \\
&\quad +\left|f\left(\sqrt[N]{z^N+x^N}\right)g(y)-f\left(\sqrt[N]{x^N+y^N+z^N}\right)\right| \\
&\quad +\left|f\left(\sqrt[N]{z^N+x^N}\right)-f(z)g(x)\right||g(y)| \\
&\le (2+|g(y)|)\epsilon
\end{aligned}$$

for all $x, y, z \in \mathbb{R}$. It follows from (3.5) that

$$\left|g\left(\sqrt[N]{x^N+y^N}\right)-g(x)g(y)\right| \le \left(\frac{2+M_g}{|f(z)|}\right)\epsilon \tag{3.6}$$

for all $x, y, z \in \mathbb{R}$. Taking the infimum of the right hand side of (3.6) with respect to z we have

$$\left|g\left(\sqrt[N]{x^N+y^N}\right)-g(x)g(y)\right| \le \left(\frac{2+M_g}{M_f}\right)\epsilon \tag{3.7}$$

for all $x, y \in \mathbb{R}$. Thus, we get (3.2). Now, putting $x = 0$ in (3.1) we have

$$|f(y)-f(0)g(y)| \le \epsilon \tag{3.8}$$

for all $y \in \mathbb{R}$. Thus, we get (3.3). Since g is a bounded, in view of (3.8), f is also bounded. Secondly, we assume that g is unbounded. Choosing a sequence $y_n \in \mathbb{R}$, $n = 1, 2, 3, \ldots$, such that $|g(y_n)| \to \infty$ as $n \to \infty$, putting $y = y_n$, $n = 1, 2, 3, \ldots$, in (3.1), dividing the result by $|g(y_n)|$ we have

$$\left|\frac{f\left(\sqrt[N]{x^N+y_n^N}\right)}{g(y_n)}-f(x)\right| \le \frac{\epsilon}{|g(y_n)|} \tag{3.9}$$

for all $x \in \mathbb{R}$. Letting $n \to \infty$ we have

$$f(x) = \lim_{n\to\infty}\frac{f\left(\sqrt[N]{x^N+y_n^N}\right)}{g(y_n)} \tag{3.10}$$

for all $x \in \mathbb{R}$. Multiplying both sides of (3.10) by $g(y)$ and using (3.1) and (3.10) we have

$$g(y)f(x) = \lim_{n\to\infty}\frac{g(y)f\left(\sqrt[N]{x^N+y_n^N}\right)}{g(y_n)} \tag{3.11}$$

$$= \lim_{n\to\infty} \frac{f\left(\sqrt[N]{x^N + y^N + y_n^N}\right) + R(x,y,y_n)}{g(y_n)}$$

$$= \lim_{n\to\infty} \frac{f\left(\sqrt[N]{\left(\sqrt[N]{x^N + y^N}\right)^N + y_n^N}\right)}{g(y_n)} + \lim_{n\to\infty} \frac{R(x,y,y_n)}{g(y_n)}$$

$$= f\left(\sqrt[N]{x^N + y^N}\right) + \lim_{n\to\infty} \frac{R(x,y,y_n)}{g(y_n)}$$

for all $x, y \in \mathbb{R}$, where $R(x,y,y_n) = g(y) f\left(\sqrt[N]{x^N + y_n^N}\right) - f\left(\sqrt[N]{x^N + y^N + y_n^N}\right)$. Using (3.1) we have

$$\lim_{n\to\infty} \frac{|R(x,y,y_n)|}{g(y_n)} \le \lim_{n\to\infty} \frac{\epsilon}{g(y_n)} = 0 \tag{3.12}$$

for all $x, y \in \mathbb{R}$. Thus, it follows from (3.10)∼(3.12) that

$$f\left(\sqrt[N]{x^N + y^N}\right) = f(x)g(y) \tag{3.13}$$

for all $x, y \in \mathbb{R}$. Now, the proof is complete. □

In particular, if $g = f$ in (3.1), we have the following.

Corollary 3.2. *Let $\epsilon \ge 0$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy the functional inequality*

$$\left| f(x)f(y) - f\left(\sqrt[N]{x^N + y^N}\right) \right| \le \epsilon \tag{3.14}$$

for all $x, y \in \mathbb{R}$. Then either f is a bounded function satisfying

$$|f(x)| \le \frac{1}{2}\left(1 + \sqrt{1 + 4\epsilon}\right) \tag{3.15}$$

for all $x \in \mathbb{R}$, or f satisfies the functional equation

$$f\left(\sqrt[N]{x^N + y^N}\right) = f(x)f(y) \tag{3.16}$$

for all $x, y \in \mathbb{R}$, where $f : \mathbb{R} \to \mathbb{R}$.

Proof. First, we assume that f is bounded. Using the triangle inequality with (3.14) and letting $M := \sup_{x\in\mathbb{R}} |f(x)|$ we have

$$|f(x)f(y)| \le \left| f\left(\sqrt[N]{x^N + y^N}\right) \right| + \epsilon \le M + \epsilon \tag{3.17}$$

for all $x, y \in \mathbb{R}$. Taking the supremum of the left hand side of (3.17) with respect to y we have

$$|f(x)|M \le M + \epsilon$$

for all $x \in \mathbb{R}$, which implies

$$M\left(|f(x)| - 1\right) \le \epsilon \tag{3.18}$$

for all $x \in \mathbb{R}$. The inequality (3.18) holds for all $x \in \mathbb{R}$ such that $|f(x)| \le 1$. If $|f(x)| > 1$, then we have

$$|f(x)|\,(|f(x)| - 1) \le \epsilon \tag{3.19}$$

for all $x \in \mathbb{R}$. Fixing x and solving the quadratic inequality (3.19) we get (3.15). Now, the remaining part of the proof is the similar as that of Theorem 3.1 (see (3.9)∼(3.13)) we get (3.16). This completes the proof. □

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