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# COINCIDENCE AND FIXED POINT RESULTS FOR GENERALIZED WEAK CONTRACTION MAPPING ON $b$-METRIC SPACES

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**Abstract.** In this paper, we introduce the modification of a generalized $(\Psi, L)-$weak contraction and we prove some coincidence point results for self-mappings $G, T$ and $S$, and some fixed point results for some maps by using a $(c)-$comparison function and a comparison function in the sense of a $b$-metric space.

## 1. Introduction

Bakhtin [6] and Czerwik [11] introduced the notion of $b-$metric spaces as a generalization of the notion of metric spaces. The idea of $b-$metric spaces has weaker than the triangular inequality axiom.

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Also, many authors gave some fixed point theorems in the notion of metric spaces, for example see [1, 2, 4, 5, 7, 8, 9, 15, 22, 24, 25, 30, 31, 33, 34, 35, 36, 37, 38, 39, 40]. Also, for some work on $b-$metric, we refer the reader to [3, 10, 12, 16, 17, 18, 19, 20, 21, 23, 26, 27, 28, 32].

Now, we present the definition of the $b$-metric space.

**Definition 1.1.** ([6, 11]) Let $X$ be a nonempty set and $s \geq 1$ be a real number. A function $d : X \times X \to [0, \infty)$ is called a $b-$metric if it satisfies the following properties for each $x, y, z \in X$.
- (b1) $d(x, y) = 0$ iff $x = y$.
- (b2) $d(x, y) = d(y, x)$.
- (b3) $d(x, z) \leq s\,[d(x, y) + d(y, z)]$.

In this case, the pair $(X, d)$ is said to be a $b$-metric space.

The definitions of a Cauchy and a convergent sequence, as well as, the complete $b-$metric space are given as follows:

**Definition 1.2.** ([13]) Let $(X, d)$ be a $b-$metric space. A sequence $\{x_n\}$ on $X$ is said to be
1. Cauchy if $d(x_n, y_n) \to 0$ as $n, m \to \infty$,
2. convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} x_n = x$.

**Definition 1.3.** ([13]) The $b-$metric $(X, d)$ is said to be complete if every Cauchy sequence in $X$ is convergent.

Kamran [14] defined a new generalized metric space, called an extended $b-$metric space as follows.

**Definition 1.4.** Let $X$ be a nonempty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$ is called an extended $b-$metric if for all $x, y, z \in X$ the following conditions are satisfied
- $(d_\theta 1)$ $d_\theta(x, y) = 0$ iff $x = y$;
- $(d_\theta 2)$ $d_\theta(x, y) = d_\theta(y, x)$;
- $(d_\theta 3)$ $d_\theta(x, z) \leq \psi(x, z)\,[d_\theta(x, y) + d_\theta(y, z)]$.

The pair $(X, d_\theta)$ is called an extended $b-$metric space.

In the following definition, Shatanawi [29] define a $(c)-$comparison function with base s.

**Definition 1.5.** ([29]) Let $s$ be a constant $s \geq 1$. A map $\Psi : [0, +\infty) \to [0, +\infty)$ is called a $(c)-$comparison function with base $s$ if $\Psi$ satisfies the following:

(i) $\Psi$ is monotone increasing,

(ii) $\sum\limits_{n=0}^{\infty} s^n \Psi^n(st)$ converges for all $t \geq 0$.

If $\psi$ is a (c)-comparison function, then for all $t > 0$ we have $\psi(t) < t$ and $\psi(0) = 0$.

Before starting to get our main results, we formulate the following new definitions. Then we give formulate and prove some our new results:

**Definition 1.6.** A single-valued mapping $f : X \to X$ is called a Ćirić strong almost contraction if there exists $\delta \in [0, 1)$, $L \geq 0$ and for $s \geq 1$ such that

$$d(f_x, f_y) \leq \frac{\delta}{s} \max\left\{sd(x, y), sd(x, f_x), sd(y, f_y), \frac{1}{2}\left[f(x, f_y) + d(y, f_x)\right]\right\} + Ld(y, f_x)$$

for all $x, y \in X$.

**Definition 1.7.** Let $(X, d)$ be a $b-$metric space. A mapping $T$ is called a modification of $(\delta, L)$-weak contraction if $\delta \in [0, 1)$ and $L \geq 0$ be such that

$$d(Tx, Ty) \leq \frac{\delta}{s} d(x, y) + Ld(y, Tx). \tag{1.1}$$

By using the symmetry condition of the $b-$metric space, then condition (1.1) is equivalent to

$$d(Tx, Ty) \leq \frac{\delta}{s} d(x, y) + Ld(x, Ty). \tag{1.2}$$

Moreover, by (1.1) and (1.2), the modification of the $(\delta, L)$-weak contraction condition of the mapping $T$ can be replaced by the following condition:

$$d(Tx, Ty) \leq \frac{\delta}{s} d(x, y) + L \min\{d(y, Tx), d(x, Ty)\}.$$

**Definition 1.8.** Let $(X, d)$ be a $b-$metric space. A map T is called modification of $(\Psi, L)-$weak contraction if $\Psi$ is a comparison function and $L \geq 0$ is such that

$$d(Tx, Ty) \leq \frac{1}{s} \Psi(sd(x, y)) + Ld(y, Tx). \tag{1.3}$$

Using the symmetry condition of the $b-$metric space, then (1.3) is equivalent to

$$d(Tx, Ty) \leq \frac{1}{s} \Psi(sd(x, y)) + Ld(x, Ty). \tag{1.4}$$

Thus by (1.3) and (1.4), the modification of $(\Psi, L)-$weak contraction condition of the mapping $T$ with respect to $G$ can be replaced by the following condition:

$$d(Tx, Ty) \leq \frac{1}{s}\Psi(sd(x, y)) + L\min\{d(y, Tx), d(x, Ty)\}.$$

**Remark 1.9.** Assume that $x_n \to z$ as $n \to +\infty$ in a $b-$metric space $(X, d)$ such that $d(z, z) = 0$. Then $\lim\limits_{n\to+\infty} d(x_n, y) = d(z, y)$ for every $y \in X$.

**Theorem 1.10.** *Let $(X, d)$ be a complete $b-$metric space and $T : X \to X$ be a modification of $(\Psi, L)-$weak contraction. Then $T$ has a unique fixed point.*

*Proof.* Start $x_0 \in X$, we construct a sequence $(x_n)$ in $X$ such that $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Since $T$ is a modification of $(\Psi, L)-$weak contraction, we have

$$d(Tx_{n-1}, Tx_n) \leq \frac{1}{s}\Psi(sd(x_{n-1}, x_n) + Ld(x_n, Tx_{n-1}) = \frac{1}{s}\Psi(sd(x_{n-1}, x_n).$$

So

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \frac{1}{s}\Psi(sd(x_{n-1}, x_n)).$$

Induction on $n$ implies that

$$d(x_n, x_{n+1}) \leq \frac{1}{s}\Psi^n(sd(x_0, x_1))$$

for all $n \in \mathbb{N}$. Triangle inequality implies that for $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} s^k d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{\infty} s^k d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{\infty} \frac{1}{s}\Psi^k(sd(x_0, x_1)). \end{aligned}$$

Since $\Psi$ is a $(c)-$comparison function, $\sum\limits_{k=n}^{\infty} s^k\Psi^k(sd(x_0, x_1))$ is convergent and so $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, $\{x_n\}$ converges with respect to $\tau_d$ to a point $z \in X$; that is, $\lim\limits_{n\to\infty} d(x_n, z) = d(z, z) = 0$. Since $x_n = Tx_{n-1}$, we conclude that $Tx_n \to z$.

Now, we claim that $d(z, Tz) = 0$. Now,

$$\begin{aligned} d(z,Tz) &\leq s\left[d(z,Tx_n) + d(Tx_n,Tz)\right] \\ &= s\left[d(z,x_{n+1}) + d(Tx_n,Tz)\right] \\ &\leq s\left[d(z,x_{n+1}) + \frac{1}{s}\psi(sd(x_n,z)) + Ld(z,x_{n+1})\right] \\ &\leq s\left[d(z,x_{n+1}) + d(x_n,z) + Ld(z,x_{n+1})\right]. \end{aligned}$$

Letting $n \to \infty$, we obtain

$$d(z, Tz) = 0$$

and hence $z = Tz$. To prove the uniqueness of the fixed point, we assume there are two distinct fixed points of $T$, say $z$ and $w$. So $d(z, w) > 0$. So

$$\begin{aligned} 0 &< d(z,w) = d(Tz,Tw) \\ &\leq \frac{1}{s}\Psi(sd(z,w)) + L_1 d(z,Tz) \\ &= \frac{1}{s}\Psi(sd(z,w)) \\ &< d(z,w), \end{aligned}$$

which is a contradiction. Therefore $T$ has a unique fixed point. □

In this paper, we introduce the notion of a modification of generalized $(s, L)-$weak contraction and a modification of a generalized $(\psi, L)-$weak contraction mapping in $b-$metric spaces.

First of all, we prove fixed point result for two mapping $S$ and $T$ and some fixed point results for a mapping $T$. our results generalize Theorem 1.10.

## 2. The main result

We start our work by formulating the following definitions:

**Definition 2.1.** Let $(X, d)$ be a $b-$metric space and $G, T, S : X \to X$ be three mappings such that $TX \subseteq GX$ and $SX \subseteq GX$. We call the pair $(T, S)$ a modification of generalized $(s, L)-$weak contraction if there exists $L \geq 0$ such that

$$\begin{aligned} d(Tx,Sy) \leq \frac{1}{s}\max\Big\{&sd(Gx,Gy), sd(Gx,Tx), sd(Gy,Ty), \\ &\frac{1}{2}\left(d(Gx,Sy) + d(Tx,Gx)\right)\Big\} + L\min\{d(Gx,Sy), d(Tx,Gy)\} \end{aligned} \tag{2.1}$$

for all $x, y \in X$.

**Definition 2.2.** Let $(X,d)$ be a $b-$metric space and $T,S:X\to X$ be two mappings. We call the pair $(T,S)$ a modification of generalized $(\Psi,L)-$weak contraction if there exists $L\geq 0$ such that

$$d(Tx,Sy)\leq\frac{1}{s}\Psi\Big(\max\Big\{sd(Gx,Gy),sd(Gx,Tx),sd(Gy,Ty),\frac{1}{2}\left(d(Gx,Sy)+d(Tx,Gx)\right)\Big\}\Big)+L\min\{d(Gx,Sy),d(Tx,Gy)\}\tag{2.2}$$

for all $x,y\in X$.

**Theorem 2.3.** *Let $(X,d)$ be a complete $b-$metric space and $G,T,S:X\to X$ be mappings such that the pair $(T,S)$ is a modification of generalized $(\Psi,L)-$ weak contraction. If $\Psi$ is a $(c)-$comparison function and $GX$ is a complete subspace of $X$, then $G,T$ and $S$ have a coincidence point.*

*Proof.* Choose $Gx_0\in X$. Put $Gx_1=Tx_0$. Again, put $Gx_2=Sx_1$. Continuing this process, we construct a sequence $(Gx_n)$ in $X$ such that $Gx_{2n+1}=Tx_{2n}$ and $Gx_{2n+2}=Sx_{2n+1}$. Suppose that $d(Gx_n,Gx_{n+1})=0$ for some $n\in\mathbb{N}$. Without loss of generality, we assume $n=2k$ for some $k\in\mathbb{N}$. Thus $d(Gx_{2k},Gx_{2k+1})=0$. Now, by (2.2), we have

$$\begin{aligned}
&d(Gx_{2k+1},Gx_{2k+2})\\
&=d(Tx_{2k},Sx_{2k+1})\\
&\leq\frac{1}{s}\Psi(\max\{sd(Gx_{2k},Gx_{2k+1}),sd(Gx_{2k},Tx_{2k}),\\
&\qquad sd(Gx_{2k+1},Sx_{2k+1}),\frac{1}{2}\left[d(Gx_{2k},Sx_{2k+1})+d(T_{2k},Gx_{2k+1})\right]\}\\
&\quad+L\min\{d(Tx_{2k},Gx_{2k+1}),d(Gx_{2k},Sx_{2k+1})\}\\
&=\frac{1}{s}\Psi(\max\{sd(Gx_{2k},Gx_{2k+1}),\\
&\qquad\frac{1}{2}[d(Gx_{2k},Gx_{2k+2})+d(Gx_{2k+1},Gx_{2k+1})]\}\\
&\quad+L\min\{d(Gx_{2k+1},Gx_{2k+1}),d(Gx_{2k},Gx_{2k+2})\}\\
&\leq\frac{1}{s}\Psi(\max\{sd(Gx_{2k},Gx_{2k+1}),\\
&\qquad\frac{s}{2}[d(Gx_{2k},Gx_{2k+1})+d(Gx_{2k+1},Gx_{2k+2})]\}\\
&\leq\frac{1}{s}\Psi(\max\{sd(Gx_{2k},Gx_{2k+1},sd(Gx_{2k+1},Gx_{2k+2})\}.\\
&=\frac{1}{s}\Psi(sd(Gx_{2k+1},Gx_{2k+2})).
\end{aligned}$$

Since $\Psi(t)<t$ for all $t>0$, we conclude that $d(Gx_{2k+1},Gx_{2k+2})=0$. By $(b1)$ and $(b2)$ of the definition of $b-$metric spaces, we have $Gx_{2k+1}=Gx_{2k+2}$. So

$Gx_{2k} = Gx_{2k+1} = Gx_{2k+2}$. Therefore $Gx_{2k} = Tx_{2k} = Sx_{2k}$ and hence $x_k$ is a coincidence point of $G$, $T$ and $S$. Thus, we may assume that $d(Gx_n, Gx_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. Given $n \in \mathbb{N}$. If $n$ is even, then $n = 2t$ for some $t \in \mathbb{N}$. By (2.2), we have

$$\begin{aligned} d(Gx_{2t}, Gx_{2t+1}) &= d(Gx_{2t+1}, Gx_{2t}) \\ &= d(Tx_{2t}, Sx_{2t-1}) \\ &\leq \frac{1}{s}\Psi(\max\{sd(Gx_{2t}, Gx_{2t-1}), sd(Gx_{2t}, Tx_{2t}), \\ &\quad sd(Gx_{2t-1}, Sx_{2t-1}), \\ &\quad \frac{1}{2}\left[d(Gx_{2t}, Sx_{2t-1}) + d(Tx_{2t}, Gx_{2t-1})\right]\}) \\ &\quad + L\min\{d(Gx_{2t}, Sx_{2t-1}), d(Tx_{2t}, Gx_{2t-1})\} \\ &= \frac{1}{s}\Psi(\max\{sd(Gx_{2t}, Gx_{2t-1}), sd(Gx_{2t}, Gx_{2t+1}), \\ &\quad \frac{1}{2}\left[d(Gx_{2t}, Gx_{2t}) + d(Gx_{2t+1}, Gx_{2t-1})\right]\}) \\ &\quad + L\min\{d(Gx_{2t}, Gx_{2t}), d(Gx_{2t+1}, Gx_{2t-1})\}. \end{aligned}$$

Using $(b4)$ of the definition of $b-$metric spaces, we reach to

$$\begin{aligned} d(Gx_{2t}, Gx_{2t+1}) &\leq \frac{1}{s}\Psi(\max\{sd(Gx_{2t}, Gx_{2t-1}), sd(Gx_{2t}, Gx_{2t+1}), \\ &\quad \frac{s}{2}\left[d(Gx_{2t-1}, Gx_{2t}) + d(Gx_{2t}, Gx_{2t+1})\right]\}) \\ &\leq \frac{1}{s}\Psi(\max\{sd(Gx_{2t}, Gx_{2t-1}), sd(Gx_{2t}, Gx_{2t+1})\}. \end{aligned} \tag{2.3}$$

If $\max\{sd(Gx_{2t}, Gx_{2t-1}), sd(Gx_{2t}, Gx_{2t+1})\} = sd(Gx_{2t}, Gx_{2t+1})$, then (2.3) yields a contradiction. Thus,

$$\max\{sd(Gx_{2t}, Gx_{2t-1}), sd(Gx_{2t}, Gx_{2t+1})\} = sd(Gx_{2t}, Gx_{2t-1})$$

and hence

$$d(Gx_{2t}, Gx_{2t+1}) \leq \frac{1}{s}\Psi(sd(Gx_{2t}, Gx_{2t-1})). \tag{2.4}$$

If $n$ is odd, then $n = 2t + 1$ for some $t \in \mathbb{N} \cup \{0\}$. By similar arguments as above, we can show that

$$d(Gx_{2t+1}, Gx_{2t+2}) \leq \frac{1}{s}\Psi(sd(Gx_{2t}, Gx_{2t+1})). \tag{2.5}$$

By (2.4) and (2.5), we have

$$d(Gx_n, Gx_{n+1}) \leq \frac{1}{s}\Psi(sd(Gx_{n-1}, Gx_n)). \tag{2.6}$$

By repeating (2.6) in $n$-times, we get $d(Gx_n, Gx_{n+1}) \leq \frac{1}{s}\Psi^n(sd(Gx_0, Gx_1))$. For $n, m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(Gx_n, Gx_m) &\leq \sum_{i=n}^{m-1} s^i d(Gx_i, Gx_{i+1}) \\ &\leq \sum_{i=n}^{m-1} s^i \psi^i(sd(Gx_0, Gx_1)) \\ &\leq \sum_{i=n}^{\infty} s^i \psi^i(sd(Gx_0, Gx_1)). \end{aligned}$$

Since $\Psi$ is $(c)-$comparison, we have $\sum\limits_{i=n}^{\infty} s^i\Phi^i(d(Gx_0, Gx_1))$ is convergent and hence $\lim\limits_{n\to+\infty} \sum\limits_{i=n}^{\infty} s^i\Phi^i(d(Gx_0, Gx_1)) = 0$. So, $\lim\limits_{n,m\to+\infty} d(Gx_n, Gx_m) = 0$. Thus $\{Gx_n\}$ is a Cauchy sequence in $GX$. Since $GX$ is complete, there exists $z \in GX$ such that $Gx_n \to Gz$ with $d(Gz, Gz) = 0$. So,

$$\lim_{n,m\to+\infty} d(Gx_n, Gx_m) = \lim_{n\to\infty} d(Gx_n, Gz) = d(Gz, Gz) = 0. \tag{2.7}$$

Now, we prove that $Sz = Tz$. Since $d(Gx_{2n+1}, Gz) \to d(Gz, Gz) = 0$ and $d(Gx_{2n+2}, Gz) \to d(Gz, Gz) = 0$, by Remark 1.9, we get

$$\lim_{n\to+\infty} d(Gx_{2n+1}, Sz) = d(Gz, Sz) \tag{2.8}$$

and

$$\lim_{n\to+\infty} d(Gx_{2n+2}, Sz) = d(Gz, Tz). \tag{2.9}$$

By using (2.2), we have

$$\begin{aligned} d(Gx_{2n+1}, Sz) &= d(Tx_{2n}, Sz) \\ &\leq \frac{1}{s}\Psi(\max\{sd(Gx_{2n}, Gz), sd(Gx_{2n}, Tx_{2n}), sd(Gz, Sz), \\ &\qquad \frac{1}{2}\left[d(Tx_{2n}, Gz) + d(Gx_{2n}, Sz)\right]\}) \\ &\quad + L\min\left\{d(Tx_{2n}, Gz), d(Gx_{2n}, Sz)\right\} \\ &\leq \frac{1}{s}\psi(\max\{sd(Gx_{2n}, Gz), sd(Gx_{2n}, Gx_{2n+1}), sd(Gz, Sz), \\ &\qquad \frac{1}{2}\left[d(Gx_{2n+1}, Gz) + d(Gx_{2n}, Sz)\right]\}) \\ &\quad + L\min\left\{d(Gx_{2n+1}, Gz), d(Gx_{2n}, Sz)\right\}. \end{aligned}$$

On letting $n \to +\infty$ in the above inequality and using (2.7) and (2.8), we get that $d(Gz, Sz) \leq \frac{1}{s}\psi(sd(Gz, Sz))$. Since $\psi(t) < t$ for all $t > 0$, we conclude that $d(Gz, Sz) = 0$. By using $(b1)$ and $(b2)$ of the definition of $b-$metric spaces, we get that $Sz = Gz$. By similar arguments as above, we may show that $Tz = Gz$. so $z$ is a coincidence point of $G, T$ and $S$ □

**Theorem 2.4.** *Let $(X, d)$ be a complete $b-$metric space and $T, S : X \to X$ be two mappings such that*

$$d(Tx, Sy) \leq \frac{1}{s}\Phi\left(\max\left\{sd(x,y), sd(x,Tx), sd(y,Sy), \frac{1}{2}\left[d(Tx,y)+d(x,Sy)\right]\right\}\right) + L\min\left\{d(x,Tx), d(x,Sy), d(Tx,y)\right\} \quad (2.10)$$

*for all $x, y \in X$. If $\Psi$ is a $(c)-$comparison function, then the common fixed point of $T$ and $S$ is unique.*

*Proof.* By taking $G = i$ the identity map on $X$, then Theorem 2.3 implies that $i, T$ have a coincidence point; that is, there is $z \in X$ such that $z = iz = Tz = Sz$. So $z$ is a common fixed point of $T$ and $S$. To prove the uniqueness of the common fixed point of $T$ and $S$, we let $u, v$ be two common fixed points of $T$ and $S$. Then $Tu = Su = u$ and $Tv = Sv = v$.

Now, we will show that $u = v$. By (2.10), we have

$$\begin{aligned} d(u,v) &= d(Tu, Sv) \\ &\leq \frac{1}{s}\psi\left(\max\left\{sd(u,v), sd(u,Tu), sd(v,Sv), \frac{1}{2}\left[d(Tu,v)+d(v,Tu)\right]\right\}\right) \\ &\quad + L\min\left\{d(u,Tu), d(Tu,v), d(v,Tu)\right\} \\ &\leq \frac{1}{s}\psi\left(\max\left\{sd(u,v), sd(u,Tu), sd(v,v), \frac{1}{2}\left[d(Tu,v)+d(v,u)\right]\right\}\right) \\ &\quad + L\min\left\{d(u,u), d(u,v), d(v,u)\right\} \\ &= \frac{1}{s}\psi(sd(u,v)). \end{aligned}$$

Since $\psi(t) < t$ for all $t > 0$, we conclude that $d(u, v) = 0$. By $(b1)$ and $(b2)$ of the definition of $b-$metric spaces, we get that $u = v$. □

**Corollary 2.5.** *Let $(X, d)$ be a complete $b-$metric space and $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq \frac{1}{s}\Psi\left(\max\left\{sd(x,y), sd(x,Tx), sd(y,Ty), \frac{1}{2}\left[d(Tx,y)+d(x,Sy)\right]\right\}\right) + L\min\left\{d(x,Tx), d(x,Ty), d(Tx,y)\right\}$$

*for all $x, y \in X$. If $\Psi$ is a $(c)-$comparison function, then $T$ has a unique fixed point.*

**Corollary 2.6.** *Let $(X,d)$ be a $b-$metric space and $T,S:X\to X$ be two mappings such that*

$$\begin{aligned} d(Tx,Ty) \leq \frac{1}{s}\Psi(\max\{&sd(Sx,Sy), sd(Sx,Tx), sd(Sy,Ty), \\ &\frac{1}{2}\left[d(Tx,Sy)+d(Sx,Ty)\right]\}) \\ &+ L\min\left\{d(Sx,Ty), d(Sy,Tx)\right\} \end{aligned}$$

*for all $x,y\in X$. Also, suppose that*

(1) *$TX\subseteq SX$, and*
(2) *$SX$ is a complete subspace of the $b-$metric space $X$.*

*If $\Psi$ is a $(c)-$comparison function, then $T$ and $S$ have a unique coincidence point.*

**Corollary 2.7.** *Let $(X,d)$ be a complete $b-$metric space and $T:X\to X$ be a mapping. Suppose there exist two non-negative numbers $k$ and $l$ such that*

$$\begin{aligned} d(Tx,Ty) &\leq k\max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(Tx,y)+d(x,Ty)\right]\right\} \\ &\quad +L\min\left\{d(x,Tx), d(x,Ty), d(Tx,y)\right\} \end{aligned}$$

*for all $x,y\in X$. If $k\in[0,1)$, then $T$ has a unique fixed point.*

**Corollary 2.8.** *Let $(X,d)$ be a $b-$metric space and $T,S:X\to X$ be two mapping. Suppose there exist two non-negative numbers $k$ and $l$ such that*

$$\begin{aligned} &d(Tx,Ty) \\ &\leq k\max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}\left[d(Tx,Sy)+d(Sx,Ty)\right]\right\} \\ &\quad + L\min\left\{d(Sx,Ty), d(Sy,Tx)\right\} \end{aligned}$$

*for all $x,y\in X$. Also, suppose that*

(1) *$TX\subseteq SX$, and*
(2) *$SX$ is a complete subspace of the $b-$metric space $X$.*

*If $k\in[0,1)$, then $T$ and $S$ have a coincidence point.*

**Corollary 2.9.** *Let $(X,d)$ be a $b-$metric space and $T,S:X\to X$ be two mappings. Suppose that there exist a $(c)-$comparison function $\Psi$ and $L\geq 0$ such that*

$$\begin{aligned} d(Tx,Ty) \leq \frac{1}{s}\Psi(\max\{&sd(Sx,Sy), sd(Sx,Tx), sd(Sy,Ty), \\ &\frac{1}{2}\left[d(Tx,Sy)+d(Sx,Ty)\right]\}) \\ &+ L\min\left\{d(Tx,Sx), d(Sx,Ty), d(Sy,Tx)\right\} \end{aligned}$$

*for all $x, y \in X$. Also, suppose that*

(1) $TX \subseteq SX$, *and*
(2) $SX$ *is a complete subspace of the* $b-$*metric space* $X$.

*Then the point of coincidence of $T$ and $S$ is unique; that is, if $Tu = Su$ and $Tv = Sv$, then $Tu = Tv = Sv = Su$.*

The $(c)-$comparison function in Theorems 2.3 and 2.4 can be replaced by a comparison function if we formulated the contractive condition to a suitable form. For this instance, we have the following result

**Theorem 2.10.** *Let $(X, d)$ be a complete $b-$metric space and $G, T : X \to X$ be mappings such that $TX \subseteq GX$ and*

$$\begin{aligned} d(Tx, Ty) \leq \frac{1}{s}\Psi\left(\max\left\{sd(Gx, Gy), sd(Gx, Tx), sd(Gy, Ty)\right\}\right) \\ + L\min\left\{d(Gx, Tx), d(Gx, Ty), d(Gy, Tx)\right\} \end{aligned} \tag{2.11}$$

*for all $x, y \in X$. If $\Psi$ is a comparison function and $GX$ is a complete subspace of $X$, then $G$ and $T$ have a coincidence point.*

*Proof.* Choose $Gx_0 \in X$. Put $Gx_1 = Tx_0$. Again, put $Gx_2 = Tx_1$. Continuing the same process, we can construct a sequence $\{Gx_n\}$ in X such that $Gx_{n+1} = Tx_n$. If $d(Gx_k, Gx_{k+1}) = 0$ for some $k \in \mathbb{N}$, then by the definition of $b-$metric spaces, we have $Gx_k = Gx_{k+1} = Tx_k$, that is, $Gx_k$ is a coincidence point of $G$ and $T$. Thus, we assume that $d(Gx_n, Gx_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. By (2.11), we have

$$\begin{aligned} & d(Gx_n, Gx_{n+1}) \\ & = d(Tx_{n-1}, Tx_n) \\ & \leq \frac{1}{s}\Psi(\max\{sd(Gx_{n-1}, Gx_n), sd(Gx_{n-1}, Tx_{n-1}), sd(Gx_n, Tx_n)\}) \\ & \quad + L\min\{d(Gx_{n-1}, Tx_n), d(Gx_{n-1}, Tx_n), d(Gx_n, Tx_{n-1})\} \\ & = \frac{1}{s}\Psi(\max\{sd(Gx_{n-1}, Gx_n), sd(Gx_n, Gx_{n+1})\}) \\ & \quad + L\min\{d(Gx_{n-1}, Tx_{n+1}), d(Gx_n, Gx_n)\} \\ & = \frac{1}{s}\Psi(\max\{sd(Gx_{n-1}, Gx_n), sd(Gx_n, Gx_{n+1})\}). \end{aligned}$$

If

$$\max\{sd(Gx_{n-1}, Gx_n), sd(Gx_n, Gx_{n+1})\} = sd(Gx_n, Gx_{n+1}),$$

then

$$d(Gx_n, Gx_{n+1}) \leq \frac{1}{s}\Psi(sd(Gx_n, Gx_{n+1})) < d(Gx_n, Gx_{n+1}),$$

a contradiction. Thus,

$$\max\{sd(Gx_{n-1},Gx_n),sd(Gx_n,Gx_{n+1})\}=sd(Gx_{n-1},Gx_n)$$

and hence

$$d(Gx_n,Gx_{n+1})\le\frac{1}{s}\Psi(sd(Gx_{n-1},Gx_n))\text{ for all }n\in\mathbb{N}. \tag{2.12}$$

Repeating (2.12) in $n$-times, we get that

$$d(Gx_n,Gx_{n+1})\le\frac{1}{s}\Psi^n(sd(Gx_0,Gx_1)).$$

Now, we will prove that $\{Gx_n\}$ is a Cauchy sequence in $GX$. For this, given $\epsilon>0$, since $\frac{1}{(2+L)}(\epsilon-\Phi(\epsilon))>0$ and $\lim_{n\to+\infty}\Phi^n(sd(Gx_0,Gx_1))=0$, there exists $k\in\mathbb{N}$ such that $d(Gx_n,Gx_{n+1})<\frac{1}{s(2+L)}(\epsilon-\Phi(\epsilon))$ for all $n\ge k$. Now, given $m,n\in\mathbb{N}$ with $m>n$. Claim: $d(Gx_n,Gx_m)<\epsilon$ for all $m>n>k$. We prove our claim by induction on $m$. Since $k+1>k$, we have

$$d(Gx_k,Gx_{k+1})\le\frac{1}{s\,(2+L)}(\epsilon-\Phi(\epsilon))<\epsilon.$$

The last inequality proves our claim for $m=k+1$. Assume that our claim holds for $m=k$.

Now, we prove our claim for $m=k+1$, we have

$$\begin{aligned}d(Gx_n,Gx_{k+1})&\le s\left[d(Gx_n,Gx_{n+1})+d(Gx_{n+1},Gx_{k+1})\right]\\&=s\left[d(Gx_n,Gx_{n+1})+d(Tx_n,Tx_k)\right].\end{aligned} \tag{2.13}$$

By (2.11), we have

$$\begin{aligned}d(Tx_n,Tx_k)&\le\frac{1}{s}\Psi(\max\{sd(Gx_n,Gx_k),sd(Gx_n,Tx_n),sd(Gx_k,Tx_k)\})\\&\quad+L\min\{d(Gx_n,Tx_n),d(Gx_n,Tx_k),d(Gx_k,Tx_n)\}\\&=\frac{1}{s}\Psi(\max\{sd(Gx_n,Gx_k),sd(Gx_n,Gx_{n+1}),sd(Gx_k,Gx_{k+1})\})\\&\quad+L\min\{d(Gx_n,Gx_{n+1}),d(Gx_n,Gx_{k+1}),d(Gx_k,Gx_{n+1})\}\\&\le\frac{1}{s}\Psi(\max\{sd(Gx_n,Gx_k),sd(Gx_n,Gx_{n+1}),sd(Gx_k,Gx_{k+1})\})\\&\quad+Ld(Gx_n,Gx_{n+1}).\end{aligned}$$

If

$$\max\{sd(Gx_n,Gx_k),sd(Gx_n,Gx_{n+1}),sd(Gx_k,Gx_{k+1})\}=sd(Gx_n,Gx_k),$$

then (2.13) implies that

$$\begin{aligned}
d(Gx_n, Gx_{k+1}) &\leq s\left[d(Gx_n, Gx_{n+1}) + \frac{1}{s}\Psi(sd(Gx_n, Gx_k)) + Ld(Gx_n, Gx_{n+1})\right] \\
&< \left[\frac{1+L}{s\,(2+L)}(\epsilon - \Phi(\epsilon)) + \frac{1}{s}\Phi(\epsilon)\right] s \\
&< \epsilon.
\end{aligned}$$

If

$$\max\{sd(Gx_n, Gx_k), sd(Gx_n, Gx_{n+1}), sd(Gx_k, Gx_{k+1})\} = sd(Gx_n, Gx_{n+1}),$$

then (2.13) implies that

$$\begin{aligned}
&d(Gx_n, Gx_{k+1}) \\
&\leq s\left[d(Gx_n, Gx_{n+1}) + \frac{1}{s}\Psi(sd(Gx_n, Gx_{n+1})) + Ld(Gx_n, Gx_{n+1})\right] \\
&< (2+L)sd(Gx_n, Gx_{n+1}) \\
&< \frac{\epsilon - \Phi(\epsilon)}{\epsilon} \\
&< \epsilon.
\end{aligned}$$

If

$$\max\{sd(Gx_n, Gx_k), sd(Gx_n, Gx_{n+1}), sd(Gx_k, Gx_{k+1})\} = sd(Gx_k, Gx_{k+1}),$$

then (2.13) implies that

$$\begin{aligned}
&d(Gx_n, Gx_{k+1}) \\
&\leq s\left[d(Gx_n, Gx_{n+1}) + \frac{1}{s}\Psi(sd(Gx_k, Gx_{k+1})) + Ld(Gx_n, Gx_{n+1})\right] \\
&< (s+L)d(Gx_n, Gx_{n+1}) + sd(Gx_k, Gx_{k+1}) \\
&< \frac{s+L}{s\,(2+L)}(\epsilon - \Phi(\epsilon)) + \frac{s}{s\,(2+L)}(\epsilon - \Phi(\epsilon)) \\
&< \epsilon.
\end{aligned}$$

Thus $\{Gx_n\}$ is a Cauchy sequence in $X$. Since $GX$ is complete, $\{Gx_n\}$ converges, with respect to $\tau_p$, to a point $Gz$ for some $z \in X$ such that

$$\lim_{n,m\to+\infty} d(Gx_n, Gx_m) = \lim_{n\to+\infty} d(Gx_n, Gz) = d(Gz, Gz) = 0. \tag{2.14}$$

Now, assume that $d(Gz, Tz) > 0$. By using ($b4$) of the definition of $b$-metric spaces and (2.11), we have

$$\begin{aligned}
&d(Gz, Tz)\\
&\leq s\left[d(Gz, Gx_{n+1}) + d(Gx_{n+1}, Tz)\right]\\
&= s\left[d(Gz, Gx_{n+1}) + d(Tx_n, Tz)\right]\\
&\leq s[d(Gz, Gx_{n+1}) + \frac{1}{s}\Psi(\max\{sd(Gx_n, Gz), sd(Gx_n, Tx_n), sd(Gz, Tz)\})\\
&\quad + L\min\{d(Gx_n, Tx_n), d(Gx_n, Tz), d(Gx_n, Tz)\}]\\
&= s[d(z, Gx_{n+1}) + \frac{1}{s}\Psi(\max\{sd(Gx_n, z), sd(Gx_n, Gx_{n+1}), sd(z, Tz)\})\\
&\quad + L\min\{d(Gx_n, Gx_{n+1}), d(Gx_n, Tz), d(Gx_{n+1}, Sz)\}]. \qquad (2.15)
\end{aligned}$$

Since

$$\lim_{n,m\to+\infty} d(Gx_n, Gx_{n+1}) = \lim_{n\to+\infty} d(Gx_n, Gz) = 0$$

and $d(Gz, Tz) > 0$, we can choose $n_0 \in \mathbb{N}$ such that

$$\max\{sd(Gx_n, Gz), sd(Gx_n, Gx_{n+1}), sd(Gz, Tz)\} = sd(Gz, Tz)$$

for all $n \geq n_0$. Thus (2.15) becomes

$$\begin{aligned}
d(Gz, Tz) \leq s[&d(Gz, Gx_{n+1}) + \frac{1}{s}\Psi(sd(Gz, Tz))\\
&+ L\min\{d(Gx_n, Gx_{n+1}), d(Gx_n, Tz), d(Gx_{n+1}, Tz)\}],
\end{aligned}$$

for all $n \geq n_0$. On letting $n \to +\infty$ in the above inequality and using (2.14), we get that

$$d(Gz, Tz) \leq \frac{1}{s}\Psi(sd(Gz, Tz)) < d(Gz, Tz),$$

a contradiction. Thus $d(z, Tz) = 0$. By using ($b1$) and ($b2$) of the definition of a $b$-metric space, we get that $Gz = Tz$, that is, $z$ is a coincidence point of $G$ and $T$. □

**Corollary 2.11.** *Let $(X, d)$ be a $b-$metric space and $T : X \to X$ be a mapping. Suppose there exist a comparison function $\Psi$ and $L \geq 0$ such that*

$$\begin{aligned}
d(Tx, Ty) \quad \leq \quad & \frac{1}{s}\Psi\left(\max\{sd(x, y), sd(x, Tx), sd(y, Ty)\}\right)\\
&+L\min\{d(x, Tx), d(x, Ty), d(y, Tx)\}
\end{aligned}$$

*for all $x, y \in X$. Then $T$ has unique fixed point.*

*Proof.* By taking $i = G$, the identity function on $X$. Then from Theorem 2.10, we conclude that $i$ and $T$ have a coincidence point $z \in X$. So $z = ix = Tx$. So $x$ is a fixed point of $T$. One can easily show that from the contractive condition, the fixed point of $T$ is unique. □

## 3. Example

**Example 3.1.** Let $X = [0, +\infty)$. Consider the complete $b$-metric space $d : X \times X \to [0, +\infty)$, $d(x, y) = (x-y)^2$ with constant $s = 2$. Define the mappings $G, T, S : X \to X$ by $Gx = x$, $Tx = \frac{1}{3}x$ and $Sx = \frac{1}{6}x$, and define $\Psi : [0, +\infty) \to [0, +\infty)$ by $\Psi(t) = \frac{1}{4}$. Then

(1) $\Psi$ is a continuous $(c)$-comparison function.
(2) $T, S$ and $\Psi$ satisfy the following inequality:

$$\begin{aligned} d(Tx, Sy) \leq \frac{1}{s}\Psi(\max\{&sd(Gx, Gy), sd(Gx, Tx), sd(Gy, Sy), \\ &\frac{1}{2}\left[d(Tx, Gy) + d(Gx, Sy)\right]\}) \\ &+ L\min\left\{d(Gx, Tx), d(Gx, Sy), d(Tx, Gy)\right\}. \end{aligned}$$

In fact, it is clear that $\Psi$ is a nondecreasing continuous function. Now, let $t \in [0. +\infty)$. Then,

$$\Psi^n(st) = \Psi^n(2t) = \frac{1}{4^n}(2t).$$

Thus

$$\begin{aligned} \sum_{n=0}^{\infty} s^n \Psi^n(st) &= \sum_{n=0}^{\infty} \frac{2^n}{4^n}(2t) \\ &= 2t\sum_{n=0}^{\infty}\frac{1}{2^n} \\ &< +\infty. \end{aligned}$$

So $\Psi$ is a $(c)$-comparison function.

To show (2), let $x, y \in X$. Then

$$d(Tx, Sy) = d\left(\frac{1}{3}x, \frac{1}{6}y\right) = \left(\frac{1}{3}x - \frac{1}{6}y\right)^2 = \frac{1}{9}\left(x - \frac{1}{2}y\right)^2.$$

Now, we have 3 cases:

Case I: $x = \frac{1}{2}y$. Here, we have

$$\begin{aligned} d(Tx, Sy) = 0 \leq \frac{1}{s}\Psi(\max\{&sd(Gx, Gy), sd(Gx, Tx), sd(Gy, Sy), \\ &\frac{1}{2}\left[d(Tx, Gy) + d(Gx, Sy)\right]\}) \\ &+ L\min\left\{d(Gx, Tx), d(Gx, Sy), d(Tx, Gy)\right\}. \end{aligned}$$

Case II: $x > \frac{1}{2}y$. Here, we have

$$\begin{aligned}
d(Tx,Sy) &= \frac{1}{9}\left(x-\frac{1}{2}y\right)^2 \leq \frac{x^2}{6} \\
&= \frac{1}{2}(2)\left(\frac{2}{3}x\right)^2\left(\frac{1}{4}\right) \\
&= \frac{1}{2}\Psi\left(2\left(x-\frac{1}{3}x\right)^2\right) \\
&= \frac{1}{2}\Psi\left(2d\left(x,\frac{1}{3}x\right)\right) \\
&= \frac{1}{s}\Psi\left(sd\left(Gx,Tx\right)\right) \\
&\leq \frac{1}{s}\Psi(\max\{sd(Gx,Gy), sd(Gx,Tx), sd(Gy,Sy), \\
&\qquad \frac{1}{2}\left[d(Tx,Gy)+d(Gx,Sy)\right])\} \\
&\quad + L\min\left\{d(Gx,Tx), d(Gx,Sy), d(Tx,Gy)\right\}.
\end{aligned}$$

Case III: $x < \frac{1}{2}y$. Here, we have

$$\begin{aligned}
d(Tx,Sy) &= \frac{1}{9}\left(x-\frac{1}{2}y\right)^2 \leq \frac{y^2}{36} \\
&\leq \left(\frac{25}{36}\right)\left(\frac{y^2}{4}\right) \\
&= \frac{1}{2}\Psi\left(2\left(\frac{25}{36}\right)y^2\right) \\
&= \frac{1}{2}\Psi\left(2\left(y-\frac{1}{6}y\right)^2\right) \\
&= \frac{1}{2}\Psi\left(2d\left(y,\frac{1}{6}y\right)\right) \\
&= \frac{1}{s}\Psi\left(sd\left(Gy,Sx\right)\right) \\
&= \frac{1}{s}\Psi(\max\{sd(Gx,Gy), sd(Gx,Tx), sd(Gy,Sy), \\
&\qquad \frac{1}{2}\left[d(Tx,Gy)+d(Gx,Sy)\right]\}) \\
&\quad + L\min\left\{d(Gx,Tx), d(Gx,Sy), d(Tx,Gy)\right\}.
\end{aligned}$$

Hence we know that $G, T, S$ and $\Psi$ satisfy all hypotheses of Theorem 2.4. So $T$ and $S$ have a unique common fixed point.

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