Nonlinear Functional Analysis and Applications
Vol. 26, No. 1 (2021), pp. 165-175
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.01.12
http://nfaa.kyungnam.ac.kr/journal-nfaa

BEST APPROXIMATIONS FOR MULTIMAPS ON ABSTRACT CONVEX SPACES

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Abstract. In this article we derive some best approximation theorems for multimaps in abstract convex metric spaces. We are based on generalized KKM maps due to Kassay-Kolumbán, Chang-Zhang, and studied by Park, Kim-Park, Park-Lee, and Lee. Our main results are extensions of a recent work of Mitrović-Hussain-Sen-Radenović on G-convex metric spaces to partial KKM metric spaces. We also recall known works related to single-valued maps, and introduce new partial KKM metric spaces which can be applied our new results.

1. Introduction

In 2020, Mitrović et al. [7] obtained a best approximation theorem for multimaps in G-convex spaces. As applications, they derived results on the best approximations in hyperconvex and normed spaces. Their results generalize many existing results in the literature.

However, they are based on G-convex spaces due to the present author, and G-convex spaces are obsolete now; see [8]. In fact, since the usefulness of G-convex spaces was known, there have appeared scores of useless imitations, modifications, and so-called generalizations. This is why we derived abstract convex spaces and (partial) KKM spaces for new era of the KKM theory. Since

[0]Received August 27, 2020. Revised September 13, 2020. Accepted October 6, 2020.

[0]2020 Mathematics Subject Classification: 46A03, 47H10, 49J53, 54C60, 54H25, 91A11, 91B02.

[0]Keywords: Abstract convex space, KKM map, (partial) KKM space, KKM classes $\mathfrak{KC}$, $\mathfrak{KO}$, best approximation.

2006, we have tried to establish a large number of KKM theoretic results for abstract convex spaces.

Our aim in the present article is to extend the best approximation theorems of Mitrović et al. [7] to abstract convex metric spaces, to recall known works related to single-valued maps, and to introduce new spaces which can be applied our new results.

This article is organized as follows: Section 2 is a routine preliminary on terminology of abstract convex spaces. There we introduce the KKM maps with respect to a multimap, $\mathfrak{KC}$-maps [resp. $\mathfrak{KO}$-maps], and partial KKM spaces. We add a routine diagram showing typical subclasses of abstract convex spaces.

In Section 3, we introduce generalized KKM maps in the sense of Kassay-Kolumbán [2] in 1990 and Chang-Zhang [1] in 1991. We show that such maps on a partial KKM spaces can be regarded usual KKM maps on another partial KKM spaces. Actually, the basic results of Kim-Park [4] on G-convex spaces adopted by Mitrović et al. [7] are extended to partial KKM spaces.

Section 4 deals with generalized versions of the main results of Mitrović et al. [7]. In fact, their main results on G-convex spaces are extended to partial KKM spaces.

In Section 5, our previous best approximation theorem for single-valued maps on partial KKM metric spaces in [11] is shown to be a consequence of the main results in Section 4.

Finally, Section 6 deals with examples of partial KKM metric spaces that can be applied by our new results.

2. Abstract convex spaces

For the concepts on abstract convex spaces, KKM spaces and the KKM classes $\mathfrak{KC}$, $\mathfrak{KO}$, we follow [9, 10] with some modifications and the references therein:

Definition 2.1. Let E be a topological space, D a nonempty set, $\langle D \rangle$ the set of all nonempty finite subsets of D, and $\Gamma : \langle D \rangle \multimap E$ a multimap with nonempty values $\Gamma_A := \Gamma(A)$ for each $A \in \langle D \rangle$. The triple $(E, D; \Gamma)$ is called an *abstract convex space* whenever the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to some $D' \subset D$ if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$.

In case $E = D$, let $(E;\Gamma) := (E, E;\Gamma)$.

Definition 2.2. Let $(E, D;\Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \;\; A \in \langle D \rangle,$$

then G is called a *KKM map with respect to* F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{KC}(E, Z) := \{F : E \multimap Z \mid F \;\text{ is a } \mathfrak{KC}\text{−map}\}.$$

Similarly, $\mathfrak{KO}(E, Z)$ is defined.

Definition 2.3. The *partial KKM principle* for an abstract convex space $(E, D;\Gamma)$ is the statement $1_E \in \mathfrak{KC}(E, E)$, that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$, that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a (*partial*) *KKM space* if it satisfies the (partial) KKM principle, resp.

There are plenty of examples of KKM spaces; see [13] and the references therein.

Now we have the following diagram for subclasses of abstract convex spaces $(E, D;\Gamma)$:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde type convex space

$\Longrightarrow$ Horvath space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space

$\Longrightarrow$ KKM space $\Longrightarrow$ Partial KKM space

$\Longrightarrow$ Abstract convex space.

Note that Horvath spaces are new class including c-spaces due to Horvath; see [12].

3. Generalized KKM maps in abstract convex spaces

Recall that Kassay-Kolumbán in 1990 [2] and Chang-Zhang in 1991 [1] introduced the concept of generalized KKM maps. Since then many authors adopted various concepts of generalized KKM maps and applied them to extend or refine well-known previous results. In fact, it has been followed by Chang-Ma in 1993, Yuan in 1995, Cheng in 1997, Tan in 1997, Lin-Chang in 1998, Lee-Cho-Yuan in 1999, Kirk-Sims-Yuan in 2000 and many authors for various classes of abstract convex spaces; see [10]. All of those authors applied their results on KKM type theorems and others to extend or refine well-known previous results in the KKM theory; for example, variational or quasi-variational inequalities, fixed point theorems, the Ky Fan type minimax inequalities, the von Neumann type minimax or saddle point theorems, Nash equilibrium problems and others.

In our previous review [10], we gave a unified account for such maps in abstract convex spaces unifying the previous works of Kim and Park [4], Lee [6], Park and Lee [15]. We were mainly concerned with results closely related to the KKM type theorems and characterizations of generalized KKM maps on various types of abstract convex spaces. In short, we showed that most of the generalized KKM maps can be reduced to the usual KKM maps in our abstract convex spaces. Some related topics were also added there.

Inspired by recent works on generalized KKM maps, we introduce the following definition:

Definition 3.1. Let $(X, D; \Gamma)$ be an abstract convex space and Y be a nonempty set such that, for each $A \in \langle Y \rangle$, there exists a function $\sigma_A : A \to D$. Then a new abstract convex space $(X, A; \Gamma^A)$ induced by Γ and A is defined by the following

$$\Gamma^A(J) := \Gamma(\sigma_A(J)) \quad \text{for each } J \subset A.$$

Moreover, a multimap $T : Y \multimap X$ (called a *generalized KKM map*) reduces to a KKM map on $(X, A; \Gamma^A)$ for each $A \in \langle Y \rangle$ satisfying $\Gamma^A(J) \subset T(J)$ for each $J \subset A$.

The following characterization of generalized KKM maps in [10] extends Theorem 2 of Park-Lee [15], which was stated for G-convex spaces:

Theorem 3.2. *Let $(X, D; \Gamma)$ be a partial KKM space [resp. KKM space], Y a nonempty set, and $T : Y \multimap X$ a map with closed [resp. open] values.*

(i) *If T is a generalized KKM map, then the family of its values has the finite intersection property.*
(ii) *The converse holds whenever $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$.*

For the completeness, we add its proof here.

Proof. (i) Let $T : A \multimap X$ be a KKM map having closed [resp. open] values on $(X, A; \Gamma^A)$, that is,

$$\Gamma^A(J) \subset T(J) \quad \forall J \subset A.$$

Let $A = \{y_i\}_{i=1}^n$, $z_i = \sigma_A(y_i) \in D$, and $G(z_i) = T(y_i)$ for each $i = 1, \ldots, n$. Then

$$\Gamma(\Gamma^A(J)) \subset G(\Gamma^A(J)) \quad \forall J \subset A.$$

Hence $G : \Gamma^A(A) \multimap X$ is a KKM map with closed [resp. open] values on $(X, A; \Gamma|_{\langle\sigma_A(A)\rangle})$ which is a (partial) KKM space. Hence $\{G(z_i)\}_{i=1}^n = \{T(y_i)\}_{i=1}^n$ has the finite intersection property.

(ii) Suppose that $X = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in X$. For any $A \in \langle Y\rangle$, by assumption, we have an $x^* \in \bigcap_{y\in A} T(y) \neq \emptyset$. Define a function $\sigma_A : A \to D = X$ by $\sigma_A(y) = x^*$ for all $y \in A$. Then for any nonempty subset J of A, we have

$$\Gamma_{\sigma_A(J)} = \Gamma_{\{x^*\}} = \{x^*\} \subset \bigcap_{y\in A} T(y) \subset T(J).$$

Therefore, T is a generalized KKM map. □

Motivated by Theorem 3.2, we obtained the following [10, Theorem D]:

Theorem 3.3. *Let $(X, D; \Gamma)$ be a partial KKM space [resp. KKM space], Y a nonempty set such that $T : Y \multimap X$ a generalized KKM map with closed [resp. open] values [that is. there exist a function $\sigma_A : A \to D$ for each $A \in \langle Y\rangle$ such that $T : Y \multimap X$ is a KKM map as in Definition]. Then $\{T(y) : y \in Y\}$ has the finite intersection property.*

Example 3.4. We list examples of generalized KKM maps or related matters appeared in the literature in the chronological order.

(1) A generalized KKM map $F : Y \multimap X$ reduces to a KKM map if $Y = D$ and σ is chosen to be the identity function 1_A for each $A \in \langle D\rangle$. This is the case originally considered by Knaster-Kuratowski-Mazurkiewicz in 1929 and Ky Fan in 1961.
(2) Kassay and Kolumbán [2] first considered the concept of generalized KKM maps. See Park, MR1168056 (93e:46008) in 1993.
(3) Let X and Y be convex subsets of topological vector spaces E and F, resp. A map $G : X \multimap F$ is called a generalized KKM map by Chang and Zhang [1], if for any finite set $\{x_1, \cdots, x_n\} \subset X$, there exists a finite set $\{y_1, \cdots, y_n\} \subset F$ such that any finite subset $\{y_{i_1}, \cdots, y_{i_k}\} \subset \{y_1, \cdots, y_n\}$, $1 \leq k \leq n$, we have $\mathrm{co}\{y_{i_1}, \cdots, y_{i_k}\} \subset \bigcup_{j=1}^k G(x_{i_j})$. Note that any KKM map $G : X \multimap E$ is a generalized KKM map, and a counterexample ensuring the converse does not hold was given in [1].

Theorem 3.3 reduces to the following due to Kim-Park [4, Theorem 3]:

Corollary 3.5. *Let $(X, D; \Gamma)$ be a G-convex space, I a nonempty set, and $F : I \multimap X$ a map with closed [resp. open] values. If F is a generalized KKM map, then the family of its values has the finite intersection property (More precisely, for each $N \in \langle I \rangle$, there exists an $N' \in \langle D \rangle$ such that $\Gamma_{N'} \cap \bigcap_{z \in N} F(z) \neq \emptyset$).*

Corollary 3.5 was quoted as [7, Theorem 1]. From now on, in this Section, we follow the notation in [7] except $\Omega = \Gamma$.

From Theorem 3.3, we immediately have the following:

Theorem 3.6. *Let $(X; \Gamma)$ be a partial KKM space, S a nonempty set, and $\Phi : S \multimap X$ a generalized KKM map with closed values. If there exists a nonempty compact subset L of X such that $\bigcap_{t \in T} \Phi(t) \subset L$ for some $T \in \langle S \rangle$ then $\bigcap_{s \in S} \Phi(s) \neq \emptyset$.*

Mitrović et al. stated the following consequence of Corollary 3.5 as [7, Theorem 2]:

Corollary 3.7. *Let (X, Ω) be a G-convex space, S a nonempty set and $\Phi : S \multimap X$ a generalized KKM map with closed values. If there exists a nonempty compact subset L of X such that $\bigcap_{t \in T} \Phi(t) \subset L$ for some $T \in \langle S \rangle$ then $\bigcap_{s \in S} \Phi(s) \neq \emptyset$.*

Note that the compactness or coercivity condition for the whole intersection property in Theorem 3.6 and Corollary 3.7 can be replaced by a number of more general conditions in our KKM theory on abstract convex spaces.

4. Best approximations on partial KKM spaces

Our aim in this section is to improve the best approximation theorems of Mitrović et al. [7]. We need some preliminary from [7].

A multimap Φ is *continuous* if it is u.s.c. and l.s.c.

Denote $\mathrm{Bd}(S)$ the *boundary* of the set S.

Let (X, d) be a metric space, $r \in \mathbb{R}^+ \cup \{0\}$ and $\emptyset \neq S \subset X$. We denote the *r-parallel set* of S by

$$S + r = \bigcup \{B(s, r) : s \in S\},$$

where $B(s, r) = \{t \in X : d(s, t) \leq r\}$ is the closed ball.

For nonempty subsets S and T of X, we define

$$d(S, T) = \inf\{d(s, t) : s \in S,\ t \in T\}.$$

We call a set $K \subset X$ is *metrically convex* if for any $x,\ y \in K$ and positive numbers p_i and p_j such that $d(x, y) \leq p_i + p_j$, there exists $z \in K$ such that $z \in B(x, p_i) \cap B(y, p_j)$.

Applying Theorem 3.6, we can prove the following new best approximation theorem for partial KKM spaces:

Theorem 4.1. *Let $\Phi : S \multimap X$ be a continuous multimap with compact values such that*

(1) $\Phi(x) + r$ *is* $\Gamma - convex$ *for all* $x \in S,\ r \geq 0$

and $g : S \to S$ is a continuous onto map, where (X, Γ) a partial KKM space with metric d and S a nonempty Γ-convex subset of X. If there exists a nonempty compact subset K of X such that

$$\bigcap_{y \in M} \{x \in S : d(g(x), \Phi(x)) \leq d(g(y), \Phi(x))\} \subset K \ for\ some\ M \in \langle S \rangle,$$

then there exists $v_0 \in S$ such that

$$d(g(v_0), \Phi(v_0)) = \inf_{x \in S} d(x, \Phi(v_0)).$$

If S is metrically convex and $g(v_0) \notin \Phi(v_0)$, then $v_0 \in \mathrm{Bd}(S)$.

Proof. Just follow the proof of [7, Theorem 3] word by word and use Theorem 3.6 instead of [7, Theorem 2]. □

Next results follow from Theorem 4.1.

Corollary 4.2. *Let $\Phi : S \multimap X$ be a continuous multimap with compact values such that condition (1) is satisfied and $g : S \to S$ is a continuous onto map, where $(X; \Gamma)$ a partial KKM space with metric d and S a nonempty Γ-convex set contained in a compact subset K of X. Then there exists $v_0 \in S$ such that*

$$d(g(v_0), \Phi(v_0)) = \inf_{x \in S} d(g(x), \Phi(v_0)).$$

If K is metrically convex and $g(v_0) \notin \Phi(v_0)$, then $v_0 \in \mathrm{Bd}(S)$.

Corollary 4.3. *Let the metric space $(X; \Gamma)$ be a partial KKM space with metric d, S a nonempty Γ-convex set contained in a compact subset K of X, $\Phi : S \multimap X$ is a continuous multimap with compact values such that condition (1) is satisfied. Then there exists $v_0 \in S$ such that*

$$d(v_0, \Phi(v_0)) = \inf_{x \in S} d(x, \Phi(v_0)).$$

(In particular, if $\Phi(S) \subset S$, then v_0 is a fixed point of Φ.)

If K is metrically convex and $v_0 \notin \Phi(v_0)$, then $v_0 \in \mathrm{Bd}(S)$.

Remark 4.4. (1) For G-convex spaces instead of partially KKM spaces, Theorem 4.1, Corollaries 4.2 and 4.3 reduce Theorem 3, its Corollaries 1 and 2 of [7], resp.

(2) There is an example of partial KKM space which is not KKM space, hence not a G-convex space, given by Kulpa and Szymanski [5] in 2014. Therefore Theorem 4.1, Corollaries 4.2 and 4.3 are proper generalizations of Theorem 3, its Corollaries 1 and 2 of [7], resp.

5. For single-valued maps

Recently, we defined the following in [14]:

Definition 5.1. An *abstract convex metric space* $(E, D; \Gamma, d)$ or simply a *metric space* consists of a metric space (E, d), a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, such that the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

When $E = D$, let $(E; \Gamma, d) := (E, D; \Gamma, d)$.

Similarly, we can define (partial) KKM metric spaces and abstract convex uniform spaces, resp.

In our previous work [11], we considered as follows: Let A be a nonempty bounded subset of a metric space (M, d). Then we define the following as in Khamsi [3]:

(i) $\mathrm{BI}(A) = \mathrm{ad}(A) := \bigcap \{B \subset M : B$ is a closed ball in Msuch that $A \subset B\}$.
(ii) $\mathcal{A}(M) := \{A \subset M : A = \mathrm{BI}(A)\}$, i.e., $A \in \mathcal{A}(M)$ iff A is an intersection of closed balls. In this case we will say A is an *admissible* subset of M.
(iii) A is called *subadmissible*, if for each $N \in \langle A \rangle$, $\mathrm{BI}(N) \subset A$. Obviously, if A is an admissible subset of M, then A must be subadmissible.

Note that, in Section 4, we gave quite rare best approximation theorems for multimaps, and that they can be reduced to the case $\Phi = f$ is a single valued map.

The following is a consequence of Corollary 4.3:

Theorem 5.2. *Let $(X, S; \Gamma, d)$ be a partial KKM metric space, S a nonempty Γ-convex set contained in a compact subset K of X, $f : S \to X$ is a continuous map. Then there exists $v_0 \in S$ such that*

$$d(v_0, f(v_0)) = \inf_{x \in S} d(x, f(v_0)).$$

(In particular, if $f(S) \subset S$, then v_0 is a fixed point of f.)

Proof. If the condition

(1) $B(f(x), r)$ is Γ-convex for all $x \in S$ and $r \geq 0$

is satisfied, then clearly Theorem 5.2 follows from Corollary 4.3. However, the condition (1) simply tells that

(2) each $B(f(x), r)$ is subadmissible for all $x \in S$ and $r \geq 0$,

and this holds for any metric spaces. Hence we have done the proof. □

For $S = K$, Theorem 5.2 reduces to the following best approximation theorem in [11, Theorem 4.3]:

Corollary 5.3. *Let $(M, X; \Gamma, d)$ be a partial KKM metric space where $X \subset M$ is a compact subadmissible subset. Let $f : X \to M$ be continuous. Then there exists $y_0 \in X$ such that*

$$d(y_0, f(y_0)) = \min_{x \in X} d(x, f(y_0)).$$

(In particular, if $f(X) \subset X$, then y_0 is a fixed point of f.)

Here we give the original proof in [11] in order to compare with the method of [7]:

Proof. Consider the map $G : X \multimap M$ defined by

$$G(x) := \{y \in M : d(y, f(y)) \leq d(x, f(y))\}.$$

Since f is continuous, $G(x)$ is closed for any $x \in X$. We claim that G is a KKM map on the partial KKM metric space. Indeed, assume not. Then there exist $A = \{x_1, \ldots, x_n\} \in \langle X \rangle$ and $y \in \mathrm{BI}(A)$ such that $y \notin G(A)$. This clearly implies

$$d(x_i, f(y)) < d(y, f(y)) \quad \text{for } i = 1, \ldots, n.$$

Let $\varepsilon > 0$ such that $d(x_i, f(y)) \leq d(y, f(y)) - \varepsilon$ for each i. Hence $x_i \in B(f(y), d(y, f(y)) - \varepsilon)$ for each i. Therefore, we have

$$\mathrm{BI}(A) \subset B(f(y), d(y, f(y)) - \varepsilon),$$

which implies $y \in B(f(y), d(y, f(y)) - \varepsilon)$. Clearly this gets us our contradiction which completes the proof of our claim. By the compactness of X, we deduce that $G(x)$ is compact for any $x \in X$. Therefore, there exists $y_0 \in \bigcap_{x \in X} G(x)$. This clearly implies $d(y_0, f(y_0)) \leq d(x, f(y_0))$ for any $x \in X$ which implies $d(y_0, f(y_0)) = \min_{x \in X} d(x, f(y_0))$ and the proof is complete. □

Remark 5.4. (1) Note that Corollary 5.3 reduces to [7, Lemma 4] when $(M; \Gamma, d)$ is a hyperconvex metric space. We followed Khamsi's proof faithfully.

(2) Corollary 5.3 follows from Corollary 4.3 exactly by adopting (M, X, f) instead of $(X, S = K, \Phi)$ there. This implies the validity and applicability of Theorem 4.1, Corollaries 4.2 and 4.3.

(3) Note that Corollary 5.3 generalizes the well-known 1961 best approximation theorem of Ky Fan for normed vector spaces.

6. Examples of applicable spaces

In our previous review [10], we gave a unified account for generalized KKM maps in abstract convex spaces. Moreover, we gave a large number of partial KKM spaces in [13]. Therefore, our results in this paper can be applied to such spaces.

In the end of [7], some applications of Corollaries 4.2 and 4.3 are introduced such as works of Prolla in 1983, Carbone in 1992, Khamsi in 1996, Yuan in 1999, and Amini-Harandi and Farajadeh in 2008.

In the present section, we will not recall them. However, since partial KKM metric spaces play a major roll in Section 4, we introduce some of various types of such spaces:

(1) **Normed vector spaces**: Ky Fan in 1961 obtained a best approximation theorem as a generalization of the Schauder fixed point theorem. Since his theorem is very useful, many authors tried to obtain its extensions.
(2) **Hyperconvex metric spaces**: Horvath in 1993 noticed that hyperconvex metric spaces are c-spaces, and hence, KKM metric spaces. Therefore, they can be applied results in Section 4. In fact, Khamsi in 1996 obtained some KKM theoretic results of them.
(3) **Hyperbolic metric spaces**: These are KKM metric spaces due to Kirk in 1982 and Reich-Shafrir in 1990. Therefore, they can be applied results in Section 4.
(4) **Complete continuous midpoint metric spaces**: These KKM metric spaces are due to Horvath in 2009 and have many concrete examples.
(5) **Metric spaces with global nonpositive curvature (NPC)**: These are due to Niculescu-Rovenţa in 2009 and also have many concrete examples.
(6) **A partial KKM space that is not a KKM space** due to Kulpa-Szymanski in 2014 is a metric space.

There are some other (partial) KKM metric spaces which can be applied our best approximation theorems. Some details on spaces (1) – (6); see our forthcoming work [14].

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