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FIXED POINT THEOREMS IN COMPLEX VALUED CONVEX METRIC SPACES

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Abstract. Our purpose in this paper is to introduce the concept of complex valued convex metric spaces and introduce an analogue of the Picard-Ishikawa hybrid iterative scheme, recently proposed by Okeke [24] in this new setting. We approximate (common) fixed points of certain contractive conditions through these two new concepts and obtain several corollaries. We prove that the Picard-Ishikawa hybrid iterative scheme [24] converges faster than all of Mann, Ishikawa and Noor [23] iterative schemes in complex valued convex metric spaces. Also, we give some numerical examples to validate our results.

1. Introduction

Many real life problems in science and engineering are generally functional equations. These equations can be written as fixed point equations. Consequently, scientists can investigate the existence of fixed points of such functional equations. Once the existence of fixed point of a mapping is proved,

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then one of the immediate challenge is how to find the value of the fixed point. One of the most efficient method developed by mathematicians to solve this problem is the fixed point iterative method. Several authors have developed many iteration processes to approximate the fixed point of some mappings (see, e.g. [1], [2], [24], [26]). The speed of convergence is a very important consideration in preferring an iteration process over another iteration process.

Fixed point theory has become an important tool which have been applied in the study of theoretical subjects, which are directly applicable in different applied fields of science. Other areas of applications includes optimization problems, control theory, economics and a host of others.

In 2011, Azam *et al.* [5] introduced the notion of complex valued metric spaces. They established the existence of fixed point for a pair of mappings satisfying rational inequality. Their results is intended to define rational expressions which are meaningless in cone metric spaces. Although complex valued metric spaces form a special class of cone metric spaces (see, e.g. [4], [28]), yet the definition of cone metric spaces rely on the underlying Banach space which is not a division ring. Consequently, rational expressions are not meaningful in cone metric spaces, this means that results involving mappings satisfying rational expressions cannot be generalized to cone metric spaces. Interested readers may see the following references for further studies of papers in this direction of research ([10]-[14], [30], [33]).

The relationship between the geometry of Banach spaces and fixed point theory is very strong and have attracted the attention of well-known mathematicians over the years (see, e.g. [6], [27]). Geometric properties play crucial roles in metric fixed point theory, in which convexity hypothesis and other geometric properties of Banach spaces are utilized (see, e.g. [16], [29]) and the references therein.

In 1970, Takahashi [36] introduced the concept of convexity in metric spaces. Motivated by the results of Takahashi [36], several authors have proved some interesting results in literature ([3], [6], [8], [15], [20]-[22], [27], [29], [34], [37]) and the references therein.

Motivated by the results above, we introduce the concept of complex valued convex metric spaces and introduce an analogue of the Picard-Ishikawa hybrid iterative scheme in this new framework. We approximate (common) fixed points of certain contractive conditions through these two new concepts and obtain several corollaries. We compare the rate of convergence of some iterative sequences generated by a generalized nonlinear mapping satisfying rational inequality. We provide some numerical examples to validate our analytical results. Our results generalize, extend and unify several known results, including the results of [24], [25], [35] among others.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers, for the rest of this paper, we will adopt the partial order " $\precsim$ " defined on $\mathbb{C}$ in [5].

Definition 2.1. ([5]). Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies:

(1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Definition 2.2. ([36]) Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y). \tag{2.1}$$

A metric space X together with the convex structure W is called a convex metric space, denoted by (X, d, W).

From the definition of convex structure W on X, it is obvious that

$$d(u, W(x, y, \lambda)) \geq (1 - \lambda)d(u, y) - \lambda d(u, x), \tag{2.2}$$

for each $x, y, u \in X$ and $\lambda \in [0, 1]$.

A nonempty subset C of the convex metric space X is said to be convex if $W(x, y, \lambda) \in C$ whenever $(x, y, \lambda) \in C \times C \times [0, 1]$. Takahashi [36] proved that open spheres $B(x, r) = \{y \in X : d(y, x) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(y, x) \leq r\}$ are convex. It is known that every normed space is a convex metric space. However, the converse is not true. There are many examples of convex metric spaces which are not embedded in any normed space (see, e.g. [6], [36]).

Remark 2.3. ([6]) Every normed space is a convex metric space, where a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for each $x, y, z \in X$ and $\alpha, \beta, \gamma \in I = [0, 1]$ with $\alpha + \beta + \gamma = 1$. Indeed

$$\begin{aligned} d(u, W(x, y, z; \alpha, \beta, \gamma)) &= \|u - (\alpha x + \beta y + \gamma z)\| \\ &\leq \alpha\|u - x\| + \beta\|u - y\| + \gamma\|u - z\| \\ &= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z), \end{aligned}$$

for each $u \in X$. But there exists some convex metric spaces which can not be embedded into normed space.

Example 2.4. ([6]) Let $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\}$. For $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$, we define a mapping $W : X^3 \times I^3 \to X$ by

$$W(x, y, z; \alpha, \beta, \gamma) = (\alpha x_1 + \beta y_1 + \gamma z_1, \alpha x_2 + \beta y_2 + \gamma z_2, \alpha x_3 + \beta y_3 + \gamma z_3)$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 y_1 + x_2 y_2 + x_3 y_3|.$$

Then we can show that (X, d, W) is a convex metric space, but it is not a normed space.

Example 2.5. ([6]) Let $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. For each $x = (x_1, x_2)$, $y = (y_1, y_2) \in Y$ and $\lambda \in I$. We define a mapping $W : Y^2 \times I \to Y$ by

$$W(x, y, \lambda) = \left(\lambda x_1 + (1 - \lambda) y_1, \frac{\lambda x_1 x_2 + (1 - \lambda) y_1 y_2}{\lambda x_1 + (1 - \lambda) y_1} \right)$$

and define a metric $d : Y \times Y \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (Y, d, W) is a convex metric space, but it is not a normed space.

Motivated by the results above, we next introduce the concept of complex valued convex metric spaces as follows:

Definition 2.6. Let (X, d) be a complex valued metric space. A mapping $W : X \times X \times [0, 1] \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \precsim \lambda d(u, x) + (1 - \lambda) d(u, y). \tag{2.3}$$

A complex valued metric space X together with the convex structure W is called a complex valued convex metric space, denoted by (X, d, W).

From the definition of convex structure W on X, it is obvious that

$$d(u, W(x, y, \lambda)) \succsim (1 - \lambda) d(u, y) - \lambda d(u, x), \tag{2.4}$$

for each $x, y, u \in X$ and $\lambda \in [0, 1]$.

Next, we give the following examples of complex valued convex metric spaces.

Example 2.7. Suppose $X = \mathbb{C}$ is the set of all complex numbers. For each $z_1, z_2 \in \mathbb{C}$, where $z_1 = x_1 + i y_1$, $z_2 = x_2 + i y_2$ and $\alpha \in [0, 1]$, we define $W : \mathbb{C} \times \mathbb{C} \times [0, 1] \to \mathbb{C}$ by

$$W(z_1, z_2, \alpha) = \alpha(x_1 + x_2) + i(1 - \alpha)(y_1 + y_2).$$

Define a metric $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ by

$$d(z_1, z_2) = |x_1 - x_2| + i|y_1 - y_2|.$$

Then we can show that $(\mathbb{C}, d, W)$ is a complex valued convex metric space.

Example 2.8. Suppose $X = \mathbb{C}$ is the set of all complex numbers. For each $z_1, z_2 \in \mathbb{C}$, where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $\alpha \in [0, 1]$, we define $W : \mathbb{C} \times \mathbb{C} \times [0, 1] \to \mathbb{C}$ by

$$W(z_1, z_2, \alpha) = \alpha(x_1 + x_2) + i(1 - \alpha)(y_1 + y_2).$$

Define a metric $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ by

$$d(z_1, z_2) = e^{ik}|z_1 - z_2|,$$

where $k \in [0, \frac{\pi}{2}]$. Then we can show that $(\mathbb{C}, d, W)$ is a complex valued convex metric space.

Let (X, d, W) be a complex valued convex metric space and $T : X \to X$ is a mapping on X. A point $p \in X$ is said to be a fixed point of T if $Tp = p$. In this paper, we denote the set of all fixed points of T by $F(T) := \{p \in X : Tp = p\}$.

In 1975, Dass and Gupta [9] extended the Banach contraction mapping principle by using mappings satisfying contractive condition of the rational type in the framework of complete metric spaces. Similarly, Jaggi [17], Jaggi and Dass [18] studied the existence of fixed points for mappings satisfying contractive conditions of the rational type. Motivated by these results, we study the following contractions in the framework of complex valued convex metric spaces. Let (X, d, w) be a complex valued convex metric space and $T : X \to X$ be a continuous mapping satisfying the following contractive condition:

$$\begin{aligned} d(Tx, Ty) \precsim\ & k_1.\tfrac{d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} + k_2.\tfrac{d(x,Tx)d(y,Ty)}{d(x,y)} \\ & + k_3.\tfrac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} + hd(x, y), \end{aligned} \tag{2.5}$$

where $k_1, k_2, k_3, h \in [0, 1)$ such that $k_1 + k_2 + k_3 + h < 1$ and for each $x, y \in X$ such that $x \neq y$.

Similarly, we shall study the following nonlinear mappings which could be seen as an analogue of the mappings introduced by Olatinwo [31] in the framework of complex valued convex metric spaces. Let $T : X \to X$ be a mapping satisfying

$$d(Tx, Ty) \precsim \frac{\varphi(d(x, Tx)) + ad(x, y)}{1 + Md(x, Tx)}, \tag{2.6}$$

for each $x, y \in X$, $a \in [0, 1)$, $M \geq 0$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function such that $\varphi(0) = 0$.

We next present an analogues of the Mann iterative scheme, Ishikawa iterative scheme and the Noor iterative scheme [23] in complex valued convex metric spaces. Given $x_1 \in C$, we compute the sequence $\{x_n^{(1)}\}$ as follows:

The Mann iterative sequence $\{x_n\}$ is given by

$$\begin{cases} x_1^{(1)} = x \in C, \\ x_{n+1}^{(1)} = W(Tx_n^{(1)}, x_n^{(1)}, \alpha_n), \ n \geq 1, \end{cases} \tag{2.7}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

The Ishikawa iterative sequence $\{x_n^{(2)}\}$ is given by

$$\begin{cases} x_1^{(2)} = x \in C, \\ y_n^{(2)} = W(Tx_n^{(2)}, x_n^{(2)}, \beta_n), \\ x_{n+1}^{(2)} = W(Ty_n^{(2)}, x_n^{(2)}, \alpha_n), \ n \geq 1, \end{cases} \tag{2.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

The Noor iterative sequence $\{x_n^{(3)}\}$ is given by

$$\begin{cases} x_1^{(3)} = x \in C, \\ z_n^{(3)} = W(Tx_n^{(3)}, x_n^{(3)}, \gamma_n), \\ y_n^{(3)} = W(Tz_n^{(3)}, x_n^{(3)}, \beta_n), \\ x_{n+1}^{(3)} = W(Ty_n^{(3)}, x_n^{(3)}, \alpha_n), \ n \geq 1, \end{cases} \tag{2.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

In 2013, Khan [19] introduced the Picard-Mann hybrid iterative process which is known to be faster than all of Picard, Mann and Ishikawa iterations. We now give an analogue of the Picard-Mann hybrid iterative process $\{x_n^{(4)}\}$ in the framework of complex valued convex metric spaces as follows:

$$\begin{cases} x_1^{(4)} = x \in C, \\ y_n^{(4)} = W(Tx_n^{(4)}, x_n^{(4)}, \alpha_n), \\ x_{n+1}^{(4)} = Ty_n^{(4)}, \ n \geq 1, \end{cases} \tag{2.10}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In a similar fashion, Okeke [24] recently introduced the Picard-Ishikawa hybrid iterative process $\{x_n\}_{n=0}^{\infty}$ as follows: for any fixed x_1 in D, construct the sequence $\{x_n\}$ by

$$\begin{cases} x_1 = x \in C, \\ u_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ v_n = (1 - \alpha_n)x_n + \alpha_n Tu_n, \\ x_{n+1} = Tv_n, \ n \geq 1, \end{cases} \tag{2.11}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$. The author proved that this new hybrid iterative process converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, Noor [23], Picard-Mann [19] and Picard-Krasnoselskii [26] iterative processes.

Next, we introduce an analogue of the Picard-Ishikawa hybrid iterative scheme (2.11) in the framework of complex valued convex metric spaces.

$$\begin{cases} x_1 = x \in C, \\ u_n = W(Tx_n, x_n, \beta_n), \\ v_n = W(Tu_n, x_n, \alpha_n), \\ x_{n+1} = Tv_n, \ n \geq 1, \end{cases} \tag{2.12}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$.

Next, we modify iterative scheme (2.12) to obtain the following iterative scheme for three mappings:

$$\begin{cases} x_1 = x \in C, \\ u_n = W(T_3x_n, x_n, \beta_n), \\ v_n = W(T_2u_n, x_n, \alpha_n), \\ x_{n+1} = T_1v_n, \ n \geq 1, \end{cases} \tag{2.13}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0, 1)$.

Remark 2.9. Note that the iterative scheme (2.13) contains several iterative schemes for different choices of mappings or ambient space. (2.13) reduces to

(1) (2.12) if $T_1 = T_2 = T_3 = T$.
(2) (2.11) if the ambient space is real.
(3) (2.10) if $T_1 = T_2 = T$ and $T_3 = I$, the identity mapping.

Definition 2.10. ([7]) Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to a, respectively b. Assume there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}. \tag{2.14}$$

(1) If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a faster than the sequence $\{b_n\}_{n=0}^{\infty}$ to b.
(2) If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Suppose that for two fixed point iterative processes $\{x_n\}$ and $\{y_n\}$ converging to the same fixed point z of T, the error estimates $d(x_n, z) \leq a_n$ and $d(y_n, z) \leq b_n$ for all $n \geq 1$, are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive real numbers converging to zero. Then, in view of above definition the following concept appears to be very natural (see, [15], [32]).

Definition 2.11. ([32]) If $\{a_n\}$ converges faster than $\{b_n\}$, then we say that the fixed point iterative sequence $\{x_n\}$ converges faster than the fixed point iterative sequence $\{y_n\}$ to z.

It has been observed that the comparison of the rate of convergence in the above definition depends on the choice of sequences $\{a_n\}$ and $\{b_n\}$ which are error bounds of $\{x_n\}$ and $\{y_n\}$, respectively. This method of comparison of the rate of convergence of two fixed point iterative sequences seems to be unclear (see, [15], [32]).

In 2013, Phuengrattana and Suantai [32] modified this concept as follows: Suppose $\{x_n\}$ and $\{y_n\}$ are two iterative sequences converging to the same fixed point z of T, then we say that $\{x_n\}$ converges faster than $\{y_n\}$ to z if

$$\lim_{n\to\infty} \frac{d(x_n, z)}{d(y_n, z)} = 0. \tag{2.15}$$

Lemma 2.12. ([5]) *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \longrightarrow 0$ as $n \to \infty$.*

Lemma 2.13. ([5]) *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \longrightarrow 0$ as $n \to \infty$.*

3. Convergence analysis of some fixed point iterations in complex valued convex metric spaces

In this section, we prove some convergence theorems for some generalized nonlinear mappings satisfying contractive conditions (2.5) and (2.6) in the framework of complex valued convex metric spaces. First, we prove the following theorem for three mappings satisfying contractive condition (2.6). This will approximate common fixed points needless to say important in convex minimization problems.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Suppose $T_i : X \to X$, $(i = 1, 2, 3)$ are three nonlinear mapping satisfying contractive condition* (2.6) *such that*

$$F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset.$$

Let $\{x_n\}$ be a fixed point iterative sequence generated by (2.13) *with sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges to a unique common fixed point p of T_i, $(i = 1, 2, 3)$.*

Proof. Using (2.3), (2.6) and (2.13), we obtain the following estimate for $p \in \cap_{i=1}^{3} F(T_i)$.

$$\begin{aligned} d(u_n,p) &= d(W(T_3x_n,x_n,\beta_n),p) \\ &\precsim \beta_n d(T_3x_n,p)+(1-\beta_n)d(x_n,p) \\ &\precsim \beta_n.\tfrac{d(p,T_3p)+ad(x_n,p)}{1+Md(p,T_3p)}+(1-\beta_n)d(x_n,p) \\ &= \beta_n(\varphi(0)+ad(x_n,p))+(1-\beta_n)d(x_n,p) \\ &= a\beta_n d(x_n,p)+(1-\beta_n)d(x_n,p) \\ &= (1-\beta_n(1-a))d(x_n,p). \end{aligned} \tag{3.1}$$

This gives the following estimate for $p\in\cap_{i=1}^3 F(T_i)$.

$$\begin{aligned} d(v_n,p) &= d(W(T_2u_n,x_n,\alpha_n),p) \\ &\precsim \alpha_n d(T_2u_n,p)+(1-\alpha_n)d(x_n,p) \\ &\precsim \alpha_n.\tfrac{\varphi(d(p,T_2p))+ad(u_n,p)}{1+Md(p,T_2p)}+(1-\alpha_n)d(x_n,p) \\ &= \alpha_n.(\varphi(0)+ad(u_n,p))+(1-\alpha_n)d(x_n,p) \\ &= a\alpha_n d(u_n,p)+(1-\alpha_n)d(x_n,p) \\ &\precsim a\alpha_n(1-\beta_n(1-a))d(x_n,p)+(1-\alpha_n)d(x_n,p) \\ &= (1-\alpha_n(1-a(1-\beta_n(1-a))))d(x_n,p). \end{aligned} \tag{3.2}$$

Thus we have

$$\begin{aligned} d(x_{n+1},p) &= d(T_1v_n,p) \\ &\precsim \tfrac{\varphi(d(p,T_1p))+ad(v_n,p)}{1+Md(p,T_1p)} \\ &= \varphi(0)+ad(v_n,p) \\ &\precsim a(1-\alpha_n(1-a(1-\beta_n(1-a))))d(x_n,p) \\ &\precsim (1-\alpha_n(1-a(1-\beta_n(1-a))))d(x_n,p) \\ &\vdots \\ &\precsim \textstyle\prod_{k=1}^n[1-\alpha_k(1-a(1-\beta_k(1-a)))]d(x_1,p), \end{aligned} \tag{3.3}$$

where $[1-\alpha_k(1-a(1-\beta_k(1-a)))]\in(0,1)$ since $\alpha_k,\beta_k\in(0,1)$ for all $k\in\mathbb{N}$ and $a\in[0,1)$.

It is well known in classical analysis that $1-x\leq e^{-x}$ for all $x\in[0,1]$. Using this facts together with inequality (3.3), we have

$$d(x_{n+1},p)\precsim\frac{d(x_1,p)}{e^{(1-a(1-\beta_k(1-a)))\sum_{k=1}^n\alpha_k}}. \tag{3.4}$$

This implies that

$$|d(x_{n+1},p)|\leq\frac{|d(x_1,p)|}{|e^{(1-a(1-\beta_k(1-a)))\sum_{k=1}^n\alpha_k}|}\longrightarrow 0 \quad\text{as}\quad n\to\infty. \tag{3.5}$$

Therefore, by Lemma 2.12 we have

$$\lim_{n\to\infty}d(x_n,p)=0, \tag{3.6}$$

as desired.

Next, we show that p is a unique common fixed point of T_i, $(i = 1, 2, 3)$. Suppose there exists another fixed point $p^* \in \cap_{i=1}^{3} F(T_i)$. Then by (2.6) we have

$$\begin{aligned} d(T_i p, T_i p^*) &\precsim \tfrac{\varphi(d(p,T_i p)) + a d(p,p^*)}{1 + M d(p,T_i p)} \\ &= \varphi(0) + a d(p, p^*) \\ &= a d(p, p^*). \end{aligned} \tag{3.7}$$

This implies that $|d(T_i p, T_i p^*)| \leq a|d(p, p^*)|$. This is a contradiction, and hence $p = p^*$. This completes the proof. □

In view of the Remark 2.1, although the following is a corollary to our Theorem 3.1, yet it is new in the literature.

Corollary 3.2. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition (2.6) such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a fixed point iterative sequence generated by (2.12) where sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges to a unique fixed point p of T.*

Next, we obtain the following several corollaries as consequences of the above theorem in wake of the Remark 2.1.

Corollary 3.3. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition (2.6) such that $F(T) \neq \emptyset$. Let $\{x_n^{(1)}\}$ be a fixed point iterative sequence generated by (2.7), where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n^{(1)}\}$ converges to a unique fixed point p of T.*

Corollary 3.4. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition (2.6) such that $F(T) \neq \emptyset$. Let $\{x_n^{(2)}\}$ be a fixed point iterative sequence generated by (2.8), where sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n^{(2)}\}$ converges to a unique fixed point p of T.*

Corollary 3.5. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition (2.6) such that $F(T) \neq \emptyset$. Let $\{x_n^{(3)}\}$ be a fixed point iterative sequence generated by (2.9), where sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n^{(3)}\}$ converges to a unique fixed point p of T.*

Corollary 3.6. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition (2.6) such that $F(T) \neq \emptyset$. Let $\{x_n^{(4)}\}$ be a fixed point iterative sequence generated by (2.10), where sequence $\{\alpha_n\}$ is in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n^{(4)}\}$ converges to a unique fixed point p of T.*

Next, we prove the following theorem for nonlinear mappings satisfying contractive condition (2.5).

Theorem 3.7. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition (2.5) such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a fixed point iterative sequence generated by (2.12), where sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges to a unique fixed point p of T.*

Proof. Using (2.3), (2.5) and (2.12), then we obtain the following estimate for $p \in F(T)$.

$$\begin{aligned} d(u_n, p) &= d(W(Tx_n, x_n, \beta_n), p) \\ &\precsim \beta_n d(Tx_n, p) + (1 - \beta_n) d(x_n, p) \\ &\precsim \beta_n \{k_1 . \tfrac{d(x_n, Tx_n)(1 + d(p, Tp))}{1 + d(x_n, p)} + k_2 . \tfrac{d(p, Tp) d(x_n, Tx_n)}{d(x_n, p)} \\ &\quad + k_3 . \tfrac{d(p, Tp) d(x_n, Tx_n)}{d(x_n, p) + d(p, Tx_n) + d(x_n, Tp)} + h d(x_n, p)\} + (1 - \beta_n) d(x_n, p) \\ &= \beta_n h d(x_n, p) + (1 - \beta_n) d(x_n, p) \\ &= (1 - \beta_n(1 - h)) d(x_n, p). \end{aligned} \tag{3.8}$$

Using (2.3), (2.5), (2.12) and (3.8), then we obtain the following estimate for $p \in F(T)$.

$$\begin{aligned} d(v_n, p) &= d(W(Tu_n, x_n, \alpha_n), p) \\ &\precsim \alpha_n d(Tu_n, p) + (1 - \alpha_n) d(x_n, p) \\ &\precsim \alpha_n \{k_1 . \tfrac{d(u_n, Tu_n)(1 + d(p, Tp))}{1 + d(u_n, p)} + k_2 . \tfrac{d(p, Tp) d(u_n, Tu_n)}{d(u_n, p)} \\ &\quad + k_3 . \tfrac{d(p, Tp) d(u_n, Tu_n)}{d(u_n, p) + d(p, Tu_n) + d(u_n, Tp)} + h d(u_n, p)\} + (1 - \alpha_n) d(x_n, p) \\ &= \alpha_n h d(u_n, p) + (1 - \alpha_n) d(x_n, p) \\ &\precsim \alpha_n h (1 - \beta_n(1 - h)) d(x_n, p) + (1 - \alpha_n) d(x_n, p) \\ &= (1 - \alpha_n(1 - h(1 - \beta_n(1 - h)))) d(x_n, p). \end{aligned} \tag{3.9}$$

Using (2.3), (2.5), (2.12) and (3.9), then we obtain the following estimate for $p \in F(T)$.

$$\begin{aligned} d(x_{n+1},p) &= d(Tv_n,p) \\ &\precsim k_1.\tfrac{d(v_n,Tv_n)(1+d(p,Tp))}{1+d(v_n,p)} + k_2.\tfrac{d(p,Tp)d(v_n,Tv_n)}{d(v_n,p)} \\ &\quad +k_3.\tfrac{d(p,Tp)d(v_n,Tv_n)}{d(v_n,p)+d(p,Tv_n)+d(v_n,Tp)} + hd(v_n,p) \\ &= hd(v_n,p) \\ &\precsim h(1-\alpha_n(1-h(1-\beta_n(1-h))))d(x_n,p) \\ &\precsim (1-\alpha_n(1-h(1-\beta_n(1-h))))d(x_n,p) \\ &\vdots \\ &\precsim \textstyle\prod_{k=1}^{n}[1-\alpha_k(1-h(1-\beta_k(1-h)))]d(x_1,p), \end{aligned} \tag{3.10}$$

where $[1-\alpha_k(1-h(1-\beta_k(1-h)))] \in (0,1)$ since $\alpha_k, \beta_k \in (0,1)$ for all $k \in \mathbb{N}$ and $h \in [0,1)$. It is well known in classical analysis that $1-x \leq e^{-x}$ for all $x \in [0,1]$. Using this facts together with inequality (3.10), we have

$$d(x_{n+1},p) \precsim \frac{d(x_1,p)}{e^{(1-h(1-\beta_k(1-h)))\sum_{k=1}^{n}\alpha_k}}. \tag{3.11}$$

This implies that

$$|d(x_{n+1},p)| \leq \frac{|d(x_1,p)|}{|e^{(1-h(1-\beta_k(1-h)))\sum_{k=1}^{n}\alpha_k}|} \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.12}$$

Therefore, by Lemma 2.12 we have

$$\lim_{n\to\infty} d(x_n,p) = 0, \tag{3.13}$$

as desired.

Next, we show that p is a unique fixed point of T. Suppose there exists another fixed point p^* of T. Then by (2.5) we have

$$\begin{aligned} d(Tp,Tp^*) &\precsim k_1.\tfrac{d(p^*,Tp^*)(1+d(p,Tp))}{1+d(p,p^*)} + k_2.\tfrac{d(p,Tp)d(p^*,Tp^*)}{d(p,p^*)} \\ &\quad +k_3.\tfrac{d(p,Tp)d(p^*,Tp^*)}{d(p,p^*)+d(p,Tp^*)+d(p^*,Tp)} + hd(p,p^*) \\ &= hd(p,p^*). \end{aligned} \tag{3.14}$$

This implies that $|d(Tp,Tp^*)| \leq h|d(p,p^*)|$. This is a contradiction. Hence, $p = p^*$ and the proof is complete. □

Next, we give the following theorem for three mappings satisfying contractive condition (2.5).

Theorem 3.8. *Let C be a nonempty closed and convex subset of a complex valued convex metric space (X,d,W). Suppose $T_i : X \to X$, $(i = 1,2,3)$ are three nonlinear mapping satisfying contractive condition* (2.5) *such that*

$$F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset.$$

Let $\{x_n\}$ be a fixed point iterative sequence generated by (2.13) *where sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in* $(0,1)$ *such that $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then $\{x_n\}$ converges to a unique common fixed point p of T_i,* $(i = 1, 2, 3)$.

Proof. The proof of Theorem 3.8 follows similar lines as in the proof of Theorem 3.1. □

Next, we obtain the following corollaries as consequences of Theorem 3.7.

Corollary 3.9. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition* (2.5) *such that $F(T) \neq \emptyset$. Let $\{x_n^{(1)}\}$ be a fixed point iterative sequence generated by* (2.7)*, where $\{\alpha_n\}$ is a sequence in* $[0,1]$ *such that $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then $\{x_n^{(1)}\}$ converges to a unique fixed point p of T.*

Corollary 3.10. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition* (2.5) *such that $F(T) \neq \emptyset$. Let $\{x_n^{(2)}\}$ be a fixed point iterative sequence generated by* (2.8)*, where sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in* $[0,1]$ *such that $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then $\{x_n^{(2)}\}$ converges to a unique fixed point p of T.*

Corollary 3.11. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition* (2.5) *such that $F(T) \neq \emptyset$. Let $\{x_n^{(3)}\}$ be a fixed point iterative sequence generated by* (2.9)*, where sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in* $[0,1]$ *such that $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then $\{x_n^{(3)}\}$ converges to a unique fixed point p of T.*

Corollary 3.12. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition* (2.5) *such that $F(T) \neq \emptyset$. Let $\{x_n^{(4)}\}$ be a fixed point iterative sequence generated by* (2.10)*, where $\{\alpha_n\}$ is a sequence in* $[0,1]$ *such that $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then $\{x_n^{(4)}\}$ converges to a unique fixed point p of T.*

Our next theorem considers the rate of convergence of various iterative schemes mentioned in this paper.

Theorem 3.13. *Suppose that C is a nonempty closed and convex subset of a complex valued convex metric space (X, d, W). Let $T : X \to X$ be a nonlinear mapping satisfying contractive condition* (2.6) *such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$,*

$\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n = \infty$ satisfying the following conditions:

(A1) $0 < \xi < \alpha_n < 1$,

(A2) $0 < \alpha_n < \frac{1}{1+a}$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$.

Then the Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ in (2.12) converges faster to a unique fixed point p of T than all of Mann iteration scheme $\{x_n^{(1)}\}$ in (2.7), Ishikawa iteration scheme $\{x_n^{(2)}\}$ in (2.8) and Noor iteration scheme $\{x_n^{(3)}\}$ in (2.9), provided that all the iteration schemes have the same initial guess x_1.

Proof. By Theorem 3.1, Corollary 3.3 - Corollary 3.5, we have that $\{x_n^{(1)}\}$, $\{x_n^{(2)}\}$ and $\{x_n^{(3)}\}$ converges to a unique fixed point p of T.

Next, we have the following estimate by using inequality (2.4) in (2.9)

$$\begin{aligned} d(x_{n+1}^{(3)},p) &\gtrsim (1-\alpha_n)d(x_n^{(3)},p) - \alpha_n d(Ty_n^{(3)},p) \\ &\gtrsim (1-\alpha_n)d(x_n^{(3)},p) - \alpha_n\left[\frac{\varphi(d(p,Tp)) + ad(y_n^{(3)},p)}{1+Md(p,Tp)}\right] \\ &= (1-\alpha_n)d(x_n^{(3)},p) - \alpha_n[\varphi(0) + ad(y_n^{(3)},p)] \\ &= (1-\alpha_n)d(x_n^{(3)},p) - \alpha_n ad(W(Tz_n^{(3)},x_n^{(3)},\beta_n),p) \\ &\gtrsim (1-\alpha_n(1-a(1-\beta_n(1-a))))d(x_n^{(3)},p) \\ &\quad -\alpha_n\beta_n a^2(1-\beta_n(1-a))d(x_n^{(3)},p) \\ &\gtrsim (1-\alpha_n(1+a))d(x_n,p) \\ &\;\vdots \\ &\gtrsim \textstyle\prod_{k=1}^{n}(1-\alpha_n(1+a))d(x_1,p). \end{aligned} \tag{3.15}$$

Following the same lines of proof, we obtain the following inequalities for the Mann iteration $\{x_n^{(1)}\}$ and the Ishikawa iteration $\{x_n^{(2)}\}$,

$$d(x_{n+1}^{(i)},p) \gtrsim \prod_{k=1}^{n}(1-\alpha_k(1+a))d(x_n^{(i)},p),\ (i=1,2). \tag{3.16}$$

Combinning (3.15) with (3.16), we obtain

$$d(x_{n+1}^{(i)},p) \gtrsim \prod_{k=1}^{n}(1-\alpha_k(1+a))d(x_n^{(i)},p),\ (i=1,2,3). \tag{3.17}$$

Using condition (A1) in (3.3), we have

$$d(x_{n+1},p) \precsim (1-\xi(1-a))d(x_1,p). \tag{3.18}$$

From (3.17) and (3.18), we have

$$\frac{d(x_{n+1},p)}{d(x_{n+1}^{(i)},p)} \precsim \frac{(1-\xi(1-a))^n}{\prod_{k=1}^{n}(1-\alpha_k(1+a))}. \tag{3.19}$$

Let

$$\lambda_n = \frac{(1-\xi(1-a))^n}{\prod_{k=1}^{n}(1-\alpha_k(1+a))}. \tag{3.20}$$

Therefore, by assumption (A2) we have

$$\begin{aligned}\lim_{n\to\infty}\frac{\lambda_{n+1}}{\lambda_n} &= \lim_{n\to\infty}\frac{(1-\xi(1-a))^{n+1}}{\prod_{k=1}^{n+1}(1-\alpha_k(1+a))d(x_1,p)}\cdot\frac{\prod_{k=1}^{n}(1-\alpha_k(1+a))d(x_1,p)}{(1-\xi(1-a))^n}\\ &= \lim_{n\to\infty}\frac{(1-\xi(1-a))}{(1-\alpha_{n+1}(1+a))}\\ &= (1-\xi(1-a))<1. \end{aligned} \tag{3.21}$$

By ratio test we have that $\sum_{n=1}^{\infty}\lambda_n<\infty$. This means that $\lim_{n\to\infty}\lambda_n=0$. Hence, the Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ in (2.12) converges faster to a unique fixed point p of T than all of Mann iteration scheme $\{x_n^{(1)}\}$ in (2.7), Ishikawa iteration scheme $\{x_n^{(2)}\}$ in (2.8) and Noor iteration scheme $\{x_n^{(3)}\}$ in (2.9). The proof of Theorem 3.13 is completed. □

4. Numerical examples

In this section, we provide a number of numerical examples to validate our analytical results. We compare the speed of convergence of various iterative schemes discussed in this paper, viz: the Mann iterative scheme $\{x_n^{(1)}\}$ in (2.7), the Ishikawa iterative scheme $\{x_n^{(2)}\}$ in (2.8), the Noor iterative scheme $\{x_n^{(3)}\}$ in (2.9), the Picard-Mann hybrid iterative scheme $\{x_n^{(4)}\}$ in (2.10) and the Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ in (2.12).

In the figures below, we denote the Mann iterative scheme $\{x_n^{(1)}\}$ by M, the Ishikawa iterative scheme $\{x_n^{(2)}\}$ by I, the Noor iterative scheme $\{x_n^{(3)}\}$ by N, the Picard-Mann hybrid iterative scheme $\{x_n^{(4)}\}$ by PM and the Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ by PI. All the codes were written in Matlab (R2010a) and run on PC with Intel(R) Core(TM) i3-4030U CPU @ 1.90 GHz.

Example 4.1. Let $T: X\to X$ be a mapping such that $Tx=\frac{x}{4}$, with $a=\frac{1}{2}$ and $X=[0,\infty)$. Suppose the starting point $x_1=x_1^{(1)}=x_1^{(2)}=x_1^{(3)}=x_1^{(4)}=10$ and the number of iterations for each iterative scheme is $n=100$. With respect to Theorem 3.13, we present the following numerical examples:

Case I: Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ in (2.12) versus Mann iterative scheme $\{x_n^{(1)}\}$ in (2.7). Choose $\alpha_n=\frac{1}{10n+1}$ and $\beta_n=\frac{1}{5n+1}$. Figure 1 below compares the rate of convergence of $\{x_n\}$ and $\{x_n^{(1)}\}$.

Case II: Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ in (2.12) versus Ishikawa iterative scheme $\{x_n^{(2)}\}$ in (2.8). Choose $\alpha_n=\frac{1}{10n+1}$ and $\beta_n=\frac{1}{5n+1}$. Figure 2 below compares the rate of convergence of $\{x_n\}$ and $\{x_n^{(2)}\}$.

Case III: Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ in (2.12) versus Noor iterative scheme $\{x_n^{(3)}\}$ in (2.9). Choose $\alpha_n = \frac{1}{10n+1}$, $\beta_n = \frac{1}{5n+1}$ and $\gamma_n = \frac{1}{7n+2}$. Figure 3 below compares the rate of convergence of $\{x_n\}$ and $\{x_n^{(3)}\}$.

Case IV: Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ in (2.12) versus Picard-Mann hybrid iterative scheme $\{x_n^{(4)}\}$ in (2.10). Choose $\alpha_n = \frac{1}{10n+1}$ and $\beta_n = \frac{1}{5n+1}$. Figure 4 below compares the rate of convergence of $\{x_n\}$ and $\{x_n^{(4)}\}$.

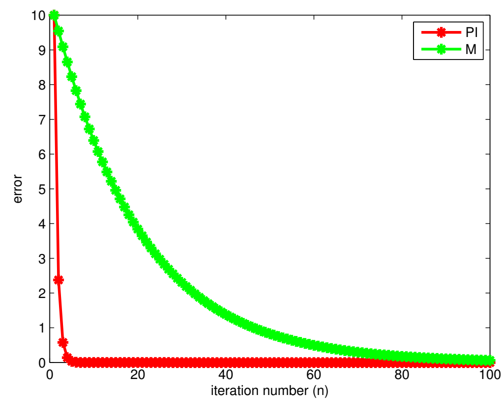

Figure 1. Error versus iteration number (n)

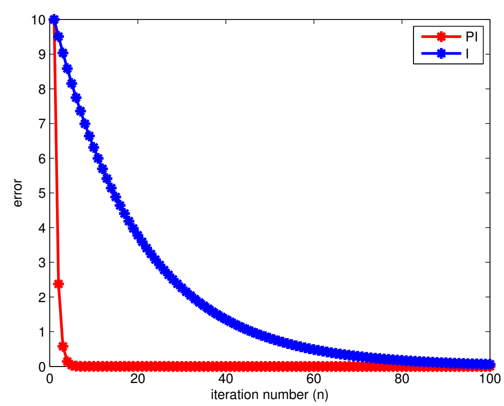

Figure 2. Error versus iteration number (n)

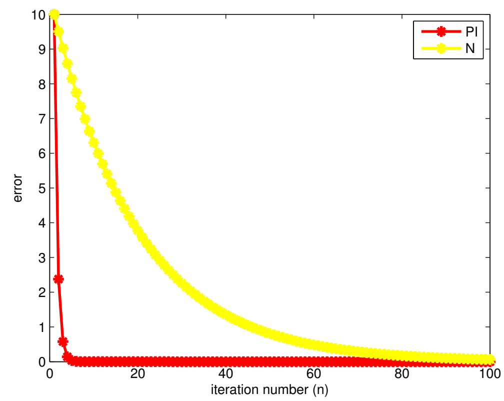

FIGURE 3. Error versus iteration number (n)

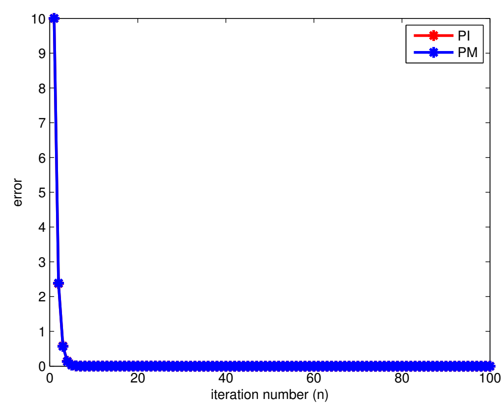

FIGURE 4. Error versus iteration number (n)

Remark 4.2. From Figure 1, Figure 2 and Figure 3, we see that the Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ converges faster than all of Mann iterative scheme $\{x_n^{(1)}\}$, the Ishikawa iterative scheme $\{x_n^{(2)}\}$ and the Noor iterative scheme $\{x_n^{(3)}\}$ to the fixed point $p = 0$ of T. In Figure 4, we see that the Picard-Ishikawa hybrid iterative scheme $\{x_n\}$ and the Picard-Mann hybrid iterative scheme $\{x_n^{(4)}\}$ have the same rate of convergence.

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