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GENERALIZED PSEUDO-DIFFERENTIAL OPERATORS INVOLVING FRACTIONAL FOURIER TRANSFORM

B. B. Waphare[1] and P. D. Pansare[2]

[1]MIT Arts, Commerce and Science College, Alandi (D), Pune, India
e-mail: balasahebwaphare@gmail.com

[2]Research Scholar, Savitribai Phule Pune University, Pune, India
e-mail: pradippansare@gmail.com

Abstract. Generalized pseudo-differential operators (PDO) involving fractional Fourier transform associate with the symbol $a(x, y)$ whose derivatives satisfy certain growth condition is defined. The product of two generalized pseudo-differential operators is shown to be a generalized pseudo-differential operator.

1. Introduction

If $\phi \in L_1(\mathbb{R})$ then the Fourier transform of a function is defined by

$$\widetilde{\phi}(y) = (\mathcal{F}(\phi))(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} \phi(x) dx, \ y \in \mathbb{R} \tag{1.1}$$

and if $\widetilde{\phi} \in L_1(\mathbb{R})$ then the inverse Fourier transform is given by

$$\phi(x) = (\mathcal{F}^{-1}(\widetilde{\phi}))(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} \widetilde{\phi}(y) dy, \ x \in \mathbb{R}. \tag{1.2}$$

The one dimensional fractional Fourier transform ([5], [6]) with parameters α, β of $\phi(x)$ denoted by $(\mathcal{F}_{\alpha,\beta}(\phi))(t) = \widehat{\phi}_{\alpha,\beta}(t)$ is given by

$$(\mathcal{F}_{\alpha,\beta}(\phi))(t) = \widehat{\phi}_{\alpha,\beta}(t) = \int_{\mathbb{R}} K_{\alpha,\beta}(x, t) \phi(x) dx, \tag{1.3}$$

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[0]Corresponding author: B .B. Waphare(balasahebwaphare@gmail.com).

where $K_{\alpha,\beta}(x,t) = \begin{cases} C_{\alpha,\beta} e^{i(x^2+t^2)\cot(\alpha-\beta)^{(\alpha+\beta)} - ixt\csc(\alpha-\beta)} & \text{if} \quad (\alpha-\beta) \neq n\pi, \\ \frac{1}{(2\pi)^{\alpha+\beta}} e^{-ixt} & \text{if} \quad (\alpha-\beta) = \frac{\pi}{2}, \end{cases}$
for n is an integer and

$$C_{\alpha,\beta} = (2\pi isin(\alpha-\beta))^{-(\alpha+\beta)} e^{i(\alpha-\beta)^{(\alpha+\beta)}} = \left(\frac{1-icot(\alpha-\beta)}{2\pi}\right)^{\alpha+\beta}.$$

It is inverted by

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{K_{\alpha,\beta}(x,t)} (\mathcal{F}_{\alpha,\beta}(\phi))(t)dt, \tag{1.4}$$

where

$$\overline{K_{\alpha,\beta}(x,t)} = C'_{\alpha,\beta} e^{-i(x^2+t^2)\cot(\alpha-\beta)^{(\alpha+\beta)}} + ixt\csc(\alpha-\beta) \tag{1.5}$$

and

$$\begin{aligned} C'_{\alpha,\beta} &= \frac{(2\pi i\sin(\alpha-\beta))^{\alpha+\beta}}{\sin(\alpha-\beta)} e^{-i(\alpha-\beta)^{(\alpha+\beta)}} \\ &= \frac{(2\pi i)^{\alpha+\beta}(\sin(\alpha-\beta))^{\alpha+\beta}}{\sin(\alpha-\beta)} (\cos(\alpha-\beta) - i\sin(\alpha-\beta))^{(\alpha+\beta)} \\ &= \frac{(2\pi i)^{\alpha+\beta}(\sin(\alpha-\beta))^{2(\alpha+\beta)}}{\sin(\alpha-\beta)(\sin(\alpha-\beta))^{\alpha+\beta}} (\cos(\alpha-\beta) - i\sin(\alpha-\beta))^{(\alpha+\beta)} \\ &= \frac{(2\pi i)^{\alpha+\beta}(\sin(\alpha-\beta))^{2(\alpha+\beta)}}{\sin(\alpha-\beta)} (\cot(\alpha-\beta) - i)^{\alpha+\beta} \\ &= \frac{(2\pi)^{\alpha+\beta}(\sin(\alpha-\beta))^{2(\alpha+\beta)}}{\sin(\alpha-\beta)} (1 + i\cot(\alpha-\beta))^{\alpha+\beta} \\ &= \frac{(\sin(\alpha-\beta))^{2(\alpha+\beta)}}{\sin(\alpha-\beta)} \left[2\pi(1 + i\cot(\alpha-\beta))\right]^{\alpha+\beta}. \end{aligned}$$

Definition 1.1. A tempered distribution $\phi \in H^s(\mathbb{R})$ (the Sobolev space), $s \in \mathbb{R}$, if its fractional Fourier transform $\mathcal{F}_{\alpha,\beta}\phi$ corresponding to a locally integrable function $(\mathcal{F}_{\alpha,\beta}(\phi))(t)$ over $\mathbb{R}$ such that

$$\| \phi \|_{H^s} = \left(\int_{\mathbb{R}} \left|(1+|t|^2)^{s/2}\mathcal{F}_{\alpha,\beta}(\phi)(t)\right|^2 dt\right)^{1/2} < \infty. \tag{1.6}$$

This space is complete with respect to the norm $\| \phi \|_{H^s}$.

2. Properties of fractional Fourier transform

First we recall the definition of the Schwartz space $S(\mathbb{R})$.

Definition 2.1. The space S, the so called space of smooth functions of rapid descent is defined as follows: ϕ is member of S if and only if it is complex valued C^∞ function on $\mathbb{R}$ and for every choice of b and c of non negative integers, it satisfies

$$\Gamma_{b,c}(\phi) = \sup_{x\in\mathbb{R}} \mid x^b D^c \phi(x) \mid < \infty. \tag{2.1}$$

Proposition 2.2. *Let $K_{\alpha,\beta}(x,t)$ be the kernel of fractional Fourier transform and $\Delta_x^r = \left(\frac{d}{dx} - ix\cot(\alpha-\beta)\right)^r$. Then*

$$\Delta_x^r K_{\alpha,\beta}(x,t) = (-it\csc(\alpha-\beta))^r K_{\alpha,\beta}(x,t),$$

for all $r \in \mathbb{N}_0$.

Proof.

$$\begin{aligned} \frac{d}{dx}\left[K_{\alpha,\beta}(x,y)\right] &= C_{\alpha,\beta}\frac{d}{dx}\left[e^{i\left[(x^2+t^2)\cos(\alpha-\beta)\right]^{(\alpha+\beta)} - ixt\csc(\alpha-\beta)}\right] \\ &= K_{\alpha,\beta}(x,t)i(x\cot(\alpha-\beta) - t\csc(\alpha-\beta)). \end{aligned}$$

So that

$$\left(\frac{d}{dx} - ix\cot(\alpha-\beta)\right)K_{\alpha,\beta}(x,t) = (-it\csc(\alpha-\beta))K_{\alpha,\beta}(x,t).$$

Continuing in this way, we get

$$\left(\frac{d}{dx} - ix\cot(\alpha-\beta)\right)^r K_{\alpha,\beta}(x,t) = (-it\csc(\alpha-\beta))^r K_{\alpha,\beta}(x,t).$$

□

Proposition 2.3. *For all $\phi \in S(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} \Delta_x^r K_{\alpha,\beta}(x,t)\phi(x)dx = \int_{\mathbb{R}} K_{\alpha,\beta}(x,t)(\Delta_x')^r\phi(x)dx,$$

for all $r \in \mathbb{N}_0$, where $\Delta_x' = -(\frac{d}{dx} + ix\cot(\alpha-\beta))$.

Proof. First we prove

$$\int_{\mathbb{R}} \Delta_x K_{\alpha,\beta}(x,t)\phi(x)dx = \int_{\mathbb{R}} K_{\alpha,\beta}(x,t)(\Delta_x')\phi(x)dx.$$

Using integration by parts, we have

$$\begin{aligned} &\int_{\mathbb{R}} \left(\frac{d}{dx} - ix\cot(\alpha-\beta)\right)K_{\alpha,\beta}(x,t)\phi(x)dx \\ &= -\int_{\mathbb{R}} K_{\alpha,\beta}(x,t)\left(\frac{d}{dx} + ix\cot(\alpha-\beta)\right)\phi(x)dx. \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} \Delta_x K_{\alpha,\beta}(x,t)\phi(x)dx = \int_{\mathbb{R}} K_{\alpha,\beta}(x,t)(\Delta'_x)\phi(x)dx.$$

In general, we have

$$\int_{\mathbb{R}} \Delta_x^r K_{\alpha,\beta}(x,t)\phi(x)dx = \int_{\mathbb{R}} K_{\alpha,\beta}(x,t)(\Delta'_x)^r\phi(x)dx.$$

□

Proposition 2.4. *Let $\phi \in S$. Then*

$$\left(\mathcal{F}_{\alpha,\beta}(\Delta'_x)^r\phi(x)\right)(t) = (-it\csc(\alpha,\beta))^r \left(\mathcal{F}_{\alpha,\beta}\phi(x)\right)(t)$$

for all $r \in \mathbb{N}_0$.

Proof. By using Proposition 2.2 and Proposition 2.3, we have

$$\begin{aligned}\left(\mathcal{F}_{\alpha,\beta}(\Delta'_x)^r\phi(x)\right)(t) &= \int_{\mathbb{R}} K_{\alpha,\beta}(x,t)(\Delta'_x)^r\phi(x)dx\\ &= \int_{\mathbb{R}} \Delta_x^r K_{\alpha,\beta}(x,t)\phi(x)dx\\ &= (it\csc(\alpha-\beta))^r\int_{\mathbb{R}} K_{\alpha,\beta}(x,t)\phi(x)dx\\ &= (-it\csc(\alpha,\beta))^r\left(\mathcal{F}_{\alpha,\beta}\phi(x)\right)(t).\end{aligned}$$

□

3. Product of two generalized pseudo differential operators

Sobolev space $H^s(\mathbb{R})$ is defined by (1.6). The variant of this, denoted by $H^s(\mathbb{R}\times\mathbb{R})$, where $s \in \mathbb{R}$ defined by [2] as follows:

A tempered distribution $\phi \in S'(\mathbb{R}^2)$ is said to belong to $H^s(\mathbb{R}\times\mathbb{R})$, if $\left(\mathcal{F}_{\alpha,\beta}\phi\right)(t,\eta)$ is locally integrable on $\mathbb{R}^2$ and

$$\left[(1+|t|^2)(1+|\eta|^2)\right]^{s/2}\left(\mathcal{F}_{\alpha,\beta}\phi\right)(t,\eta) \in L^2(\mathbb{R}^2).$$

A norm in this space is defined as

$$\begin{aligned}\parallel\phi\parallel_{H^s(\mathbb{R}\times\mathbb{R})} &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |(1+|t|^2)^{s/2}(1+|\eta|^2)^{s/2}\left(\mathcal{F}_{\alpha,\beta}\phi\right)(t,\eta)|^2 dt d\eta\right)^{1/2}\\ &< \infty, \quad \phi \in S'(\mathbb{R}\times\mathbb{R}).\end{aligned} \tag{3.1}$$

Definition 3.1. The function $a(x,y) : C^\infty(\mathbb{R}\times\mathbb{R}) \mapsto \mathbb{C}$ belongs to the class S^m if and only if for all $q, \upsilon, b \in \mathbb{N}_0$, there exist $D_{b,\upsilon,m} > 0$ such that

$$(1+|x|)^q|D_x^b D_y^\upsilon a(x,y)| \leq D_{b,\upsilon,m}(1+|y|)^{m-\upsilon}. \tag{3.2}$$

Lemma 3.2. *For any symbol $a \in S^m, m \in \mathbb{R}$ and $l > 1 \in \mathbb{N}$, there exists a positive constant C_m such that*

$$|(\mathcal{F}_{\alpha,\beta}a)(t,\eta)| \leq C_m(1+|\eta|)^m(1+t^2\csc^2(\alpha-\beta))^{-l/2}. \tag{3.3}$$

Proof. We know that

$$(\mathcal{F}_{\alpha,\beta}a)(t,\eta) = C_{\alpha,\beta}\int_{\mathbb{R}} e^{(i(x^2+t^2)\cot(\alpha-\beta))^{(\alpha+\beta)}-ixt\csc(\alpha-\beta)}a(x,\eta)dx.$$

So that

$$(1+it\csc(\alpha-\beta))^l(\mathcal{F}_{\alpha,\beta}a)(t,\eta) = \int_{\mathbb{R}} K_{\alpha,\beta}(x,t)(1-\Delta'_x)^l a(x,\eta)dx \tag{3.4}$$

for all $\eta, t \in \mathbb{R}, l > 1 \in \mathbb{N}$. Now

$$\begin{aligned}(1-\Delta'_x)^l a(x,\eta) &= \sum_{r=0}^{l}\binom{l}{r}(-1)^r(\Delta'_x)^r a(x,\eta)\\ &= \sum_{r=0}^{l}\binom{l}{r}(-1)^r\sum_{k=0}^{r}P(x,k)D_x^k a(x,\eta)\\ &= \sum_{r=0}^{l}\binom{l}{r}(-1)^r\sum_{k=0}^{r}\sum_{l_1=0}^{k}d_{l_1}x^{l_1}D_x^k a(x,\eta),\end{aligned}$$

where $P(x,k)$ is a polynomial of maximum degree r. Using (3.2) in above equation, we have

$$\left|(1-\Delta'_x)^l a(x,\eta)\right| \leq \sum_{r=0}^{l}\binom{l}{r}\sum_{k=0}^{r}\sum_{l_1=0}^{k}|d_{l_1}|D_{k,m}(1+|\eta|)^m(1+|x|)^{-l}. \tag{3.5}$$

Therefore by (3.4) and (3.5), we get

$$\begin{aligned}&|(1+it\csc(\alpha-\beta))|^l\,|(\mathcal{F}_{\alpha,\beta}a)(t,\eta)|\\ &\leq |C_{\alpha,\beta}|\sum_{r=0}^{l}\binom{l}{r}\sum_{k=0}^{r}\sum_{l_1=0}^{k}|d_{l_1}|D_{k,m}(1+|\eta|)^m \times \int_{\mathbb{R}}(1+|x|)^{-l}dx.\end{aligned}$$

Since integral is convergent for large value of l, there exists a constant $C_m > 0$ depending on $l, \alpha, \beta, k, r, m$ such that

$$|(\mathcal{F}_{\alpha,\beta}a)(t,\eta)| \leq C_m(1+|\eta|)^m(1+t^2\csc^2(\alpha-\beta))^{-l/2}.$$

□

A linear partial differential operator $A(x, \Delta'_x)$ on $\mathbb{R}$ is given by

$$A(x, \Delta'_x) = \sum_{r=0}^{m} a_r(x)(\Delta'_x)^r, \tag{3.6}$$

where the coefficient $a_r(x)$ are functions defined on $\mathbb{R}$ and $\Delta'_x = -(\frac{d}{dx} + ix\cot(\alpha - \beta))$. If we replace $(\Delta'_x)^r$ in (3.6) by monomial $(-it\csc(\alpha-\beta))^r$ in $\mathbb{R}$, then we obtain the so called symbol

$$A(x, t) = \sum_{r=0}^{m} a_r(x)(-it\csc(\alpha - \beta))^r. \tag{3.7}$$

In order to get another representation of the operator $A(x, \Delta'_x)$, let $\phi \in S(\mathbb{R})$, by (1.3), (1.4) and Proposition 2.4, we have

$$\begin{aligned}(A(x, \Delta'_x)\phi)(x) &= \sum_{r=0}^{m} a_r(x)\mathcal{F}^{-1}_{\alpha,\beta}\mathcal{F}_{\alpha,\beta}(\Delta'_x)^r\phi(x) \\ &= \sum_{r=0}^{m} a_r(x)\mathcal{F}^{-1}_{\alpha,\beta}(-it\csc(\alpha-\beta))^r\left(\mathcal{F}_{\alpha,\beta}\phi(x)\right)(t) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \overline{K_{\alpha,\beta}(x,t)}A(x,t)(\mathcal{F}_{\alpha,\beta}\phi)(t)dt,\end{aligned}$$

where $\overline{K_{\alpha,\beta}(x,t)}$ is defined by (1.5). If we replace the symbol $A(x, t)$ by more general symbol $a(x, t)$ which is no longer polynomial in t, we get the generalized pseudo-differential operator $A_{a,\alpha,\beta}$ defined below.

Definition 3.3. Let $a(x, y)$ be a complex valued function belonging to the space $\mathbb{C}^\infty(\mathbb{R}\times\mathbb{R})$, and its derivatives satisfy certain growth conditions (3.2). Then the generalized pseudo-differential operator $A_{a,\alpha,\beta}$ associated with the symbol $a(x, y)$ is defined by

$$(A_{a,\alpha,\beta}\phi) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{K_{\alpha,\beta}(x,y)}a(x,y)\left(\mathcal{F}_{\alpha,\beta}\phi\right)(y)dy, \tag{3.8}$$

where $\left(\mathcal{F}_{\alpha,\beta}\phi\right)(y)$ is defined in (1.3).

For pseudo-differential operator involving Fourier transform, we may refer to [3, 4].

Theorem 3.4. *For any symbol $a(x, y) \in S^m$, the associated operator $(A_{a,\alpha,\beta}\phi)(x)$ can be represented by*

$$\begin{aligned}(A_{a,\alpha,\beta}\phi)(x) &= \left(\frac{C'_{\alpha,\beta}}{2\pi}\right)^2 \int_{\mathbb{R}}\int_{\mathbb{R}} \left(e^{\left[-i(x^2+t^2+y^2+\eta^2)\cot(\alpha-\beta)\right]^{(\alpha+\beta)}}\right) \\ &\quad \times e^{i(xt+y\eta)}\left(\mathcal{F}_{\alpha,\beta}a\right)(t,\eta)\left(\mathcal{F}_{\alpha,\beta}\phi\right)(\eta)dtd\eta,\end{aligned} \tag{3.9}$$

where $\phi \in S(R)$.

Proof. Proof can be completed by making use of (1.3) and (1.4) in (3.8). □

Corollary 3.5. *For any symbol* $a \in S^m$ *and* $\phi \in S(\mathbb{R})$,

$$\mathcal{F}_{\alpha,\beta}\left[A_{a,\alpha,\beta}\phi\right](t,\eta) = \left(\mathcal{F}_{\alpha,\beta}a\right)(t,\eta)\left(\mathcal{F}_{\alpha,\beta}\phi\right)(\eta), \tag{3.10}$$

where the fractional Fourier transform is taken with respect to all the variables x *and* y.

Definition 3.6. Let $\sigma(x,t) \in S^{m_1}$ and $\tau(y,t) \in S^{m_2}$. Then the product of two generalized pseudo-differential operators $B_{\tau,\alpha,\beta}$ and $A_{\sigma,\alpha,\beta}$ associated with symbol $\tau(y,t)$ and $\sigma(x,t)$ respectively is defined by

$$\left(B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi\right)(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{K_{\alpha,\beta}(x,t)}\tau(y,t)\mathcal{F}_{\alpha,\beta}\left[A_{\sigma,\alpha,\beta}\right](t,y)dt, \tag{3.11}$$

if the integral is convergent.

Theorem 3.7. *Let* $\sigma(x,t) \in S^{m_1}$ *and* $\tau(y,t) \in S^{m_2}$. *Then the product of two pseudo-differential operators* $B_{\tau,\alpha,\beta}$ *and* $A_{\sigma,\alpha,\beta}$ *is again a generalized pseudo-differential operator whose symbol is in* $S^{m_1+m_2}$.

Proof. Let $\phi \in S(\mathbb{R})$. Then from the Definition 3.6 and (3.10), we have

$$\begin{aligned}
&\left(B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi\right)(x,y) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \overline{K_{\alpha,\beta}(x,t)}\tau(y,t)(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)(\mathcal{F}_{\alpha,\beta}\phi)(t)dt.
\end{aligned} \tag{3.12}$$

This shows that $\tau(y,t)(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)$ is symbol of the product $B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}$.

Now, we have to show that this symbol is in $S^{m_1+m_2}$.
For $a_1, b_1 \in \mathbb{N}_0$, we have

$$\begin{aligned}
&|(D_y)^{a_1}(D_t)^{b_1}\tau(y,t)(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)| \\
&\leq \sum_{a_2=0}^{a_1}\binom{a_1}{a_2}\sum_{b_2=0}^{b_1}\binom{b_1}{b_2}|(D_y)^{a_1-a_2}(D_t)^{b_1-b_2}\tau(y,t)| \\
&\quad \times |(D_y)^{a_2}(D_t)^{b_2}(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)|.
\end{aligned} \tag{3.13}$$

Now,

$$
\begin{aligned}
&(D_y)^{a_2}(D_t)^{b_2}(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)\\
&= C_{\alpha,\beta}\int_{\mathbb{R}} D_y^{a_2}\left(e^{\left[i(x^2+y^2)\cot(\alpha-\beta)\right](\alpha+\beta)-ixy\csc(\alpha-\beta)}\right) D_t^{b_2}\sigma(y,t)dx\\
&= C_{\alpha,\beta}\int_{\mathbb{R}}\sum_{a_3=0}^{a_2}\left(\begin{array}{c} a_2\\ a_3\end{array}\right) D_y^{a_2-a_3}\left(e^{\left[i(x^2+y^2)\cot(\alpha-\beta)\right](\alpha+\beta)}\right)\\
&\quad\times D_y^{a_3}\left(e^{-ixy\csc(\alpha-\beta)}\right) D_t^{b_2}\sigma(y,t)dx\\
&= C_{\alpha,\beta}\int_{\mathbb{R}}\sum_{a_3=0}^{a_2}\left(\begin{array}{c} a_2\\ a_3\end{array}\right) e^{\left[i(x^2+y^2)\cot(\alpha-\beta)\right](\alpha+\beta)}\\
&\quad\times P_{a_2-a_3}(y,(i\cot(\alpha-\beta))(\alpha+\beta))\\
&\quad\times(-ix\csc(\alpha-\beta))^{a_3}e^{-ix\csc(\alpha-\beta)}D_t^{b_2}\sigma(y,t)dx\\
&= C_{\alpha,\beta}\sum_{a_3=0}^{a_2}\left(\begin{array}{c} a_2\\ a_3\end{array}\right)\sum_{s=0}^{a_2-a_3} a_s\cot(\alpha-\beta)y^s(-i\csc(\alpha-\beta))^{a_3}\\
&\quad\times\int_{\mathbb{R}} e^{\left[i(x^2+y^2)\cot(\alpha-\beta)\right](\alpha+\beta)-ixy\csc(\alpha-\beta)}x^{a_3}D_t^{b_2}\sigma(y,t)dx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&|(D_y)^{a_2}(D_t)^{b_2}(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)|\\
&\le |C_{\alpha,\beta}|\sum_{a_3=0}^{a_2}\left(\begin{array}{c} a_2\\ a_3\end{array}\right)\sum_{s=0}^{a_2-a_3}|a_s\cot(\alpha-\beta)|\\
&\quad\times|y|^s|\csc(\alpha-\beta)|^{a_3}\int_{\mathbb{R}}|x^{a_3}||D_t^{b_2}\sigma(y,t)|dx\\
&\le |C_{\alpha,\beta}|\sum_{a_3=0}^{a_2}\left(\begin{array}{c} a_2\\ a_3\end{array}\right)\sum_{s=0}^{a_2-a_3}|a_s\cot(\alpha-\beta)|\\
&\quad\times(1+|y|)^s|\csc(\alpha-\beta)|^{a_3}\int_{\mathbb{R}}(1+|x|)^{a_3}|D_t^{b_2}\sigma(y,t)|dx\\
&\le D_{b_3,m}|C_{\alpha,\beta}|\sum_{a_3=0}^{a_2}\left(\begin{array}{c} a_2\\ a_3\end{array}\right)\sum_{s=0}^{a_2-a_3}|a_s\cot(\alpha-\beta)|\\
&\quad\times(1+|y|)^s|\csc(\alpha-\beta)|^{a_3}(1+|t|)^{m_1-b_2}\int_{\mathbb{R}}(1+|x|)^{a_3-q}dx.
\end{aligned}
$$

The x-integral being convergent for $q > a_3+1$, therefore

$$
|(D_y)^{a_2}(D_t)^{b_2}(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)| \le L(1+|y|)^s(1+|t|)^{m_1-b_2}, \tag{3.14}
$$

where L is a positive constant depending on $\alpha-\beta, a_2, a_3, , s, b_2, m$ and q.

By (3.13), (3.14) and (3.2), we have

$$
\begin{aligned}
&|(D_y)^{a_2}(D_t)^{b_2}(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)| \\
&\leq L\sum_{a_2=0}^{a_1}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\sum_{b_2=0}^{b_1}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} C_{a_1-a_2,b_1-b_2,m_2,s}(1+|t|)^{m_1+m_2-b_1} \\
&\leq L_1(1+|t|)^{m_1+m_2-b_1},
\end{aligned}
$$

where L_1 is a positive constant depending on $a_1, a_2, b_1, b_2, m_2, s$ and L. Therefore the symbol of the product $B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}$ is in $S^{m_1+m_2}$. □

Theorem 3.8. *Let $\sigma(x,t) \in S^{m_1}$ and $\tau(y,t) \in S^{m_2}$. Then for certain $C_1 > 0, m_1, m_2 \in \mathbb{R}_+$,*

$$
\| (B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi)(x,y) \|^2_{H^s(\mathbb{R}\times\mathbb{R})} \leq C_1 \| \phi \|^2_{H^{s+m_1+m_2}(\mathbb{R})},
$$

for all $\phi \in S(\mathbb{R})$.

Proof. From Definition 3.6 and (3.12), it follows that $(B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi)(x,y)$ has the fractional Fourier transform equal to $\tau(y,t)(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)$. Therefore,

$$
\begin{aligned}
&\| (B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi)(x,y) \|^2_{H^s(\mathbb{R}\times\mathbb{R})} \\
&= \| (1+|t|^2)^{s/2}(1+|y|^2)^{s/2}\mathcal{F}_{\alpha,\beta}(B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta})(t,y) \|^2_{L^2(\mathbb{R}\times\mathbb{R})} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |(1+|t|^2)^{s/2}(1+|y|^2)^{s/2}\tau(y,t) \\
&\quad \times (\mathcal{F}_{\alpha,\beta}\sigma)(y,t)(\mathcal{F}_{\alpha,\beta}\phi)(t)|^2 dydt.
\end{aligned}
$$

Since from (3.2)

$$
|\tau(y,t)| \leq C_{m_2}(1+|t|)^{m_2}(1+|y|^2)^{-l},
$$

and from (3.3)

$$
|(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)| \leq C_{m_1}(1+|t|)^{m_1}(1+y^2\csc^2(\alpha-\beta))^{-l/2}.
$$

We know that

$$
(1+|t|)^m \leq \begin{cases} 2^{m/2}(1+|t|^2)^{m/2} & \text{if} \quad m \geq 0, \\ (1+|t|^2)^{m/2} & \text{if} \quad m < 0. \end{cases}
$$

Therefore,

$$
(1+|t|)^m \leq \max(1, 2^{m/2})(1+|t|^2)^{m/2} = C_m(1+|t|^2)^{m/2},
$$

where $C_m = \max(1, 2^{m/2})$. Thus

$$\begin{aligned}
&\| (B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi)(x,y) \|^2_{H^s(\mathbb{R}\times\mathbb{R})} \\
&\leq C_{m_1}C_{m_2}\int_{\mathbb{R}}(1+|y|^2)^{s-2l}(1+y^2\csc^2(\alpha-\beta))^{-l}dy \\
&\quad\times\int_{\mathbb{R}}|C_m^2(1+|t|^2)^{\frac{s+m_1+m_2}{2}}(\mathcal{F}_{\alpha,\beta}\phi)(t)|^2dt.
\end{aligned}$$

Now we have to show that the first integral is convergent. Consider

$$\begin{aligned}
&\int_{\mathbb{R}}(1+|y|^2)^{s-2l}(1+y^2\csc^2(\alpha-\beta))^{-l}dy \\
&= \int_{|y|\leq 1}(1+|y|^2)^{s-2l}(1+y^2\csc^2(\alpha-\beta))^{-l}dy \\
&\quad + \int_{|y|>1}(1+|y|^2)^{s-2l}(1+y^2\csc^2(\alpha-\beta))^{-l}dy \\
&= I_1 + I_2,
\end{aligned}$$

where I_1 and I_2 denotes first and second integrals respectively. Since I_1 is bounded, we have to show the convergent for I_2.

$$I_2 = \int_{|y|>1}(1+|y|^2)^{s-2l}(1+y^2\csc^2(\alpha-\beta))^{-l}dy.$$

If $s-2l>0$, then

$$\begin{aligned}
I_2 &\leq \int_{|y|>1} 2^{s-2l}|y|^{2s-4l}(\sin^2(\alpha-\beta)+y^2)^{-l}(\csc(\alpha-\beta))^{-2l}dy \\
&\leq \int_{|y|>1} 2^{s-2l}|y|^{2s-4l}|y|^{-2l}(\csc(\alpha-\beta))^{-2l}dy \\
&= 2^{s-2l}(\csc(\alpha-\beta))^{-2l}\int_{|y|>1}|y|^{2s-6l}dy \\
&< \infty.
\end{aligned}$$

The y-integral is convergent by choosing $l > \frac{1+2s}{6}$.

Now, for $s-2l<0$, let $s-2l=-p$,

$$\begin{aligned}
I_2 &\leq \int_{|y|>1}\frac{\sin^{2l}(\alpha-\beta)}{(1+|y|^2)^p(\sin^2(\alpha-\beta)+y^2)^l}dy \\
&\leq \int_{|y|>1}\frac{1}{|y|^{2p}y^{2l}}dy \\
&\leq \int_{|y|>1} y^{-(2p+2l)}dy < \infty.
\end{aligned}$$

The above integral is convergent by choosing $2p + 2l > 1$. Therefore,

$$\begin{aligned}
&\| (B_{\tau,\alpha,\beta} A_{\sigma,\alpha,\beta}\phi)(x,y) \|^2_{H^s(\mathbb{R}\times\mathbb{R})} \\
&\leq C_1 \int_{\mathbb{R}} |(1+|t|^2)^{\frac{s+m_1+m_2}{2}} (\mathcal{F}_{\alpha,\beta}\phi)(t)|^2 dt \\
&\leq C_1 \| \phi \|^2_{H^{s+m_1+m_2}(\mathbb{R})},
\end{aligned}$$

where C_1 is a positive constant. This completes the proof. □

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