MAJORIZATION PROBLEMS FOR UNIFORMLY STARLIKE FUNCTIONS BASED ON RUSCHEWEYH $q$–DIFFERENTIAL OPERATOR RELATED WITH EXPONENTIAL FUNCTION

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Abstract. The main object of this present paper is to study some majorization problems for certain classes of analytic functions defined by means of $q$–calculus operator associated with exponential function.

1. INTRODUCTION

Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)
which are analytic in the open unit disk $\mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \}$. For given $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$, the Hadamard product of $f$ and $g$ is defined by

$$(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g \ast f)(z).$$

For two analytic functions $f, g \in \mathcal{A}$, we say that $f$ is subordinate to $g$, denoted by $f \prec g$, if there exists a Schwarz function $\omega(z)$ which is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z))$ for $z \in \mathbb{U}$. Note that, if the function $g$ is univalent in $\mathbb{U}$, due to Miller and Mocanu [6], we have

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

If $f$ and $g$ are analytic functions in $\mathbb{U}$, following MacGregor [5], we say that $f$ is majorized by $g$ in $\mathbb{U}$, that is $f(z) \ll g(z)$ ($z \in \mathbb{U}$) if there exists a function $\phi(z)$, analytic in $\mathbb{U}$, such that

$$|\phi(z)| < 1 \text{ and } f(z) = \phi(z)g(z) \quad (z \in \mathbb{U}).$$

It is of interest to note that the notion of majorization is closely related to the concept of quasi-subordination between analytic functions.

Now we recall here the notion of $q$-operator that is, $q$-difference operator that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of $q$-calculus was initiated by Jackson [3], recently Kanas and Răducanu [4] have used the fractional $q$-calculus operators in investigations of certain classes of functions which are analytic in $\mathbb{U}$.

Let $0 < q < 1$. For any non-negative integer $n$, the $q$-integer number $n$ is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}, \quad [0]_q = 0. \quad (1.2)$$

In general, we will denote

$$[x]_q = \frac{1 - q^x}{1 - q}$$

for a non-integer number $x$. Also the $q$-number shifted factorial is defined by

$$[n]_q! = [n]_q[n-1]_q[\ldots][2]_q[1]_q, \quad [0]_q! = 1. \quad (1.3)$$

Clearly,

$$\lim_{q \to 1^-} [n]_q = n \quad \text{and} \quad \lim_{q \to 1^-} [n]_q! = n!.$$
For $0 < q < 1$, the Jackson’s $q$-derivative operator (or $q$-difference operator) of a function $f \in A$ given by (1.1) defined as follows [3]:

$$
\mathcal{D}_q f(z) = \begin{cases} 
\frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0 \\
 f'(0) & \text{for } z = 0
\end{cases},
$$

(1.4)

$\mathcal{D}_q^0 f(z) = f(z)$, and $\mathcal{D}_q^m f(z) = \mathcal{D}_q(\mathcal{D}_q^{m-1} f(z))$, $m \in \mathbb{N} = \{1, 2, \ldots\}$. From (1.4), we have

$$
\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad (z \in U),
$$

(1.5)

where $[n]_q$ is given by (1.2).

For a function $\psi(z) = z^n$, we obtain

$$
\mathcal{D}_q \psi(z) = \mathcal{D}_q z^n = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1}
$$

and

$$
\lim_{q \to 1^-} \mathcal{D}_q \psi(z) = \lim_{q \to 1^-} ([n]_q z^{n-1}) = n z^{n-1} = \psi'(z),
$$

where $\psi'$ is the ordinary derivative.

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. The $q$-generalized Pochhammer symbol is defined by

$$
[t; n]_q = [t]_q [t+1]_q [t+2]_q \ldots [t+n-1]_q
$$

(1.6)

and for $t > 0$ the $q$-gamma function is defined by

$$
\Gamma_q(t+1) = [t]_q \Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1.
$$

(1.7)

Using the $q$-difference operator, Kannas and Raducanu [4] defined the Ruscheweyh $q$-differential operator as below: For $f \in A$,

$$
\mathcal{R}^\delta_q f(z) = f(z) * F_{q, \delta+1}(z) \quad (\delta > -1, z \in U),
$$

(1.8)

where

$$
F_{q, \delta+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n + \delta)}{(n - 1)! \Gamma_q(1 + \delta)} z^n = z + \sum_{n=2}^{\infty} \frac{[\delta + 1; n-1]_q}{[n-1]_q!} a_n z^n.
$$

(1.9)

Making use of (1.8) and (1.9), we have

$$
\mathcal{R}^\delta_q f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n + \delta)}{(n - 1)! \Gamma_q(1 + \delta)} a_n z^n \quad (z \in U).
$$

(1.10)
\[ R_q^0 f(z) = f(z), \]
\[ R_q^1 f(z) = z D_q f(z), \]
\[ R_q^m f(z) = \frac{z D_q^m (z^{m-1} f(z))}{[m]_q} \quad (m \in \mathbb{N}). \]

Also we have
\[ D_q (R_q^\delta f(z)) = 1 + \sum_{n=2}^\infty \Theta_n(q, \delta) a_n z^{n-1}, \quad (1.11) \]
where
\[ \Theta_n := \Theta_n(q, \delta) = \frac{[n]_q \Gamma_q(n + \delta)}{[n-1]_q \Gamma_q(1 + \delta)}. \quad (1.12) \]

It is easy to check that
\[ z D_q (F_{q, \delta+1}(z)) = \left(1 + \frac{[\delta]_q}{q^\delta}\right) F_{q, \delta+2}(z) - \frac{[\delta]_q}{q^\delta} F_{q, \delta+1}(z) \quad (z \in \mathbb{U}). \quad (1.13) \]

Making use of (1.8)-(1.13) and the properties of Hadamard product, we obtain the following equality
\[ z D_q (R_q^\delta f(z)) = \left(1 + \frac{[\delta]_q}{q^\delta}\right) R_q^{1+\delta} f(z) - \frac{[\delta]_q}{q^\delta} R_q^\delta f(z) \quad (z \in \mathbb{U}). \quad (1.14) \]

From (1.10), we note that
\[ \lim_{q \to 1^-} F_{q, \delta+1}(z) = \frac{z}{(1 - z)^{\delta+1}}, \quad \lim_{q \to 1^-} R_q^\delta f(z) = f(z) * \frac{z}{(1 - z)^{\delta+1}}. \]

Thus, when \( q \to 1^- \) we can say that Ruscheweyh \( q \)-differential operator reduces to the differential operator defined by Ruscheweyh [9] and (1.14) gives the well-known recurrent formula for Ruscheweyh differential operator.

Majorization problems for the class \( S^* = S^*(0) \) had been investigated by MacGregor [5], further Altintas et al. [1] investigated a majorization problem for \( S(\gamma) \) the class of starlike functions of complex order \( \gamma \ (\gamma \in \mathbb{C} \setminus \{0\}) \), and Goyal and Goswami [2] generalized these results for the class of analytic functions involving fractional operator. Very lately, Tang and Deng [12] considered majorization properties for multivalent analytic functions related to the Srivastava-Khairnar-More operator and exponential function.

In this paper, using Ruscheweyh \( q \)-differential operator defined by (1.10) and motivated by recent works of [8], we define a new subclass of uniformly starlike functions associated with \( q \)-calculus operator, which are subordinate to exponential function, and investigate a majorization problem. Further we point out some special cases of our result.
Definition 1.1. A function \( f \in \mathcal{A} \) is said to be in the class \( \mathcal{RS}_q^\delta(\beta, e^z) \), if and only if

\[
\left[ \frac{z \mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{\mathcal{R}_q^\delta f(z)} - \beta \left| \frac{z \mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{\mathcal{R}_q^\delta f(z)} - 1 \right| \right] < e^z, \tag{1.15}
\]

where \( \delta > -1, \beta > 0 \) and \( z \in \mathbb{U} \).

For \( \beta = 0 \) we have \( \mathcal{RS}_q^\delta(\beta, e^z) \equiv \mathcal{RS}_q^\delta(e^z) \):

\[
\frac{z \mathcal{D}_q(\mathcal{R}_q^\delta f(z))}{\mathcal{R}_q^\delta f(z)} < e^z
\]

where \( \delta > -1, \beta > 0 \) and \( z \in \mathbb{U} \).

Further by taking \( q \to 1^- \) and \( \delta = 0 \) we have \( \mathcal{RS}_q^\delta(e^z) \equiv \mathcal{S}^*(e^z) \):

\[
\frac{zf'(z)}{f(z)} < e^z \quad (z \in \mathbb{U}).
\]

2. A majorization problem for the class \( \mathcal{RS}_q^\delta(\beta, e^z) \)

We state the following \( q \)-analogue of the result given by Nehari (cf. [7]) and Selvakumaran et al. [10].

Lemma 2.1. If the function \( \phi(z) \) is analytic and \( |\phi(z)| < 1 \) in \( \mathbb{U} \), then

\[
|\mathcal{D}_q\phi(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}. \tag{2.1}
\]

Theorem 2.2. Let the function \( f \in \mathcal{A} \) and suppose that \( g \in \mathcal{RS}_q^\delta(\beta, e^z) \). If \( \mathcal{R}_q^\delta f(z) \) is majorized by \( \mathcal{R}_q^\delta g \) in \( \mathbb{U} \), then

\[
|\mathcal{R}_q^{\delta+1}f(z)| \leq |\mathcal{R}_q^{\delta+1}g(z)| \quad (|z| \leq r_1), \tag{2.2}
\]

where \( r_1 = r_1(\delta, \beta) \), is the smallest positive root of the equation

\[
r^2 q^\delta e^r - r^2 ([\delta]_q - ([\delta]_q + q^\delta)\beta) - q^\delta e^r - 2r q^\delta (1 + \beta) + ([\delta]_q - ([\delta]_q + q^\delta)\beta) = 0, \tag{2.3}
\]

where \([\delta]_q > ([\delta]_q + q^\delta)\beta + q^\delta e \) and \( \beta \geq 0 \).

Proof. Since \( g \in \mathcal{RS}_q^\delta(\beta, e^z) \), we find from (1.15) that

\[
\left[ \left( \frac{z \mathcal{D}_q(\mathcal{R}_q^\delta g(z))}{\mathcal{R}_q^\delta g(z)} \right) - \beta \left| \frac{z \mathcal{D}_q(\mathcal{R}_q^\delta g(z))}{\mathcal{R}_q^\delta g(z)} - 1 \right| \right] = e^{w(z)}, \tag{2.4}
\]
where \( w(z) = c_1z + c_2z^2 + c_3z^3 + \cdots \) is analytic in \( U \), with \( w(0) = 0 \) and \( |w(z)| \leq |z| \) for all \( z \in U \). Letting
\[
\varpi = \frac{zD_q(R^\delta_q g(z))}{R^\delta_q g(z)}
\]
in (2.4), we have
\[
\varpi - \beta |\varpi - 1| = e^{u(z)}
\]
which implies
\[
\varpi = \frac{e^{u(z)} - \beta e^{-i\varphi}}{1 - \beta e^{-i\varphi}}.
\]
This is, from (2.4), we get
\[
\frac{zD_q(R^\delta_q g(z))}{R^\delta_q g(z)} = \frac{e^{u(z)} - \beta e^{-i\varphi}}{1 - \beta e^{-i\varphi}}. \tag{2.5}
\]
Now, by applying the relation (1.14) in (2.5), we get
\[
\frac{R^\delta_{q+1} g(z)}{R^\delta_q g(z)} = \frac{[\delta_q - ([\delta_q + q^\delta] \beta - q^\delta e^{-i\varphi})]}{([\delta_q + q^\delta] (1 - \beta e^{-i\varphi})}
\]
which yields that
\[
\left| R^\delta_q g(z) \right| \leq \frac{([\delta_q + q^\delta] (1 + \beta)}{[\delta_q - ([\delta_q + q^\delta] \beta - q^\delta e^{-i\varphi})} \left| R^\delta_{q+1} g(z) \right|. \tag{2.7}
\]
Since \( R^\delta_q f \) is majorized by \( R^\delta_q g(z) \) in \( U \), we have
\[
R^\delta_q f(z) = \phi(z)R^\delta_q g(z).
\]
By applying \( q \)-differentiation with respect to \( z \), we get
\[
zD_q(R^\delta_q f(z)) = zD_q(\phi(z))R^\delta_q g(z) + z\phi(z)D_q(R^\delta_q g(z)). \tag{2.8}
\]
Noting the fact that Schwarz function \( \phi(z) \) satisfies the \( q \)-analogue of the result given by Nehari (cf. [7]) proved in Lemma 2.1,
\[
\left| D_q \phi(z) \right| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \tag{2.9}
\]
and using (1.14), (2.7) and (2.9) in (2.8), we have
\[
\left| R^\delta_{q+1} f(z) \right| \leq \left| \phi(z) \right| + \frac{\left| 1 - |\phi(z)|^2 \right|}{1 - |z|^2} \left| \phi(z) \right| \frac{[zq^\delta(1 + \beta)}{[\delta_q - ([\delta_q + q^\delta] \beta - q^\delta e^{-i\varphi})} \left| R^\delta_{q+1} g(z) \right|.
\]
Setting \( |z| = r \) and \( |\phi(z)| = \rho \) (\( 0 \leq \rho \leq 1 \)), the above inequality leads us to the inequality
\[
\left| R^\delta_{q+1} f(z) \right| \leq \left( \rho + \frac{1 - \rho^2}{1 - r^2} \right) \frac{r q^\delta(1 + \beta)}{[\delta_q - ([\delta_q + q^\delta] \beta - q^\delta e^{-i\varphi})} \left| R^\delta_{q+1} g(z) \right|. \tag{2.10}
\]
That is,

$$|\mathcal{R}_q^{δ+1}f(z)| \leq Θ_1(r, ρ)|\mathcal{R}_q^{δ+1}g(z)|,$$

where the function $Θ_1(r, ρ)$ is given by

$$Θ_1(r, ρ) = ρ + \frac{r(1 - ρ^2)q^δ(1 + β)}{(1 - r^2)\{[δ]_q - ([δ]_q + q^{δ}β - q^{δ}e^r)\}}.$$

In order to determine the bound of $Θ_1(r, ρ)$, we have to choose

$$r_1 = \max\{r ∈ [0, 1) : Θ_1(r, ρ) ≤ 1, ρ ∈ [0, 1]\} = \max\{r ∈ [0, 1) : Θ_2(r, ρ) ≥ 0, ρ ∈ [0, 1]\},$$

where

$$Θ_2(r, ρ) = (1 - r^2)\{[δ]_q - ([δ]_q + q^{δ}β - q^{δ}e^r)\} - r(1 + ρ)q^{δ}(1 + β).$$

Obviously, for $ρ = 1$, the function $Θ_2(r, ρ)$ takes its minimum value, namely

$$\min\{Θ_2(r, ρ) : ρ ∈ [0, 1]\} = Θ_2(r, 1) = Θ_2(r),$$

where

$$Θ_2(r) = (1 - r^2)\{[δ]_q - ([δ]_q + q^{δ}β - q^{δ}e^r)\} - 2rq^{δ}(1 + β).$$

Furthermore, if $Θ_2(0) = [δ]_q > ([δ]_q + q^{δ}β + q^{δ}e) ρ$ and $Θ_2(1) = -2q^{δ}(1 + β) < 0$, then there exists $r_1$ such that $Θ_2(r) ≥ 0$ for all $r ∈ [0, r_1]$, where $r_1 = r_1(δ, β)$, the smallest positive root of the equation (2.3). This completes the proof. □

Putting $β = 0$ and $ρ = 1$ in Theorem 2.2, we have the following corollary:

**Corollary 2.3.** Let the function $f ∈ A$ and suppose that $g ∈ \mathcal{RS}_q^{δ}(e^z)$. If $\mathcal{R}_q^{δ}f$ is majorized by $\mathcal{R}_q^{δ}g$ in $U$, then

$$|\mathcal{R}_q^{δ+1}f(z)| ≤ |\mathcal{R}_q^{δ+1}g(z)|, \quad |z| ≤ r_2,$$

(2.11)

where $r_2 = r_2(δ)$, is the smallest positive root of the equation

$$r^2q^{δ}e^r - r^{δ}q^{δ} - q^{δ}e^r - 2rq^{δ} + [δ]_q = 0.$$

(2.12)

For $β = 0, q → 1^−$ and $δ = 0$, Corollary 2.3 reduces to the following result:

**Corollary 2.4.** Let the function $f ∈ A$ be analytic and univalent in the open unit disk $U$ and suppose that $g ∈ \mathcal{S}_0^{δ}(e^z)$. If $f$ is majorized by $g$ in $U$, then

$$|f′(z)| ≤ |g′(z)|, \quad |z| ≤ r_3,$$

where $r_3$ is the smallest positive root of $r^2e^r - 2r - e^r = 0.$
3. A majorization problem for the class $\mathcal{R}(\mu, \tau)$

Due to Alitintas et al. [1], we recall the definition of the function class $\mathcal{R}(\mu, \tau)$, the class of functions $h$ of the form

$$h(z) = 1 - \sum_{n=1}^{\infty} c_n z^n \quad (c_n \geq 0 ; z \in \mathbb{U}),$$

(3.1)

which are analytic in $\mathbb{U}$ and satisfy the inequality

$$|h(z) + \mu z h'(z) - 1| < |\tau| \quad (\tau \in \mathbb{C} \setminus \{0\}; \Re(\mu) \geq 0).$$

Further we recall the following lemmas, which will be required in our investigation of the majorization problem for the class $\mathcal{R}(\mu, \tau)$.

**Lemma 3.1.** ([1]) If the function $h$ defined by (3.1) is in the class $\mathcal{R}(\mu, \tau)$, then

$$\sum_{n=1}^{\infty} c_n \leq \frac{|\tau|}{1 + \Re(\mu)}.$$ (3.2)

**Lemma 3.2.** ([1]) If the function $h$ defined by is in the class $\mathcal{R}(\mu, \tau)$, then

$$1 - \frac{|\tau|}{1 + \Re(\mu)}|z| \leq |h(z)| \leq 1 + \frac{|\tau|}{1 + \Re(\mu)}|z| \quad (z \in \mathbb{U}).$$ (3.3)

**Theorem 3.3.** Let the function $f$ and $g$ be analytic in $\mathbb{U}$ and suppose that the function $g$ is normalized and also satisfies the following inclusion property:

$$\left(\frac{z \mathcal{D}_q(\mathcal{R}^q g(z))}{\mathcal{R}^q g(z)}\right) \in \mathcal{R}(\mu, \tau).$$

If $\mathcal{R}^q f$ is majorized by $\mathcal{R}^q g$ in $\mathbb{U}$, then

$$|\mathcal{R}^q+1 f(z)| \leq |\mathcal{R}^q+1 g(z)| \quad (|z| \leq r_4),$$

(3.4)

where $r_4 = r_4(\mu, \tau, \delta)$ is the root of the cubic equation

$$q^{\delta}|\tau|r^3 - \{(q^{\delta} - [\delta]_q)(1 + \Re(\mu)) - 2|\tau|\}r^2$$

$$- [2(1 + \Re(\mu)) + q^{\delta}|\tau||r + (q^{\delta} - [\delta]_q)[1 + \Re(\mu)] = 0$$

(3.5)

which lies in the interval $(0, 1)$ and $(q^{\delta} - [\delta]_q)(1 + \Re(\mu)) > 0$.

**Proof.** For an appropriately normalized analytic function $g$ satisfying the inclusion property, we find from the assertion of Lemma 3.2 that

$$\left|\frac{z \mathcal{D}_q(\mathcal{R}^q g(z))}{\mathcal{R}^q g(z)}\right| \geq 1 - \frac{|\tau|}{1 + \Re(\mu)}r \quad (|z| = r, \ 0 < r < 1)$$

(3.6)
or, equivalently, that
\[
|R_q^\delta g(z)| \leq \frac{(q^\delta + |\delta|)(1 + \Re(\mu) - |\tau|r)}{(q^\delta - |\delta|)(1 + \Re(\mu)) - q^\delta |\tau|r} |(R_q^{\delta+1} g(z))| \quad (|z| = r, \ 0 < r < 1).
\] (3.7)

Since
\[
R_q^\delta f(z) \ll R_q^\delta g(z) \quad (z \in \mathbb{U}),
\]
there exists an analytic function \( \phi \) such that
\[
R_q^\delta f(z) = \phi(z) R_q^\delta g(z) \quad \text{and} \quad |\phi(z)| < 1.
\]

By applying \( q \)-differentiation with respect to \( z \), we get
\[
z \mathcal{D}_q (R_q^\delta f(z)) = z \mathcal{D}_q (\phi(z)) R_q^\delta g(z) + \phi(z) z \mathcal{D}_q (R_q^\delta g(z)). \] (3.8)

Thus in view of (3.7) and using (1.14), just as in the proof of Theorem 2.2, we have
\[
|\mathcal{D}_q (\phi(z))| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \quad (z \in \mathbb{U})
\]
and
\[
|\mathcal{D}_q (R_q^{\delta+1} f(z))| \leq \left( |\phi(z)| + \frac{1 - |\phi(z)|^2}{1 - |z|^2} \cdot \frac{(1 + \Re(\mu) - |\tau|r)|z|}{(q^\delta - |\delta|)(1 + \Re(\mu)) - q^\delta |\tau|r} \right) |\mathcal{D}_q (R_q^\delta g(z))|,
\]
\(|z| = r, \ 0 < r < 1\). That is,
\[
|R_q^{\delta+1} f(z)| \leq \left( |\phi(z)| + \frac{1 - |\phi(z)|^2}{1 - |z|^2} \cdot \frac{(1 + \Re(\mu) - |\tau|r)r}{(q^\delta - |\delta|)(1 + \Re(\mu)) - q^\delta |\tau|r} \right)
\times R_q^{\delta+1} g(z),
\] (3.9)
where \(|z| = r, \ 0 < r < 1\). We set \(|\phi(z)| = \rho\) and the function \( \Lambda_1(\rho, r) \) defined by
\[
\Lambda_1(\rho, r) = \rho + \frac{1 - \rho^2}{1 - r^2} \cdot \frac{(1 + \Re(\mu) - |\tau|r)r}{(q^\delta - |\delta|)(1 + \Re(\mu)) - q^\delta |\tau|r}. \] (3.10)

In order to determine the bound of \( \Lambda(\rho, r) \), we have to choose
\[
r_1 = \max\{r \in [0, 1] : \Lambda_1(\rho, r) \leq 1, \ \rho \in [0, 1]\}
= \max\{r \in [0, 1] : \Lambda_2(\rho, r) \geq 0, \ \rho \in [0, 1]\},
\]
where, for \( 0 \leq \rho \leq 1\).
\[
\Lambda_2(r, \rho) = (1 - r^2)\{(q^\delta - |\delta|)(1 + \Re(\mu)) - q^\delta |\tau|r\} - r(1 + \rho)(1 + \Re(\mu) - |\tau|r).
\]

Obviously, for \( \rho = 1\), the function \( \Lambda_2(r, \rho) \) takes its minimum value, namely
\[
\min\{\Lambda_2(r, \rho) : \rho \in [0, 1]\} = \Lambda_2(r, 1) = \Lambda_2(r),
\]
where
\[ \Lambda_2(r) = (1 - r^2)\{(q^\delta - [\delta]_q)(1 + \Re(\mu)) - q^\delta|\tau|r\} - 2r(1 + \Re(\mu) - |\tau|r). \]
Furthermore, if \( \Lambda_2(0) = (q^\delta - [\delta]_q)(1 + \Re(\mu)) > 0 \) and \( \Lambda_2(1) = -2(1 + \Re(\mu) - |\tau|) < 0 \), then there exists \( r_4 \) such that \( \Lambda_2(r) \geq 0 \) for all \( r \in [0, r_4] \), where \( r_4 = r_4(\tau, \mu, \delta) \), the smallest positive root of the equation (3.5) which completes the proof of Theorem 3.3. \( \square \)

**Remark 3.4.** Specializing the parameters \( \delta, \beta \) in (1.15) one can define the various other interesting subclasses of \( \mathcal{RS}_q^{\delta}(\beta, e^z) \), involving \( q \)-calculus operator and one can easily derive the result as in Theorem 2.2. Further as mentioned in [11] we can define new subclasses \( \mathcal{RS}_q^{\delta}(\beta, 1 + \sin z) \), \( \mathcal{RS}_q^{\delta}(\beta, \cos z) \), and \( \mathcal{RS}_q^{\delta}(\beta, z + \sqrt{1 + z^2}) \), and investigate a majorization problem for these classes.

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