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SOLUTION SETS OF SECOND-ORDER CONE LINEAR FRACTIONAL OPTIMIZATION PROBLEMS

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Abstract. We characterize the solution set for a second-order cone linear fractional optimization problem (P). We present sequential Lagrange multiplier characterizations of the solution set for the problem (P) in terms of sequential Lagrange multipliers of a known solution of (P).

1. Introduction and Preliminaries

Jeyakumar et al. [4] proved the sequential Lagrange multiplier optimality conditions for convex optimization problem, which held without any constraint qualification and which were expressed by sequences. Such optimality conditions have been studied for many kinds of convex optimization problems. In particular, Kim et al. [2] investigated sequential Lagrange multiplier optimality conditions for a semidefinite linear fractional optimization problem, which hold without any constraint qualification. Kim et al. [3] also obtained sequential Lagrange multiplier optimality conditions for a second-order cone linear

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fractional optimization problem, which hold without any constraint qualification.

Optimization problems often have multiple solutions. Mangasarian [13] presented simple and elegant characterizations of the solution set for a convex optimization problem over a convex set when one solution is known. These characterizations have been extended to various classes of optimization problems [5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17]. In particular, Jeyakumar et al. [5] gave Lagrange multiplier characterizations of the solution set of a convex optimization problem involving convex inequality constraints in terms of Lagrange multipliers of a known solution.

In this paper, we present sequential Lagrange multiplier characterizations of the solution set of a second-order cone linear fractional optimization problem in terms of sequential Lagrange multipliers of a known solution.

Recently, second-order cone optimization problems have been intensively studied [1].

In this paper, we consider the following second-order cone optimization problem:

$$\text{(P)} \qquad \text{Minimize} \quad \frac{c^T x+\alpha}{d^T x+\beta}$$
$$\text{subject to} \quad x \in K,\ a_i^T x = b_i, i = 1,\cdots,m,$$

where $c, d \in \mathbb{R}^n$, α, β are given real numbers, $a_i \in \mathbb{R}^n, i = 1, \cdots, m$ and $b_i, i = 1, \cdots, m$ are given real numbers, $K = \{x = (x_1, x_2, \cdots, x_n) \mid x_1 \geqq \sqrt{x_2^2 + x_3^2 + \cdots + x_n^2}\}$.

Let $F = \{x \in K \mid a_i^T x = b_i, i = 1, \cdots, m\}$.

2. Optimality theorems

The authors [3] established the following sequential Lagrange multiplier optimality theorem for (P), which holds without any constraint qualification;

Theorem 2.1. ([3]) *Let $\bar{x} \in F$. Then $\bar{x}$ is an optimal solution of (P) if and only if there exist $\lambda_i^l \in \mathbb{R}, i = 1, \cdots, m$ and $v_l \in K$ such that*

$$c - q(\bar{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^m \lambda_i^l a_i - v_l\Big] = 0$$

and

$$\lim_{l\to\infty} v_l^T \bar{x} = 0,$$

where $q(\bar{x}) = \frac{c^T\bar{x}+\alpha}{d^T\bar{x}+\beta}$.

The closedness of the set $\bigcup_{\lambda_i \in \mathbb{R}} \sum_{i=1}^{m} \lambda_i^l (a_i, b_i) + (-K) \times \mathbb{R}^+$ can be used as a constraint qualification for the optimal solution of (P) as in the following theorem [3];

Theorem 2.2. ([3]) *Let $\bar{x} \in F$. Suppose that $\bigcup_{\lambda_i \in \mathbb{R}} \sum_{i=1}^{m} \lambda_i (a_i, b_i) + (-K) \times \mathbb{R}^+$ is closed in $\mathbb{R}^n \times \mathbb{R}$. Then the following are equivalent:*

(i) *$\bar{x}$ is an optimal solution of (P);*

(ii) *there exist $y_i \in \mathbb{R}, i = 1, \cdots, m$ such that*

$$\sum_{i=1}^{m} y_i a_i - \frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta} d + c \in K$$

and

$$-\frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta} \beta - b^T y \geqq -\alpha;$$

(iii) *there exist $y_i \in \mathbb{R}, i = 1, \cdots, m$ such that*

$$\sum_{i=1}^{m} y_i a_i - \frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta} d + c \in K$$

and

$$\Big(\sum_{i=1}^{m} y_i a_i - \frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta} d + c\Big)^T \bar{x} = 0.$$

3. Characterizations of solution sets

Let $\bar{S}$ be the set of solutions of (P). Let $\bar{x} \in \bar{S}$. Then by Theorem 2.1, there exist a sequence $\{y_i^l\}$ in $\mathbb{R}$, $i = 1, \cdots, m$ and a sequence $\{v_l\}$ in K such that

$$c - q(\bar{x}) d + \lim_{l \to \infty} \Big[\sum_{i=1}^{m} y_i^l a_i - v_l\Big] = 0$$

and

$$\lim_{l \to \infty} v_l^T \bar{x} = 0,$$

where $q(\bar{x}) = \frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta}$.

By using the above sequences $\{y_i^l\}$ and $\{v_l\}$, we can characterize the solution set $\bar{S}$ as follows:

Theorem 3.1. *The set $\bar{S}$ of solutions of (P) is as follows:*

$$\bar{S} = \{\widetilde{x} \in F \mid c - q(\widetilde{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l a_i - v_l\Big] = 0, \lim_{l\to\infty} v_l^T \widetilde{x} = 0\},$$

where $q(\widetilde{x}) = \frac{c^T\widetilde{x}+\alpha}{d^T\widetilde{x}+\beta}$.

Proof. Let $\widetilde{x} \in \bar{S}$ be any fixed. Then $q(\bar{x}) = q(\widetilde{x})$ and so $(c - q(\bar{x})d)^T\bar{x} = (c - q(\widetilde{x})d)^T\widetilde{x}$. Since $c - q(\bar{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l a_i - v_l\Big] = 0$,

$$(c - q(\bar{x})d)^T\bar{x} + \lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l (a_i^T\bar{x}) - v_l^T\bar{x}\Big] = 0$$

and

$$(c - q(\widetilde{x})d)^T\widetilde{x} + \lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l (a_i^T\widetilde{x}) - v_l^T\widetilde{x}\Big] = 0.$$

Hence, we have $\lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l (a_i^T\bar{x}) - v_l^T\bar{x}\Big] = \lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l (a_i^T\widetilde{x}) - v_l^T\widetilde{x}\Big]$.
Since $\lim_{l\to\infty} v_l^T\bar{x} = 0$,

$$\lim_{l\to\infty}\sum_{i=1}^{m} y_i^l (a_i^T\bar{x}) = \lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l (a_i^T\widetilde{x}) - v_l^T\widetilde{x}\Big].$$

Since $\bar{x} \in \bar{S}$ and $\widetilde{x} \in \bar{S}$,

$$\lim_{l\to\infty}\sum_{i=1}^{m} y_i^l b_i = \lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l b_i - v_l^T\widetilde{x}\Big].$$

Thus $\lim_{l\to\infty} v_l^T\widetilde{x} = 0$. Hence, we have

$$\bar{S} \subset \{\widetilde{x} \in F \mid c - q(\widetilde{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l a_i - v_l\Big] = 0,\ \lim_{l\to\infty} v_l^T \widetilde{x} = 0\}.$$

The converse is true by Theorem 2.1. Consequently,

$$\bar{S} = \{\widetilde{x} \in F \mid c - q(\widetilde{x})d + \lim_{l\to\infty}\Big[\sum_{i=1}^{m} y_i^l a_i - v_l\Big] = 0,\ \lim_{l\to\infty} v_l^T \widetilde{x} = 0\}.$$

□

When $d = 0,\ \alpha = 0,\ \beta = 1$, (P) becomes the following second-order cone program (SOCP):

$$\text{(SOCP)} \qquad \text{Minimize} \quad c^T x$$
$$\text{subject to} \quad x \in K,\ a_i^T x = b_i, i = 1, \cdots, m.$$

Let $\tilde{\tilde{S}}$ be the set of solutions of (SOCP) and let $\bar{x} \in \tilde{\tilde{S}}$. Then, by Theorem 2.1 there exist a sequence $\{y_i^l\}$ in $\mathbb{R}$, $i = 1, \cdots, m$ and a sequence $\{v_l\}$ in K such that

$$c + \lim_{l \to \infty} \Big[\sum_{i=1}^{m} y_i^l a_i - v_l \Big] = 0$$

and

$$\lim_{l \to \infty} v_l^T \bar{x} = 0.$$

By using the above sequences $\{y_i^l\}$ and $\{v_l\}$, we have the following theorem from Theorem 3.1:

Theorem 3.2. *The set $\tilde{\tilde{S}}$ of soluttions of (SOCP) is as follows:*

$$\tilde{\tilde{S}} = \{\widetilde{x} \in F \mid \lim_{l \to \infty} v_l^T \widetilde{x} = 0\}.$$

Suppose that $\bigcup_{\lambda_i \in \mathbb{R}} \sum_{i=1}^{m} \lambda_i (a_i, b_i) + (-K) \times \mathbb{R}^+$ is closed in $\mathbb{R}^n \times \mathbb{R}$. Let $\bar{S}$ be the set of solutions of (P) and let $\bar{x} \in \bar{S}$. Then by Theorem 2.2, there exist $y_i \in \mathbb{R}, i = 1, \cdots, m$ and $v \in K$ such that

$$\sum_{i=1}^{m} y_i a_i - \frac{c^T \bar{x} + \alpha}{d^T \bar{x} + \beta} d + c - v = 0 \quad \text{and} \tag{3.1}$$

$$v^T \bar{x} = 0. \tag{3.2}$$

By using the above y_i and v, we can characterize the solution set $\bar{S}$ as follows;

Theorem 3.3. *We have the solution set $\bar{S}$:*

$$\bar{S} = \{\widetilde{x} \in F \mid c - q(\widetilde{x}) d + \sum_{i=1}^{m} y_i a_i - v = 0, \quad v^T \widetilde{x} = 0\},$$

where $q(\widetilde{x}) = \frac{c^T \widetilde{x} + \alpha}{d^T \widetilde{x} + \beta}$.

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