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LOCAL EXISTENCE AND EXPONENTIAL DECAY OF SOLUTIONS FOR A NONLINEAR PSEUDOPARABOLIC EQUATION WITH VISCOELASTIC TERM

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Abstract. In this paper, we investigate an initial boundary value problem for a nonlinear pseudoparabolic equation. At first, by applying the Faedo-Galerkin, we prove local existence and uniqueness results. Next, by constructing Lyapunov functional, we establish a sufficient condition to obtain the global existence and exponential decay of weak solutions.

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1. Introduction

In this paper, we consider the following Neumann-Dirichlet problem for a nonlinear pseudoparabolic equation

$$\begin{cases} u_t + Au_t + Au - \int_0^t g(t-s)Au(s)ds = f\left(x,t,u,u_x,u_t,u_{xt}\right), \\ \qquad\qquad\qquad\qquad (1<x<R,\, t>0), \\ u_x(1,t) = u(R,t) = 0, \\ u(x,0) = \tilde{u}_0(x), \end{cases} \tag{1.1}$$

where $R > 1$ is given constant and f, g, $\tilde{u}_0$ are given functions satisfying conditions specified later; $-Au \equiv u_{xx} + \frac{1}{x}u_x$ with $u = u(x,t)$ is the unknown function.

The pseudoparabolic equation

$$u_t - u_{xxt} = F\left(x,t,u,u_x,u_{xx},u_{xt}\right),\ 0<x<1,\ t>0 \tag{1.2}$$

with the initial condition $u(x,0) = \tilde{u}_0(x)$ and with the difrerent boundary conditions, has been extensively studied by many authors, see for example [1], [6]-[11], [15], [21], [30]-[32], [35], [36] among others and the references given therein.

In these works, many results about existence, asymptotic behavior, and decay of solutions were obtained.

Equations of type (1.2) with a one time derivative appearing in the highest order term are called pseudoparabolic or Sobolev equations, and arise in many areas of mathematics and physics. Mathematical study of pseudoparabolic equations goes back to works of Showalter in the seventies, since then, numerous of interesting results about linear and nonlinear pseudoparabolic equations have been obtained. We also refer to [31] for asymptotic behavior and to [32] for nonlinear problems.

An important special case of the model is the Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t + u_x + uu_x - \nu u_{xx} - \alpha^2 u_{xxt} = 0, \tag{1.3}$$

it was studied by Amick et al. in [1] with $\nu > 0$, $\alpha = 1$, $x \in \mathbb{R}$, $t \geq 0$, in which solution of (1.3) with initial data in $L^1 \cap H^2$ decays to zero in L^2 norm as $t \to +\infty$. With $\nu > 0$, $\alpha > 0$, $x \in [0,1]$, $t \geq 0$, the model has the form (1.3) was also investigated earlier by Bona and Dougalis [7], where uniqueness, global existence and continuous dependence of solutions on initial and boundary data were established and the solutions were shown to depend continuously on $\nu \geq 0$ and on $\alpha > 0$.

The Benjamin-Bona-Mahony (BBM) equation was introduced in [6], in 1972, as a model for describing long - wave behavior. Since then, the initial boundary value problems for various generalized BBM equations have been studied.

On the other hand, many people have studied the large time behaviors of solutions to the initial value problems for various generalized BBM equations, such as BBMB equations with a Burgers-type dissipative term appended, see [35].

Medeiros and Miranda [21] studied another special case, namely

$$u_t + f(u)_x - u_{xxt} = g(x,t), \tag{1.4}$$

where $u = u(x,t)$, $0 < x < 1$, and $t \geq 0$ is the time. They proved existence, uniqueness of solutions for f in C^1 and regularity in the case $f(s) = s^2/2$.

In [8], Bouziani studied the solvability of solutions for the nonlinear pseudoparabolic equation

$$u_t - \frac{\partial}{\partial x}\left(a(x,t)u_x\right) - \eta \frac{\partial^2}{\partial t \partial x}\left(a(x,t)u_x\right) = f\left(x,t,u,u_x\right), \ \alpha < x < \beta, \ 0 < t < T, \tag{1.5}$$

subject to the initial condition

$$u(x,0) = u_0(x), \ \alpha \leq x \leq \beta, \tag{1.6}$$

and the nonlocal boundary condition

$$u(\alpha,t) = \int_\alpha^\beta u(x,t)dx = 0, \tag{1.7}$$

with $u_0(\alpha) = \int_\alpha^\beta u_0(x)dx = 0$.

In [15], Dai and Huang considered the well-poseness and solvability of solutions for the nonlinear pseudoparabolic equation

$$u_t + \left(a(x,t)u_{xt}\right)_x = F\left(x,t,u,u_x,u_{xx}\right), \ \alpha < x < \beta, \ 0 < t < T, \tag{1.8}$$

with the initial condition (1.6) and the nonlocal moment boundary conditions

$$\int_\alpha^\beta u(x,t)dx = \int_\alpha^\beta xu(x,t)dx = 0, \ 0 \leq t \leq T. \tag{1.9}$$

In [30], Shang and Guo proved the existence, uniqueness, and regularities of the global strong solution and gave some conditions of the nonexistence of global solution of the nonlinear pseudoparabolic equation with Volterra integral term

$$u_t - f(u)_{xx} - u_{xxt} - \int_0^t \lambda(t-s)\left(\sigma\left(u(x,s),u_x(x,s)\right)\right)_x ds = f\left(x,t,u,u_x\right),$$
$$0 < x < 1, \ t > 0. \tag{1.10}$$

In [11], the authors Cao et al. established the global existence of classical solutions and the blow-up in a finite time for the viscous diffusion equation of higher order

$$\begin{cases} u_t + k_1 u_{xxxx} - k_2 u_{txx} - (\Phi(u_x))_x + A(u) = 0, \; 0 < x < 1, \; t > 0, \\ u(0,t) = u(1,t) = u_{xx}(0,t) = u_{xx}(1,t) = 0, \; t > 0, \\ u(x,0) = u_0(x), \; 0 < x < 1, \end{cases} \tag{1.11}$$

where $k_1 > 0$, $k_2 > 0$ and $\Phi(s)$, $A(s)$ are appropriately smooth, $u_0 \in C^{1+\beta}$ with $\beta \in (0,1)$ and $u_0(0) = u_0(1) = u_{0xx}(0) = u_{0xx}(1) = 0$.

In [36], Zhu et al. studied the exponent decay behavior and blow-up phenomena of weak solutions for a class of pseudoparabolic equations with a nonlocal term

$$\begin{cases} u_t - a\Delta u_t - \Delta u + u = bu\Phi_u + u^{p-1}u, \; (x,t) \in \Omega \times (0,+\infty), \\ u = 0, \; (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), \; x \in \Omega, \end{cases} \tag{1.12}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $p \in (1,5)$ and Φ_u is a Newtonian potential

$$\Phi_u(x) = \frac{1}{4\pi} \int_\Omega \frac{u^2(y)}{|x-y|} dy, \;\; x \in \mathbb{R}^3. \tag{1.13}$$

It is also well known that the equation (1.1) arises within frameworks of mathematical models in engineering and physical sciences on third-grade fluid flows, see [2]-[4], [16], [17], [28] and references therein. For example, the following equation of motion for the unsteady flow of third-grade fluid over the rigid plate with porous medium was investigated (see [4])

$$\rho U_t = \mu U_{yy} + \alpha_1 U_{yyt} + 6\beta_3 |U_y|^2 U_{yy} - \frac{\phi}{k}\left[\mu + \alpha_1 \frac{\partial}{\partial t} + 2\beta_3 |U_y|^2\right] U, \tag{1.14}$$

$y > 0$, $t > 0$, where U is the velocity component, ρ is the density, μ is the coefficient of viscosity, α_1 and β_3 are the material constants.

In [16], some unsteady flow problems of a second grade fluid were also considered. The flows are generated by the sudden application of a constant pressure gradient or by the impulsive motion of a boundary. Here, the velocities of the flows are described by the partial differential equations and exact analytic solutions of these differential equations are obtained. Suppose that the second grade fluid is in a circular cylinder and is initially at rest, and the fluid starts suddenly due to the motion of the cylinder parallel to its length. The axis of the cylinder is chosen as the z-axis. Using cylindrical polar coordinates, the governing partial differential equation is

$$\begin{cases} w_t = \left(\nu + \alpha \frac{\partial}{\partial t}\right)\left(w_{rr} + \frac{1}{r} w_r\right) - Nw, \; 0 < r < a, \; t > 0, \\ w(a,t) = W, \; t > 0, \\ w(r,0) = 0, \; 0 \le r < a, \end{cases} \tag{1.15}$$

where $w(r,t)$ is the velocity along the z-axis, ν is the kinematic viscosity, α is the material parameter, and N is the imposed magnetic field. In the boundary and initial conditions, W is the constant velocity at $r=a$ and a is the radius of the cylinder. In [2], Mahmood et al. considered the longitudinal oscillatory motion of second grade fluid between two infinite coaxial circular cylinders, oscillating along their common axis with given constant angular frequencies Ω_1 and Ω_2. Velocity field and associated tangential stress of the motion were determined by using Laplace and Hankel transforms. In order to find exact analytic solutions for the flow of second grade fluid between two longitudinally oscillating cylinders, the following problem was studied (see [2])

$$\begin{cases} v_t = (\mu + \alpha \dfrac{\partial}{\partial t}) \left(v_{rr} + \dfrac{1}{r} v_r \right), \, R_1 < r < R_2, \, t > 0, \\ v(R_1, t) = V_1 \sin(\Omega_1 t), \, v(R_2, t) = V_2 \sin(\Omega_2 t), \\ u(r, 0) = 0, \, R_1 \leq r \leq R_2, \end{cases} \tag{1.16}$$

where $0 < R_1 < R_2$, μ, α, V_2, Ω_1, Ω_2 are positive constants. The solutions obtained have been presented under series form in terms of Bessel functions $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$, $J_2(x)$ and $Y_2(x)$, satisfying the governing equation and all imposed initial and boundary conditions.

Besides, it is well known that pseudo-parabolic equations describe a variety of important physical processes (see [10]), such as the seepage of homogeneous fluids through a fissured rock [5] (where k is a characteristic of the fissured rock, a decrease of k corresponds to a reduction in block dimension and an increase in the degree of fissuring), the unidirectional propagation of nonlinear, dispersive, long waves [6], [34] (where u is typically the amplitude or velocity), and the aggregation of populations [19] (where u represents the population density).

We note that nonlinear parabolic problems of the form (1.1), with/without the term $u_{rr} + \frac{\gamma}{r} u_r$, were also studied in [19], [23] and references therein. In [19], by using the Galerkin and compactness method in appropriate Sobolev spaces with weight, the authors obtained the results related to the existence and asymptotic behavior of the solution of the following initial and boundary value problem for nonlinear parabolic equation

$$\begin{cases} u_t - a\,(t) \left(u_{rr} + \frac{\gamma}{r} u_r \right) + F(r, u) = f(r, t), \; 0 < r < 1, \; 0 < t < T, \\ \left| \lim\limits_{r \to 0_+} r^{\frac{\gamma}{2}} u_r(r, t) \right| < +\infty, \;\; u_r(1, t) + h(t)(u(1, t) - \bar{u}_0) = 0, \\ u(r, 0) = u_0(r). \end{cases} \tag{1.17}$$

In [23], the following nonlinear heat equation associated with Robin conditions was investigated

$$\begin{cases} u_t - \frac{\partial}{\partial x}\left[\mu(x,t)u_x\right] + f(u) = f_1(x,t),\ (x,t) \in \Omega \times (0,T), \\ u_x(0,t) = h_0 u(0,t) + g_0(t), \quad -u_x(1,t) = h_1 u(1,t) + g_1(t), \\ u\left(x,0\right) = u_0\left(x\right). \end{cases} \tag{1.18}$$

On the other hand, we note more that equations with viscoelastic term or equations with a memory condition at the boundary (can be caused by the interaction with another viscoelastic element) also have been investigated by several authors, for example we refer to [12]-[14], [22], [24], [25], [27], [29]. In [12], by assuming that the kernel g in the memory term decays exponentially, via constructing a suitable Liapunov functional and making use of the perturbed energy method, Cavalcanti et al. obtained an exponential rate of decay for the solution of the viscoelastic nonlinear wave equation

$$u_{tt} - \Delta u + f(x,t,u) + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + a(x)u_t = 0,\ \ x \in \Omega,\ \ t > 0, \tag{1.19}$$

here the damping term $a(x)u_t$ may be null for some part of the domain Ω. In [22], Messaoudi established a blow up result for solutions with negative initial energy and a global existence result for arbitrary initial data of a nonlinear viscoelastic wave equation associated with initial and Dirichlet boundary conditions.

In [27], Munoz-Rivera and Andrade dealt with the global existence and exponential decay of solutions of the nonlinear one-dimensional wave equation with a viscoelastic boundary condition. In [29], Santos studied the stability of solutions for wave equations whose boundary condition includes a integral that represents the memory effect. Here, the dissipation is strong enough to produce exponential decay of the solution, provided the relaxation function also decays exponentially. And when the relaxation function decays polynomially, the solution decays polynomially and with the same rate. In [24], the following initial boundary value problem for a nonlinear heat equation with viscoelastic was considered

$$u_t - \frac{\partial}{\partial x}\left[\mu(x,t)u_x\right] + \int_0^t g(t-s)\frac{\partial}{\partial x}\left[\mu(x,t)u_x\right]ds = f(u) + f_1(x,t), \tag{1.20}$$

$0 < x < 1$, $t > 0$, and existence, uniqueness, regularity, blow-up and exponential decay estimates were established.

Motivated by the above mentioned works, because of mathematical context, we study of the existence, uniqueness and exponential decay of solutions for (1.1). To the best of our knowledge, there are few works on the study of nonlinear pseudoparabolic equation with viscoelastic term. This paper consists of four sections. In Section 2, we present preliminaries. In Section 3, we prove the local existence and uniqueness results. In Section 4, we establish a sufficient condition to obtain the global existence and exponential decay of weak solutions.

2. Preliminaries

Throughout this paper, we set $\Omega = (1, R)$ and use $L^2 = L^2(\Omega)$ to denote the Lebesgue space with the inner product defined by $(u, v) = \int_1^R u(x)v(x)dx$, L^2-norm of a function $u \in L^2$ is denoted by $\|u\| = \sqrt{(u, u)}$. We use $H^m = H^m(\Omega)$ to denote the Sobolev spaces with the norm $\|u\|_{H^m} = \left(\sum_{i=0}^m \left\|D^i u\right\|^2\right)^{1/2}$.

Moreover, we also introduce three weighted scalar products

$$\begin{aligned} \langle u, v\rangle &= \int_1^R xu(x)v(x)dx,\ u,\ v \in L^2, \\ \langle u, v\rangle_1 &= \langle u, v\rangle + \langle u_x, v_x\rangle,\ u,\ v \in H^1, \\ \langle u, v\rangle_2 &= \langle u, v\rangle + \langle u_x, v_x\rangle + \langle u_{xx}, v_{xx}\rangle,\ u,\ v \in H^2, \end{aligned} \tag{2.1}$$

then L^2, H^1, H^2 are the Hilbert spaces with respect to the above scalar products. We denote $\|u\|_0 = \sqrt{\langle u, u\rangle}$, $u \in L^2$; $\|u\|_1 = \sqrt{\langle u, u\rangle_1}$, $u \in H^1$; $\|u\|_2 = \sqrt{\langle u, u\rangle_2}$, $u \in H^2$.

Put

$$V = \left\{v \in H^1 : v(R) = 0\right\}. \tag{2.2}$$

The symmetric bilinear form $a(\cdot, \cdot)$ is defined by

$$a(u, v) = \langle u_x, v_x\rangle,\ \text{for all } u,\ v \in V. \tag{2.3}$$

Then, we have the following lemmas.

Lemma 2.1. *The imbeddings $V \hookrightarrow C^0(\overline{\Omega})$ is compact and*

(i) $\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{R-1}\,\|v_x\|_0,\ \forall v \in V,$

(ii) $\|v\|_0 \leq \frac{\sqrt{2R}(R-1)}{2}\,\|v_x\|_0,\ \forall v \in V.$

Lemma 2.2. *The symmetric bilinear form $a(\cdot, \cdot)$ is continuous on $V \times V$ and coercive on V and*

(i) $|a(u, v)| \leq \|u\|_a \|v\|_a,\ \forall u,\ v \in V,$

(ii) $a(v, v) \geq \|v\|_a^2,\ \forall v \in V,$

where $\|v\|_a = \sqrt{a(v, v)} = \|v_x\|_0$.

Lemma 2.3. *There exists the Hilbert orthonormal base $\{w_j\}$ of L^2 consisting of the eigenfunctions w_j corresponding to the eigenvalue $\bar{\lambda}_j$ such that*

$$\begin{cases} 0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 \leq \cdots \leq \bar{\lambda}_j \leq \bar{\lambda}_{j+1} \leq \cdots,\quad \lim_{j\to+\infty} \bar{\lambda}_j = +\infty, \\ a(w_j, w) = \bar{\lambda}_j \langle w_j, w\rangle \quad \textit{for all } w \in V,\ j = 1, 2, \cdots. \end{cases} \tag{2.4}$$

Furthermore, the sequence $\{w_j/\sqrt{\bar{\lambda}_j}\}$ *is also the Hilbert orthonormal base of* V *with respect to the scalar product* $a(\cdot,\cdot)$.

On the other hand, we have w_j *satisfying the following boundary value problem*

$$\begin{cases} Aw_j \equiv -\left(w_{jxx} + \dfrac{1}{x}w_{jx}\right) = -\dfrac{1}{x}\dfrac{\partial}{\partial x}(xw_{jx}) = \bar{\lambda}_j w_j, \ in \ (1,R), \\ w_{jx}(1) = w_j(R) = 0, \ w_j \in C^\infty([1,R]). \end{cases} \tag{2.5}$$

The proof of Lemma 2.3 can be found in [[33], p.87, Theorem 7.7], with $H = L^2$ and $a(\cdot,\cdot)$ as defined by (2.3).

Lemma 2.4. *The operator* $A : V \to V'$ *in* (2.5) *is uniquely defined by Lax-Milgram's lemma, that is,*

$$a(u,v) = \langle Au, v\rangle\,, \ \textit{for all } u, \ v \in V. \tag{2.6}$$

3. Local existence

In this section, the local solution of (1.1) is established by using linear approximate method and Galerkin method. For a fixed constant $T^* > 0$, we make the following assumptions:

(H_1) $\tilde{u}_0 \in V \cap H^2$, $\tilde{u}_{0x}(1) = 0$;

(H_2) $g \in L^2(0,T^*)$;

(H_3) $f \in C^1(\bar{\Omega} \times [0,T^*] \times \mathbb{R}^4)$ satisfies the conditions:

there exists a constant σ, with $0 < \sigma < \dfrac{1}{4\tilde{R}}$ such that

$$\|D_5 f\|_{C^0(\bar{\Omega}_M)} + \|D_6 f\|_{C^0(\bar{\Omega}_M)} \le \sigma, \text{ for all } M > 0,$$

$$\|D_i f\|_{C^0(\bar{\Omega}_M)} = \sup\{|D_i f(x,t,y_1,y_2,y_3,y_4)| : (x,t,y_1,y_2,y_3,y_4) \in \bar{\Omega}_M\}, \quad i = 5,\, 6,$$

where,

$$\bar{\Omega}_M = [1,R] \times [0,T^*] \times [-\sqrt{R-1}M, \sqrt{R-1}M]^4, \ \tilde{R} = 1 + \sqrt{\frac{R}{2}}(R-1).$$

Definition 3.1. The weak solution of (1.1) is a function $u \in L^\infty(0,T;V \cap H^2)$ such that $u' \in L^\infty(0,T;V \cap H^2)$, and u satisfies the following variational equation

$$\begin{cases} \langle u'(t), w\rangle + a\left(u'(t), w\right) + a\left(u(t), w\right) \\ = \int_0^t g(t-s) a\left(u(s), w\right) ds + \langle f[u](t), w\rangle, \ \forall \ w \in V, \text{ a.e., } t \in (0,T), \\ u(0) = \tilde{u}_0, \end{cases} \tag{3.1}$$

where $f[u](x,t) = f\left(x,t,u(x,t),u_x(x,t),u'(x,t),u'_x(x,t)\right)$.

For each $T \in (0,T^*]$, we define

$$W_T = \left\{v \in L^\infty(0,T;V \cap H^2) : v' \in L^\infty(0,T;V \cap H^2)\right\},$$

then W_T is a Banach space with norm

$$\|v\|_{W_T} = \max\left\{\|v\|_{L^\infty(0,T;V\cap H^2)}\,,\ \ \left\|v'\right\|_{L^\infty(0,T;V\cap H^2)}\right\}.$$

For $M > 0$, we put

$$B_T(M) = \left\{v \in W_T : \|v\|_{W_T} \le M\right\}$$

and

$$K_M(f) = \|f\|_{C^1\left(\bar{\Omega}_M\right)} = \|f\|_{C^0\left(\bar{\Omega}_M\right)} + \sum\nolimits_{i=1}^{6} \|D_i f\|_{C^0\left(\bar{\Omega}_M\right)}\,,$$

where

$$\|f\|_{C^0(\bar{\Omega}_M)} = \sup\left\{|f(x,t,y_1,\cdots,y_4)| : (x,t,y_1,\cdots,y_4) \in \bar{\Omega}_M\right\}$$

with

$$\bar{\Omega}_M = [1,R] \times [0,T^*] \times [-\sqrt{R-1}M, \sqrt{R-1}M]^4.$$

Now, we construct the recurrent sequence $\{u_m\}$ defined by $u_0 \equiv 0$, and suppose that

$$u_{m-1} \in B_T(M). \tag{3.2}$$

We need to find $u_m \in B_T(M)$, $m \ge 1$ satisfying the linear variational problem

$$\begin{cases} \langle u_m'(t), w\rangle + a(u_m'(t), w) + a(u_m(t), w) \\ = \int_0^t g(t-s) a\left(u_m(s), w\right) ds + \langle F_m\left(t\right), w\rangle\,,\ \forall w \in V, \\ u_m(0) = \tilde{u}_0, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} F_m\left(t\right) =& f\left[u_{m-1}\right](x,t) \\ =& f\left(x,t,u_{m-1}(x,t), \nabla u_{m-1}(x,t), u_{m-1}'(x,t), \nabla u_{m-1}'(x,t)\right). \end{aligned} \tag{3.4}$$

Then we have the following theorem.

Theorem 3.2. *Let $\tilde{u}_0$, g, f satisfy the conditions (H_1)-(H_3) respectively, then there exists a recurrent sequence $\{u_m\} \subset B_T(M)$ defined by* (3.2)-(3.4).

Proof. Consider the basis $\{w_j\}$ for L^2 as in Lemma 2.3. Put

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} c_{mj}^{(k)}(t) w_j,$$

where $c_{mj}^{(k)}$ are determined via the following ordinary differential equations

$$\begin{cases} \left\langle \dot{u}_m^{(k)}(t), w_j\right\rangle + a\left(\dot{u}_m^{(k)}(t), w_j\right) + a\left(u_m^{(k)}(t), w_j\right) \\ = \int_0^t g(t-s) a\left(u_m^{(k)}(s), w_j\right) ds + \langle F_m\left(t\right), w_j\rangle\,,\ 1 \le j \le k, \\ u_m^{(k)}(0) = \tilde{u}_{0k}, \end{cases} \tag{3.5}$$

in which

$$\tilde{u}_{0k} = \sum\nolimits_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \tilde{u}_0 \quad \text{strongly in} \quad V \cap H^2. \tag{3.6}$$

Using contraction mapping principle, it is not difficult to show that the existence of approximate solution $u_m^{(k)}(t)$ to (3.5)-(3.6) in $[0,T]$.

Next, the following priori estimates show the bounds of approximate solution $u_m^{(k)}(t)$.

Multiplying the j^{th} equation of (3.5) by $\dot{c}_{mj}^{(k)}(t)$ and summing up with respect to j, afterwards, integrating in time variable from 0 to t, we get

$$\begin{aligned} X_m^{(k)}(t) &= X_m^{(k)}(0) + 2\int_0^t d\tau \int_0^\tau g(\tau - s) a\left(u_m^{(k)}(s), \dot{u}_m^{(k)}(\tau)\right) ds \\ &\quad + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds, \end{aligned} \tag{3.7}$$

where

$$X_m^{(k)}(t) = 2\int_0^t \left(\left\| \dot{u}_m^{(k)}(s) \right\|_0^2 + \left\| \dot{u}_m^{(k)}(s) \right\|_a^2 \right) ds + \left\| u_m^{(k)}(t) \right\|_a^2. \tag{3.8}$$

In (3.5) replacing $w_j = \frac{1}{\lambda_j} A w_j$ and using the hypothesis $(H_3)_i$, we obtain

$$\begin{aligned} &a\left(\dot{u}_m^{(k)}(t), w_j\right) + \left\langle A\dot{u}_m^{(k)}(t), Aw_j \right\rangle + \left\langle Au_m^{(k)}(t), Aw_j \right\rangle \\ &= \int_0^t g(t-s) \left\langle Au_m^{(k)}(s), Aw_j) \right\rangle ds + \langle F_m(t), Aw_j \rangle. \end{aligned} \tag{3.9}$$

Multiplying (3.9) by $\dot{c}_{mj}^{(k)}(t)$ and summing up with respect to j, afterwards, integrating in time variable from 0 to t, we have

$$\begin{aligned} Y_m^{(k)}(t) &= Y_m^{(k)}(0) + 2\int_0^t d\tau \int_0^\tau g(\tau - s) \left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(\tau) \right\rangle ds \\ &\quad + 2\int_0^t \left\langle F_m(s), A\dot{u}_m^{(k)}(s) \right\rangle ds, \end{aligned} \tag{3.10}$$

where

$$Y_m^{(k)}(t) = 2\int_0^t \left(\left\| \dot{u}_m^{(k)}(s) \right\|_a^2 + \left\| A\dot{u}_m^{(k)}(s) \right\|_0^2 \right) ds + \left\| Au_m^{(k)}(t) \right\|_0^2. \tag{3.11}$$

On the other hand, multiplying (3.9) by $\dot{c}_{mj}^{(k)}(t)$ and summing up with respect to j, we deduce

$$\begin{aligned} Z_m^{(k)}(t) &= \left\langle Au_m^{(k)}(t), A\dot{u}_m^{(k)}(t) \right\rangle + \int_0^t g(t-s) \left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(t) \right\rangle ds \\ &\quad + \left\langle F_m(t), A\dot{u}_m^{(k)}(t) \right\rangle, \end{aligned} \tag{3.12}$$

where

$$Z_m^{(k)}(t) = \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|A\dot{u}_m^{(k)}(t)\right\|_0^2. \tag{3.13}$$

Put

$$\begin{aligned}
S_m^{(k)}(t) =& X_m^{(k)}(t) + Y_m^{(k)}(t) + Z_m^{(k)}(t) \\
=& 2\int_0^t \left[\left\|\dot{u}_m^{(k)}(s)\right\|_0^2 + 2\left\|\dot{u}_m^{(k)}(s)\right\|_a^2 + \left\|A\dot{u}_m^{(k)}(s)\right\|_0^2\right] ds \\
&+ \left\|u_m^{(k)}(t)\right\|_a^2 + \left\|Au_m^{(k)}(t)\right\|_0^2 \\
&+ \left\|\dot{u}_m^{(k)}(t)\right\|_a^2 + \left\|A\dot{u}_m^{(k)}(t)\right\|_0^2.
\end{aligned} \tag{3.14}$$

By (3.7), (3.10) and (3.12), we obtain

$$\begin{aligned}
S_m^{(k)}(t) =& X_m^{(k)}(0) + Y_m^{(k)}(0) \\
&+ 2\int_0^t d\tau \int_0^\tau g(\tau - s) a\left(u_m^{(k)}(s), \dot{u}_m^{(k)}(\tau)\right) ds \\
&+ 2\int_0^t d\tau \int_0^\tau g(\tau - s) \left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(\tau)\right\rangle ds \\
&+ \int_0^t g(t-s)\left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(t)\right\rangle ds \\
&+ 2\int_0^t \left\langle F_m(s), A\dot{u}_m^{(k)}(s)\right\rangle ds + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds \\
&- \left\langle Au_m^{(k)}(t), A\dot{u}_m^{(k)}(t)\right\rangle + \left\langle F_m(t), A\dot{u}_m^{(k)}(t)\right\rangle \\
=& X_m^{(k)}(0) + Y_m^{(k)}(0) + \sum_{j=1}^7 I_j.
\end{aligned} \tag{3.15}$$

We shall estimate the terms of right side of (3.15) as follows

$$\begin{aligned}
I_1 =& 2\int_0^t d\tau \int_0^\tau g(\tau - s) a(u_m^{(k)}(s), \dot{u}_m^{(k)}(\tau)) ds \\
\leq& 2\int_0^t d\tau \int_0^\tau |g(\tau - s)| \sqrt{S_m^{(k)}(s)}\sqrt{S_m^{(k)}(\tau)} ds \\
\leq& 2\sqrt{T^*}\, \|g\|_{L^2(0,T^*)} \int_0^t S_m^{(k)}(\tau)\, d\tau,
\end{aligned} \tag{3.16}$$

$$
\begin{aligned}
I_2 =&2\int_0^t d\tau \int_0^\tau g(\tau - s)\left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(\tau)\right\rangle ds \\
\leq&2\int_0^t d\tau \int_0^\tau |g(\tau - s)|\sqrt{S_m^{(k)}(s)}\sqrt{S_m^{(k)}(\tau)}ds \\
\leq&2\sqrt{T^*}\,\|g\|_{L^2(0,T^*)}\int_0^t S_m^{(k)}(\tau)\,d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 =&\int_0^t g(t - s)\left\langle Au_m^{(k)}(s), A\dot{u}_m^{(k)}(t)\right\rangle ds \\
\leq&\int_0^t |g(t - s)|\sqrt{S_m^{(k)}(s)}\sqrt{S_m^{(k)}(t)}ds \\
\leq&\frac{1}{4}S_m^{(k)}(t) + \|g\|_{L^2(0,T^*)}^2\int_0^t S_m^{(k)}(s)ds.
\end{aligned}
$$

By (H_3) and (3.1), we get

$$
\begin{aligned}
I_4 =&2\int_0^t \left\langle F_m(s), A\dot{u}_m^{(k)}(s)\right\rangle ds \\
\leq&\frac{1}{2}TK_M(f)\left(R^2 - 1\right) + \int_0^t S_m^{(k)}(s)ds
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
I_5 =&2\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s)\right\rangle ds \\
\leq&\frac{R(R-1)^2}{2}\int_0^t \|F_m(s)\|_0^2\,ds + \int_0^t \left\|\dot{u}_m^{(k)}(s)\right\|_a^2 ds \\
\leq&\frac{1}{4}TK_M(f)R(R+1)(R-1)^3 + \int_0^t S_m^{(k)}(s)ds.
\end{aligned}
$$

Using Cauchy-Schwarz, we get that

$$
I_6 = -\left\langle Au_m^{(k)}(t), A\dot{u}_m^{(k)}(t)\right\rangle \leq \left\|Au_m^{(k)}(t)\right\|_0^2 + \frac{1}{4}S_m^{(k)}(t). \tag{3.18}
$$

The term $\left\|Au_m^{(k)}(t)\right\|_0^2$ is estimated as follows

$$\begin{aligned}\left\|Au_m^{(k)}(t)\right\|_0^2 \leq & \left(\|A\tilde{u}_{0k}\|_0 + \int_0^t \left\|A\dot{u}_m^{(k)}(s)\right\|_0 ds\right)^2 \\ \leq & 2\|A\tilde{u}_{0k}\|_0^2 + 2T\int_0^t \left\|A\dot{u}_m^{(k)}(s)\right\|_0^2 ds \\ \leq & 2\|A\tilde{u}_{0k}\|_0^2 + 2T^*\int_0^t S_m^{(k)}(s)ds.\end{aligned} \tag{3.19}$$

Then, it follows from (3.18)-(3.19) that

$$\begin{aligned}I_6 = & -\left\langle Au_m^{(k)}(t), A\dot{u}_m^{(k)}(t)\right\rangle \\ \leq & \frac{1}{4}S_m^{(k)}(t) + 2\|A\tilde{u}_{0k}\|_0^2 + 2T^*\int_0^t S_m^{(k)}(s)ds.\end{aligned} \tag{3.20}$$

Using Cauchy-Schwarz inquality, we have

$$I_7 = \left\langle F_m(t), A\dot{u}_m^{(k)}(t)\right\rangle \leq \|F_m(t)\|_0^2 + \frac{1}{4}S_m^{(k)}(t). \tag{3.21}$$

The function $F_m(x,t)$ can be written as follows

$$\begin{aligned}F_m(x,t) = & f(x,t,u_{m-1}(t),\nabla u_{m-1}(t),0,0) \\ & + f\left(x,t,u_{m-1}(t),\nabla u_{m-1}(t),u'_{m-1}(t),\nabla u'_{m-1}(t)\right) \\ & - f(x,t,u_{m-1}(t),\nabla u_{m-1}(t),0,0).\end{aligned} \tag{3.22}$$

Note that

$$\begin{aligned}& f(x,t,u_{m-1}(t),\nabla u_{m-1}(t),0,0) \\ & = f(x,0,u_{m-1}(0),\nabla u_{m-1}(0),0,0) \\ & \quad + \int_0^t D_2 f(x,s,u_{m-1}(s),\nabla u_{m-1}(s),0,0)\,ds \\ & \quad + \int_0^t D_3 f(x,s,u_{m-1}(s),\nabla u_{m-1}(s),0,0)\,u'_{m-1}(s)ds \\ & \quad + \int_0^t D_4 f(x,s,u_{m-1}(s),\nabla u_{m-1}(s),0,0)\,\nabla u'_{m-1}(s)ds,\end{aligned}$$

so

$$
\begin{aligned}
&\|f(\cdot,t,u_{m-1}(t),\nabla u_{m-1}(t),0,0)\|_0 \\
&\quad\leq \|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x},0,0)\|_0 \\
&\qquad + K_M(f)\int_0^t \left(\|1\|_0 + \left\|u'_{m-1}(s)\right\|_0 + \left\|\nabla u'_{m-1}(s)\right\|_0\right) ds \\
&\quad\leq \|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x},0,0)\|_0 + K_M(f)\int_0^t \left(\|1\|_0 + \tilde{R}\left\|u'_{m-1}(s)\right\|_a\right) ds \\
&\quad\leq \|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x},0,0)\|_0 + TK_M(f)\left(\sqrt{\frac{R^2-1}{2}} + \tilde{R}M\right).
\end{aligned} \tag{3.23}
$$

Applying mean value theorem to the function f, we obtain that

$$
\begin{aligned}
&f\left(x,t,u_{m-1}(t),\nabla u_{m-1}(t),u'_{m-1}(t),\nabla u'_{m-1}(t)\right) \\
&\quad - f\left(x,t,u_{m-1}(t),\nabla u_{m-1}(t),0,0\right) \\
&= D_5 f\left(x,t,u_{m-1}(t),\nabla u_{m-1}(t),\theta u'_{m-1}(t),\theta\nabla u'_{m-1}(t)\right) u'_{m-1}(t) \\
&\quad + D_6 f\left(x,t,u_{m-1}(t),\nabla u_{m-1}(t),\theta u'_{m-1}(t),\theta\nabla u'_{m-1}(t)\right) \nabla u'_{m-1}(t),
\end{aligned}
$$

where $0<\theta<1$. It turns to

$$
\begin{aligned}
&||f\left(x,t,u_{m-1}(t),\nabla u_{m-1}(t),u'_{m-1}(t),\nabla u'_{m-1}(t)\right) \\
&\quad - f\left(x,t,u_{m-1}(t),\nabla u_{m-1}(t),0,0\right)||_0 \\
&\leq \sigma\left(\left\|u'_{m-1}(t)\right\|_0 + \left\|\nabla u'_{m-1}(t)\right\|_0\right) \\
&\leq \sigma\tilde{R}M.
\end{aligned} \tag{3.24}
$$

Then, since (3.22)-(3.24), we have

$$
\|F_m(t)\|_0 \leq \|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x},0,0)\|_0 + TK_M(f)\left(\sqrt{\frac{R^2-1}{2}} + \tilde{R}M\right) + \sigma\tilde{R}M. \tag{3.25}
$$

Since (3.25), it follows from (3.21) that

$$
\begin{aligned}
I_7 &= \left\langle F_m(t), A\dot{u}_m^{(k)}(t)\right\rangle \\
&\leq 3\left[\|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x},0,0)\|_0^2 + \sigma^2\tilde{R}^2M^2\right] \\
&\quad + 3T^2K_M^2(f)\left(\sqrt{\frac{R^2-1}{2}} + \tilde{R}M\right)^2 + \frac{1}{4}S_m^{(k)}(t).
\end{aligned} \tag{3.26}
$$

Combining (3.16), (3.17), (3.20), (3.26), it implies from (3.15) that

$$
S_m^{(k)}(t) \leq D_0^{(k)} + \bar{\theta}M^2 + TD(M,T) + D_{T^*}\int_0^t S_m^{(k)}(s)\,ds, \tag{3.27}
$$

where

$$\begin{cases} \bar{\theta} = 12\sigma^2 \tilde{R}^2, \\ D_0^{(k)} = 4\left(\|\tilde{u}_{0kx}\|_0^2 + 3\|A\tilde{u}_{0k}\|_0^2 + 3\|f(\cdot,0,\tilde{u}_0,\tilde{u}_{0x},0,0)\|_0^2\right), \\ D(M,T) = \left(2 + R(R-1)^2\right)\left(R^2-1\right)K_M(f) \\ \qquad\qquad +12TK_M^2(f)\left(\sqrt{\dfrac{R^2-1}{2}} + \tilde{R}M\right)^2, \\ D_{T^*} = 4\left[2 + 2T^* + \left(4\sqrt{T^*} + \|g\|_{L^2(0,T^*)}\right)\|g\|_{L^2(0,T^*)}\right]. \end{cases} \tag{3.28}$$

Note that $0 < \sigma < \dfrac{1}{4\tilde{R}} < \dfrac{1}{2\sqrt{3}\tilde{R}}$, it yields

$$\bar{\theta} = 12\sigma^2\tilde{R}^2 < 1. \tag{3.29}$$

The convergence in (3.6) shows that there exists a constant $M > 0$ independent of k and m such that

$$D_0^{(k)} \leq \frac{1-\bar{\theta}}{2}M^2, \text{ for all } m,\ k \in \mathbb{N}. \tag{3.30}$$

By the hypothesis $(H_3)_{(ii)}$ and M chosen as above, we can choose $T \in (0, T^*]$ such that both of the following conditions are fulfilled

$$\left(\frac{1+\bar{\theta}}{2}M^2 + TD(M,T)\right)\exp\left(TD_{T^*}\right) \leq M^2 \tag{3.31}$$

and

$$k_T = 4\tilde{R}\sqrt{\sigma^2 + TK_M^2(f)}\exp\left(T^2\|g\|_{L^2(0,T^*)}^2\right) < 1. \tag{3.32}$$

By (3.31), it yields

$$S_m^{(k)}(t) \leq M^2\exp\left(-TD_{T^*}\right) + D_{T^*}\int_0^t S_m^{(k)}(s)ds.$$

Applying Gronwall's lemma, we get

$$S_m^{(k)}(t) \leq M^2\exp\left(-TD_{T^*}\right)\exp\left(tD_{T^*}\right) \leq M^2, \tag{3.33}$$

for all $t \in [0,T]$, for all m and k. Therefore, we have

$$u_m^{(k)} \in B_T(M), \text{ for all } m \text{ and } k. \tag{3.34}$$

Due to (3.34), there exists a subsequence of $\{u_m^{(k)}\}$, still denoted by $\{u_m^{(k)}\}$ such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weakly*}, \\ u_m \in B(M,T). \end{cases} \tag{3.35}$$

Passing to limit in (3.5), (3.6), we have u_m satisfying (3.3), (3.4) in $L^2(0,T)$. Theorem 3.2 is proved. □

By using Theorem 3.2 and the compact imbedding theorems, we shall prove the existence and uniqueness of weak local solution in time to the problem (1.1). First, we introduce the following space

$$W_1(T) = \left\{ v \in C^0([0,T];V) : v' \in L^2(0,T;V) \right\},$$

it is a Banach space with respect to the norm (see Lions [18])

$$\|v\|_{W_1(T)} = \|v\|_{C^0([0,T];V)} + \left\|v'\right\|_{L^2(0,T;V)}.$$

This result is presented in Theorem 3.3 below.

Theorem 3.3. *Suppose that the hypotheses* (H_1)-(H_3) *are satisfied. Then, the recurrent sequence* $\{u_m\}$ *defined by* (3.3)-(3.4) *converges strongly to a function* u *in* $W_1(T)$ *and* u *is the unique weak solution of* (1.1). *Moreover, we have the following estimate*

$$\|u_m - u\|_{W_1(T)} \leq C_T k_T^m, \ \forall \ m \in \mathbb{N}, \tag{3.36}$$

where $k_T \in [0,1)$ *is defined as in* (3.32) *and* C_T *is a constant depending only on* T, f, g, $\tilde{u}_0$ *and* k_T.

Proof. First, we prove the local existence in time of (1.1). It is necessary to prove that $\{u_m\}$ (in Theorem 3.2) is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational proplem

$$\begin{cases} \langle w_m'(t), v\rangle + a(w_m'(t), v) + a(w_m(t), v) - \int_0^t g(t-s)a(w_m(s), v)ds \\ = \langle F_{m+1}(t) - F_m(t), v\rangle, \ \forall v \in V, \\ w_m(x,0) = 0, \end{cases} \tag{3.37}$$

where

$$\begin{aligned} &F_{m+1}(t) - F_m(t) \\ &= f\left(x,t,u_m(x,t), \nabla u_m(x,t), u_m'(x,t), \nabla u_m'(x,t)\right) \\ &\quad - f\left(x,t,u_{m-1}(x,t), \nabla u_{m-1}(x,t), u_{m-1}'(x,t), \nabla u_{m-1}'(x,t)\right). \end{aligned}$$

Taking $v = w_m'$ in (3.37) and then integrating in t, we get

$$\begin{aligned} \tilde{S}_m(t) = 2\int_0^t d\tau \int_0^\tau g(\tau - s)a(w_m(s), w_m'(\tau))ds \\ + 2\int_0^t \langle F_{m+1}(s) - F_m(s), w_m'(s)\rangle ds \\ = J_1 + J_2, \end{aligned} \tag{3.38}$$

where

$$\tilde{S}_m(t) = 2\int_0^t \left(\left\|w_m'(s)\right\|_0^2 + \left\|w_m'(s)\right\|_a^2\right) ds + \|w_m(t)\|_a^2 . \tag{3.39}$$

Next, we have to estimate the integrals on right hand side of (3.38).

By the hypothesis (H_2) and using the inequalitiy $2ab \le a^2 + b^2$, $\forall a, b \in \mathbb{R}$, the first integral on on right hand side of (3.38) is estimated as follows

$$\begin{aligned} J_1 =& 2\int_0^t d\tau \int_0^\tau g(\tau - s) a(w_m(s), w_m'(\tau)) ds \\ \le& 2\int_0^t d\tau \int_0^\tau |g(\tau - s)| \, \|w_m(s)\|_a \left\|w_m'(\tau)\right\|_a ds d\tau \\ \le& 2\int_0^t d\tau \int_0^\tau |g(\tau - s)| \sqrt{\tilde{S}_m(s)} \left\|w_m'(\tau)\right\|_a ds \\ \le& \int_0^t \left\|w_m'(s)\right\|_a^2 ds + T\|g\|_{L^2(0,T^*)}^2 \int_0^t \tilde{S}_m(s) ds. \end{aligned} \tag{3.40}$$

Applying mean value theorem to the function f, we get

$$\begin{aligned} F_{m+1}(x,t) - F_m(x,t) = & D_3 f[u_m^*](t) w_{m-1}(x,t) \\ & + D_4 f[u_m^*](t) \nabla w_{m-1}(x,t) \\ & + D_5 f[u_m^*](t) w_{m-1}'(x,t) \\ & + D_6 f[u_m^*](t) \nabla w_{m-1}'(x,t), \end{aligned}$$

where

$$D_i f[u_m^*](x,t) = D_i f\left(x, t, u_m^*(x,t), \nabla u_m^*(x,t), \dot{u}_m^*(x,t), \nabla \dot{u}_m^*(x,t)\right), i = 3, ..., 6$$

and

$$u_m^* = u_{m-1} + \theta w_{m-1}, \quad 0 < \theta < 1.$$

So

$$\begin{aligned} \|F_{m+1}(t) - F_m(t)\|_0 \le& K_M(f)\left(\|w_{m-1}(t)\|_0 + \|\nabla w_{m-1}(t)\|_0\right) \\ & + \sigma\left(\left\|w_{m-1}'(t)\right\|_0 + \left\|\nabla w_{m-1}'(t)\right\|_0\right) \\ \le& \tilde{R} K_M(f) \|w_{m-1}(t)\|_a + \tilde{R}\sigma \left\|w_{m-1}'(t)\right\|_a \\ \le& \tilde{R} K_M(f) \|w_{m-1}\|_{W_1(T)} + \tilde{R}\sigma \left\|w_{m-1}'(t)\right\|_a . \end{aligned} \tag{3.41}$$

Since the above inequality, the second integral on on right-hand side of (3.38) can be estimated by

$$\begin{aligned} J_2 =&2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s)\right\rangle ds \qquad (3.42)\\ \leq& \int_0^t \left(2\tilde{R}^2 K_M^2(f)\|w_{m-1}\|_{W_1(T)}^2 + 2\tilde{R}^2\sigma^2 \left\|w_{m-1}'(s)\right\|_a^2\right) ds\\ &+\int_0^t \left\|w_m'(s)\right\|_0^2 ds\\ \leq& 2T\tilde{R}^2 K_M^2(f)\|w_{m-1}\|_{W_1(T)}^2 + 2\tilde{R}^2\sigma^2\int_0^t \left\|w_{m-1}'(s)\right\|_a^2 ds\\ &+\int_0^t \left\|w_m'(s)\right\|_0^2 ds\\ \leq& 2T\tilde{R}^2 K_M^2(f)\|w_{m-1}\|_{W_1(T)}^2 + 2\tilde{R}^2\sigma^2\|w_{m-1}\|_{W_1(T)}^2\\ &+\int_0^t \left\|w_m'(s)\right\|_0^2 ds\\ =& 2\tilde{R}^2\left(TK_M^2(f)+\sigma^2\right)\|w_{m-1}\|_{W_1(T)}^2 + \int_0^t \left\|w_m'(s)\right\|_0^2 ds. \end{aligned}$$

By (3.40), (3.42), it follows from (3.38) that

$$\tilde{S}_m(t) \leq 4\tilde{R}^2\left(TK_M^2(f)+\sigma^2\right)\|w_{m-1}\|_{W_1(T)}^2 + 2T\|g\|_{L^2(0,T^*)}^2\int_0^t \tilde{S}_m(s)ds. \qquad (3.43)$$

Using Gronwall's Lemma, we have

$$\tilde{S}_m(t) \leq 4\tilde{R}^2\left(TK_M^2(f)+\sigma^2\right)\exp\left(2T^2\|g\|_{L^2(0,T^*)}^2\right)\|w_{m-1}\|_{W_1(T)}^2.$$

This deduce that

$$\|w_m\|_{W_1(T)} \leq k_T\|w_{m-1}\|_{W_1(T)}, \ \forall m\in\mathbb{N},$$

where $k_T\in[0,1)$ is defined as in (3.32), which implies that

$$\begin{aligned} \|u_{m+p}-u_m\|_{W_1(T)} \leq& \frac{k_T^m}{1-k_T}\|u_1-u_0\|_{W_1(T)} \qquad (3.44)\\ \leq& \frac{M}{1-k_T}k_T^m, \ \forall m, \ p\in\mathbb{N}. \end{aligned}$$

The above inequality ensures that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u\in W_1(T)$ such that

$$u_m \longrightarrow u \quad \text{strongly in} \quad W_1(T). \qquad (3.45)$$

Note that $u_m \in B_T(M)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weakly*}, \\ u'_{m_j} \to u' & \text{in} \quad L^\infty(0,T;V\cap H^2) \text{ weakly*}, \\ u \in B_T(M). \end{cases} \tag{3.46}$$

We note that

$$\begin{aligned} |F_m(t) - f[u](t)| &\le K_M(f)\left(|u_{m-1}(t) - u(t)| + |\nabla u_{m-1}(t) - \nabla u(t)|\right) \\ &\quad + \sigma\left(\left|u'_{m-1}(t) - u'(t)\right| + \left|\nabla u'_{m-1}(t) - \nabla u'(t)\right|\right), \end{aligned}$$

so

$$\begin{aligned} \|F_m(t) - f[u](t)\|_0 &\le K_M(f)\left(\|u_{m-1}(t) - u(t)\|_0 + \|\nabla u_{m-1}(t) - \nabla u(t)\|_0\right) \\ &\quad + \sigma\left(\left\|u'_{m-1}(t) - u'(t)\right\|_0 + \left\|\nabla u'_{m-1}(t) - \nabla u'(t)\right\|_0\right) \\ &\le \tilde{R}K_M(f)\|u_{m-1}(t) - u(t)\|_a + \sigma\tilde{R}\left\|u'_{m-1}(t) - u'(t)\right\|_a \\ &\le \tilde{R}K_M(f)\|u_{m-1} - u\|_{W_1(T)} + \sigma\tilde{R}\left\|u'_{m-1}(t) - u'(t)\right\|_a, \end{aligned}$$

it follows that

$$\begin{aligned} &\|F_m - f[u]\|^2_{L^2(Q_T)} \\ &\le 2\tilde{R}^2\left(TK^2_M(f)\|u_{m-1} - u\|^2_{W_1(T)} + \sigma^2\left\|u'_{m-1} - u'\right\|^2_{L^2(0,T;V)}\right) \\ &\le 2\tilde{R}^2\left(TK^2_M(f)\|u_{m-1} - u\|^2_{W_1(T)} + \sigma^2\|u_{m-1} - u\|^2_{W_1(T)}\right) \\ &\le 2\tilde{R}^2\left(TK^2_M(f) + \sigma^2\right)\|u_{m-1} - u\|^2_{W_1(T)}. \end{aligned} \tag{3.47}$$

Hence, we deduce from (3.47) that

$$F_m \longrightarrow f[u] \quad \text{strongly in} \quad L^2(Q_T). \tag{3.48}$$

Letting $m = m_j \to \infty$ in (3.3), (3.4) and using (3.45), (3.46) and (3.48), we get that there exists $u \in B_T(M)$ satisfying (3.1)-(3.2). The proof of existence is completed.

Finally, we need to prove the uniqueness of solutions.

Let u_1, $u_2 \in B_T(M)$ be two weak solutions of problem (1.1). Then $u = u_1 - u_2$ satisfies the variational problem

$$\begin{cases} \langle u'(t), v\rangle + a(u'(t), v) + a(u(t), v) + \int_0^t g(t-s)\,a(u(s), v)\,ds \\ = \langle \bar{F}_1(t) - \bar{F}_2(t), v\rangle, \ \forall v \in V, \\ u(0) = 0, \end{cases} \tag{3.49}$$

where

$$\bar{F}_i(t) = f[u_i](t) = f\left(x, t, u_i, \nabla u_i, u'_i, \nabla u'_i\right), \ i = 1, 2. \tag{3.50}$$

Taking $v = u'$ and integrating in time from 0 to t, we get

$$\begin{aligned} Z\left(t\right) =& 2\int_0^t \int_0^\tau g\left(\tau - s\right) a\left(u\left(s\right), u'(\tau)\right) ds d\tau + 2\int_0^t \langle \bar{F}_1\left(s\right) - \bar{F}_2\left(s\right), u'\left(s\right)\rangle ds \\ =& I_1 + I_2, \end{aligned} \tag{3.51}$$

where

$$Z\left(t\right) = \left\|u\left(t\right)\right\|_a^2 + 2\int_0^t \left(\left\|u'\left(s\right)\right\|_0^2 + \left\|u'\left(s\right)\right\|_a^2\right) ds. \tag{3.52}$$

The integrals on the right hand side of (3.51) are computed as follows

$$\begin{aligned} I_1 =& 2\int_0^t \int_0^\tau g(\tau - s) a(u(s), u'(\tau)) ds d\tau \\ \leq& 2\int_0^t \left\|u'(\tau)\right\|_a \int_0^\tau \left|g(\tau - s)\right| \left\|u(s)\right\|_a ds d\tau \\ \leq& \int_0^t \left\|u'(\tau)\right\|_a^2 d\tau + \int_0^t \left(\int_0^\tau \left|g(\tau - s)\right| \left\|u(s)\right\|_a ds\right)^2 d\tau \\ \leq& \int_0^t \left\|u'(\tau)\right\|_a^2 d\tau + T\left\|g\right\|_{L^2(0,T^*)}^2 \int_0^t \left\|u(s)\right\|_a^2 ds \\ \leq& \int_0^t \left\|u'(\tau)\right\|_a^2 d\tau + T\left\|g\right\|_{L^2(0,T^*)}^2 \int_0^t Z(s) ds. \end{aligned} \tag{3.53}$$

Similarly (3.41), we have

$$\left\|\bar{F}_1\left(t\right) - \bar{F}_2\left(t\right)\right\|_0 \leq \tilde{R} K_M(f) \left\|u(t)\right\|_a + \sigma \tilde{R} \left\|u'(t)\right\|_a,$$

then

$$\begin{aligned} I_2 =& 2\int_0^t \langle \bar{F}_1\left(s\right) - \bar{F}_2\left(s\right), u'\left(s\right)\rangle ds \\ \leq& 2\tilde{R}^2 \int_0^t \left(K_M^2(f) \left\|u(s)\right\|_a^2 + \sigma^2 \left\|u'(s)\right\|_a^2\right) ds + \int_0^t \left\|u'(s)\right\|_0^2 ds \\ \leq& 2\tilde{R}^2 K_M^2(f) \int_0^t \left\|u(s)\right\|_a^2 ds + 2\tilde{R}^2\sigma^2 \int_0^t \left\|u'(s)\right\|_a^2 ds + \int_0^t \left\|u'(s)\right\|_0^2 ds \\ \leq& 2\tilde{R}^2 K_M^2(f) \int_0^t Z(s) ds + \tilde{R}^2\sigma^2 Z(t) + \int_0^t \left\|u'(s)\right\|_0^2 ds. \end{aligned} \tag{3.54}$$

From (3.53) and (3.54), we obtain

$$Z\left(t\right) \leq 2\tilde{R}^2\sigma^2 Z(t) + 2\left(T\left\|g\right\|_{L^2(0,T^*)}^2 + 2\tilde{R}^2 K_M^2(f)\right) \int_0^t Z(s) ds. \tag{3.55}$$

By $0<\sigma<\dfrac{1}{4\tilde{R}}\leq\dfrac{1}{2\sqrt{3}\tilde{R}}$, it yields $2\tilde{R}^2\sigma^2<\dfrac{1}{6}$. Therefore, we have

$$Z(t)\leq\frac{12}{5}\left(T\left\|g\right\|^2_{L^2(0,T^*)}+2\tilde{R}^2K_M^2(f)\right)\int_0^t Z(s)ds. \tag{3.56}$$

Using Gronwall lemma, it follows that $Z(t)\equiv 0$, that is, $u_1\equiv u_2$. The uniqueness is proved. Consquently, this completes the proof. □

4. Exponential decay of solution

In this section, the problem (1.1) is considered with $f(x,t,u,u_x,u_t,u_{xt})\equiv f(u)+F(x,t)$, then it becomes

$$\begin{cases} u_t-\left(1+\dfrac{\partial}{\partial t}\right)\left(u_{xx}+\dfrac{1}{x}u_x\right)+\int_0^t g(t-s)\left(u_{xx}(s)+\dfrac{1}{x}u_x(s)\right)ds \\ =f(u)+F(x,t),\ 1<x<R,\ t>0, \\ u_x(1,t)=u(R,t)=0, \\ u(x,0)=\tilde{u}_0(x). \end{cases} \tag{4.1}$$

First, we give the following assumptions:

(H_1') $\tilde{u}_0\in V$;
(H_3') $f\in C^1(\mathbb{R};\mathbb{R})$ such that there exist constants p, q_1, $q_2>2$, $\bar{d}_2>0$ satisfying $\int_0^u f(z)dz\leq\bar{d}_2\left(|u|^{q_1}+|u|^{q_2}\right)$, $\forall u\in\mathbb{R}$;
(H_4') $F\in L^2\left(Q_{T^*}\right)$.

Combining Theorem 3.3 and using the standard arguments of density, we obtain the following theorem.

Theorem 4.1. *Let (H_1'), (H_2), (H_3'), (H_4') hold. Then, there exist $T\in(0,T^*]$ and a unique solution of the problem (4.1) such that*

$$u\in C^0([0,T];V),\ u'\in L^2(0,T;V).$$

Next, we prove that if $\|\tilde{u}_{0x}\|_0^2-p\int_1^R xdx\int_0^{\tilde{u}_0(x)}f(z)dz>0$, with $p>2$, and if the initial energy and $\left\|F(t)\right\|_0$ are small enough, then the energy of the solution decays exponentially as $t\to+\infty$. For this purpose, we make the following assumptions:

(A_1) $g\in C^1(\mathbb{R}_+;\mathbb{R}_+)$ such that the conditions are satisfied
  (i) $g(t)>0$ for all $t\geq 0$,
  (ii) $g'(t)\leq-\chi_1 g(t)$, $\forall t\geq 0$, $\chi_1>0$,
  (iii) $L\equiv 1-\int_0^\infty g(s)ds>0$;

(A_2) $f \in C^1(\mathbb{R};\mathbb{R})$ such that there exist constants p, q_1, $q_2 > 2$, $d_2 > p$, $\bar{d}_2 > 0$ satisfying
(i) $uf(u) > 0$, $\forall u \neq 0$,
(ii) $uf(u) \leq d_2 \int_0^u f(z)dz$, $\forall u \in \mathbb{R}$,
(iii) $\int_0^u f(z)dz \leq \bar{d}_2 \left(|u|^{q_1} + |u|^{q_2}\right)$, $\forall u \in \mathbb{R}$;

(A_3) $F \in L^\infty\left(\mathbb{R}_+; L^2\right) \cap L^2(\mathbb{R}_+; L^2)$ such that there exist two constants $\bar{C}_0 > 0$, $\gamma_0 > 0$ satisfying $\|F(t)\|_0 \leq \bar{C}_0 e^{-\gamma_0 t}$, $\forall t \geq 0$.

Now, let us define the Lyapunov functional by

$$\mathcal{L}(t) = E(t) + \delta\Psi(t), \tag{4.2}$$

where δ is a positive real number which will be chosen later and

$$\Psi(t) = \frac{1}{2}\|u(t)\|_0^2 + \frac{1}{2}\|u_x(t)\|_0^2, \tag{4.3}$$

$$\begin{aligned} E(t) =& \frac{1}{2}(g \diamond u)(t) + \frac{1}{2}\left(1-\bar{g}(t)\right)\|u_x(t)\|_0^2 - \int_1^R xdx\int_0^{u(x,t)} f(z)dz \\ =& \left(\frac{1}{2}-\frac{1}{p}\right)\left[(g \diamond u)(t) + (1-\bar{g}(t))\|u_x(t)\|_0^2\right] + \frac{1}{p}I(t), \end{aligned} \tag{4.4}$$

$$\begin{aligned} I(t) &= I\left(u(t)\right) \\ &= (g \diamond u)(t) + (1-\bar{g}(t))\|u_x(t)\|_0^2 - p\int_1^R xdx\int_0^{u(x,t)} f(z)dz, \end{aligned} \tag{4.5}$$

$$\begin{aligned} (g \diamond u)(t) &= \int_0^t g(t-s)\|u_x(t)-u_x(s)\|_0^2 ds, \\ \bar{g}(t) &= \int_0^t g(s)ds. \end{aligned}$$

Lemma 4.2. *Suppose that the hypotheses* (A_1)*-*(A_3) *hold. Then*

$$E'(t) \leq -\left(1-\frac{\varepsilon_1}{2}\right)\left\|u'(t)\right\|_0^2 - \left\|u_x'(t)\right\|_0^2 - \frac{1}{2}\chi_1(g \diamond u)(t) + \frac{1}{2\varepsilon_1}\|F(t)\|_0^2, \tag{4.6}$$

for all $\varepsilon_1 > 0$.

Proof. Multiplying the equation (4.1) by $xu'(x,t)$ and integrating from 1 to R, we obtain

$$E'(t) = -\left\|u'(t)\right\|_0^2 - \left\|u_x'(t)\right\|_0^2 + \frac{1}{2}(g' \diamond u)(t) - \frac{1}{2}g(t)\|u_x(t)\|_0^2 + \langle F(t), u'(t)\rangle. \tag{4.7}$$

On the other hand

$$\begin{cases} \langle F(t), u'(t)\rangle \leq \dfrac{\varepsilon_1}{2}\left\|u'(t)\right\|_0^2 + \dfrac{1}{2\varepsilon_1}\left\|F(t)\right\|_0^2, \ \forall \varepsilon_1 > 0, \\ \dfrac{1}{2}(g' \diamond u)(t) \leq -\dfrac{1}{2}\chi_1 (g \diamond u)(t), \\ -\dfrac{1}{2} g(t)\left\|u_x(t)\right\|_0^2 \leq 0. \end{cases} \tag{4.8}$$

Then, it follows from (4.7)-(4.8) that the inequality (4.6) is valid. □

Next, we state and prove global existence of solution (4.1).

Lemma 4.3. *Assume that* (A_1)-(A_3) *hold. Let* $I(0) > 0$ *and the initial energy* $E(0)$ *satisfy*

$$L_* = L - p\bar{d}_2 R\left(\tilde{D}_1^{q_1} R_*^{q_1-2} + \tilde{D}_2^{q_2} R_*^{q_2-2}\right) > \left(1 - \frac{p}{d_2}\right) + \frac{p}{d_2}\int_0^\infty g(t)dt, \tag{4.9}$$

where

$$R_* = \sqrt{\frac{2pE_*}{(p-2)L}}, \ E_* = E(0) + \frac{1}{2}\int_0^\infty \left\|F(t)\right\|_0^2 dt, \ \tilde{D}_i = \sup_{0\neq v\in V}\frac{\|v\|_{L^{q_i}}}{\|v_x\|_0}, \ i = 1, 2. \tag{4.10}$$

Then $I(t) > 0$, *for all* $t \geq 0$.

Proof. By the continuity of $I(t)$ and $I(0) = \|\tilde{u}_{0x}\|_0^2 - p\int_1^R xdx\int_0^{\tilde{u}_0(x)} f(z)dz > 0$, there exists $T_1 > 0$ such that

$$I(t) = I\left(u(t)\right) > 0, \ \forall t \in [0, T_1]. \tag{4.11}$$

From (4.6), we get

$$\begin{aligned} E(t) &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \bar{g}(t)\right)\left\|u_x(t)\right\|_0^2 \\ &\geq \frac{(p-2)L}{2p}\left\|u_x(t)\right\|_0^2, \ \forall t \in [0, T_1]. \end{aligned} \tag{4.12}$$

On the other hand, with $\varepsilon_1 = 1$, it follows from (4.6) that

$$E(t) \leq E(0) + \frac{1}{2}\int_0^t \left\|F(s)\right\|_0^2 ds \leq E(0) + \frac{1}{2}\int_0^\infty \left\|F(s)\right\|_0^2 ds = E_*. \tag{4.13}$$

Therefore, from (4.12), we deduce that

$$\left\|u_x(t)\right\|_0^2 \leq \frac{2p}{(p-2)L}E(t) \leq \frac{2pE_*}{(p-2)L} \equiv R_*^2, \ \forall t \in [0, T_1]. \tag{4.14}$$

Since $(A_2)(iii)$ and (4.14), we have

$$\begin{aligned} p\int_1^R xdx\int_0^{u(x,t)} f(z)dz \leq & p\bar{d}_2 R\left(\|u(t)\|_{L^{q_1}}^{q_1}+\|u(t)\|_{L^{q_2}}^{q_2}\right) \\ \leq & p\bar{d}_2 R\left(\tilde{D}_1^{q_1}\|u_x(t)\|_0^{q_1-2}+\tilde{D}_2^{q_2}\|u_x(t)\|_0^{q_2-2}\right)\|u_x(t)\|_0^2 \\ \leq & p\bar{d}_2 R\left(\tilde{D}_1^{q_1}R_*^{q_1-2}+\tilde{D}_2^{q_2}R_*^{q_2-2}\right)\|u_x(t)\|_0^2 . \end{aligned} \tag{4.15}$$

Therefore

$$I(t)=(g\diamond u)(t)+L_*\|u_x(t)\|_0^2\geq 0,\ \forall t\in[0,T_1]. \tag{4.16}$$

Put

$$T_\infty=\sup\left\{T_1>0: I(t)>0,\ \forall t\in[0,T_1]\right\},$$

then we need to show that $T_\infty=+\infty$.

Suppose that $T_\infty<+\infty$, by the continuity of $I(t)$, we have $I(T_\infty)\geq 0$. If $I(T_\infty)=0$, by (4.16) we obtain

$$(g\diamond u)(T_\infty)=\|u_x(T_\infty)\|_0=0,$$

which leads to

$$\int_0^{T_\infty} g(T_\infty-s)\|u_x(s)\|_0^2\,ds=0,$$

by the continuity of function $s\longmapsto g(T_\infty-s)\|u_x(s)\|_0^2$ on $[0,T_\infty]$ and $g(T_\infty-s)>0$, for all $s\in[0,T_\infty]$, it implies that $u(s)=0$, for all $s\in[0,T_\infty]$, so $I(0)=0<I(0)$. This is a contradiction, thus $I(T_\infty)>0$. By the same arguments as above, we can deduce that there exists $\tilde{T}_\infty>T_\infty$ such that $I(t)>0$ for all $t\in[0,\tilde{T}_\infty]$. This is a contradiction to the definition of T_∞. Thus $T_\infty=+\infty$, i.e. $I(t)>0$, for all $t\geq 0$. This completes the proof. □

In next step, we prove the decay of solutions to (4.1). For this goal, we put

$$E_1(t)=(g\diamond u)(t)+\|u_x(t)\|_0^2+I(t), \tag{4.17}$$

then we need the lemmas below.

Lemma 4.4. *There exist two positive constants β_1, β_2 such that*

$$\beta_1 E_1(t)\leq\mathcal{L}(t)\leq\beta_2 E_1(t),\ \forall t\geq 0. \tag{4.18}$$

Proof. It is not difficult to see that

$$\begin{aligned}\mathcal{L}(t) =& \left(\frac{1}{2}-\frac{1}{p}\right)\left[(g \diamond u)(t) + (1-\bar{g}(t))\,\|u_x(t)\|_0^2\right] + \frac{1}{p}I(t) \\ &+ \frac{\delta}{2}\left(\|u(t)\|_0^2 + \|u_x(t)\|_0^2\right) \\ \geq& \frac{p-2}{2p}(g\diamond u)(t) + \frac{(p-2)L}{2p}\,\|u_x(t)\|_0^2 + \frac{1}{p}I(t) \\ \geq& \beta_1 E_1(t),\end{aligned} \tag{4.19}$$

where $\beta_1 = \min\left\{\frac{(p-2)L}{2p}, \frac{1}{p}\right\}$.

Similarly

$$\begin{aligned}\mathcal{L}(t) =& \left(\frac{1}{2}-\frac{1}{p}\right)\left[(g \diamond u)(t) + (1-\bar{g}(t))\,\|u_x(t)\|_0^2\right] \\ &+ \frac{1}{p}I(t) + \frac{\delta}{2}\left(\|u(t)\|_0^2 + \|u_x(t)\|_0^2\right) \\ \leq& \frac{p-2}{2p}(g\diamond u)(t) + \frac{p-2}{2p}\,\|u_x(t)\|_0^2 + \frac{1}{p}I(t) \\ &+ \frac{\delta}{2}\left(\frac{R(R-1)^2}{2}\,\|u_x(t)\|_0^2 + \|u_x(t)\|_0^2\right) \\ =& \frac{p-2}{2p}(g\diamond u)(t) \\ &+ \left[\frac{p-2}{2p} + \frac{\delta}{2}\left(\frac{R(R-1)^2}{2}+1\right)\right]\|u_x(t)\|_0^2 + \frac{1}{p}I(t) \\ \leq& \beta_2 E_1(t),\end{aligned} \tag{4.20}$$

where $\beta_2 = \max\left\{\frac{p-2}{2p} + \frac{\delta}{2}\left(\frac{R(R-1)^2}{2}+1\right), \frac{1}{p}\right\}$.
This completes the proof. $\square$

Lemma 4.5. *The functional $\Psi(t)$ satisfies the following estimation*

$$\begin{aligned}\Psi'(t) \leq& \frac{1}{2\varepsilon_2}\,\|F(t)\|_0^2 + \frac{1}{2\varepsilon_2}(g\diamond u)(t) - \frac{\delta_1 d_2}{p}I(t) \\ &- \left(\frac{d_2}{p}-\frac{\varepsilon_2}{2}\right)\bar{g}(t)\,\|u_x(t)\|_0^2 \\ &- \left[L - \frac{d_2}{p}\left(\,1-(1-\delta_1)L_*\right) - \frac{\varepsilon_2}{4}R(R-1)^2\right]\|u_x(t)\|_0^2,\end{aligned} \tag{4.21}$$

for all $\varepsilon_2 > 0$ and $\delta_1 \in (0,1)$.

Proof. By multiplying (4.1) by $xu(x,t)$ and integrating over Ω, we obtain

$$\begin{aligned}
\Psi'(t) =& -\left\|u_x(t)\right\|_0^2 + \langle f(u(t)), u(t)\rangle + \langle F(t), u(t)\rangle \\
&+ \int_0^t g(t-s)a(u(s), u(t))ds \\
=& -\left\|u_x(t)\right\|_0^2 + \langle f(u(t)), u(t)\rangle + \langle F(t), u(t)\rangle \\
&+ \bar{g}(t)\left\|u_x(t)\right\|_0^2 + \int_0^t g(t-s)a(u(s)-u(t), u(t))ds \\
=& -(1-\bar{g}(t))\left\|u_x(t)\right\|_0^2 + \langle f(u(t)), u(t)\rangle + \langle F(t), u(t)\rangle \\
&+ \int_0^t g(t-s)a(u(s)-u(t), u(t))ds.
\end{aligned} \tag{4.22}$$

Note that

$$\begin{aligned}
\int_0^t g(t-s)a(u(s)-u(t), u(t))ds \leq& \int_0^t g(t-s)\left\|u_x(t)-u_x(s)\right\|_0\left\|u_x(t)\right\|_0 ds \\
\leq& \frac{\varepsilon_2}{2}\bar{g}(t)\left\|u_x(t)\right\|_0^2 + \frac{1}{2\varepsilon_2}(g \diamond u)(t),
\end{aligned}$$

$$\begin{aligned}
\langle F(t), u(t)\rangle \leq& \frac{\sqrt{2R}(R-1)}{2}\left\|F(t)\right\|_0\left\|u_x(t)\right\|_0 \\
\leq& \frac{\varepsilon_2 R(R-1)^2}{4}\left\|u_x(t)\right\|_0^2 + \frac{1}{2\varepsilon_2}\left\|F(t)\right\|_0^2,\ \forall \varepsilon_2 > 0,
\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}
\langle f(u(t)), u(t)\rangle \leq& d_2\int_1^R xdx\int_0^{u(x,t)} f(z)dz \\
=& \frac{d_2}{p}\left[(g \diamond u)(t) + (1-\bar{g}(t))\left\|u_x(t)\right\|_0^2 - I(t)\right] \\
=& \frac{d_2}{p}\left[(g \diamond u)(t) + (1-\bar{g}(t))\left\|u_x(t)\right\|_0^2 - (1-\delta_1)I(t) - \delta_1 I(t)\right] \\
\leq& \frac{d_2}{p}(g \diamond u)(t) + \frac{d_2}{p}\left[1-\bar{g}(t)-(1-\delta_1)L_*\right]\left\|u_x(t)\right\|_0^2 - \frac{\delta_1 d_2}{p}I(t),
\end{aligned} \tag{4.24}$$

for all $\delta_1 \in (0,1)$. From (4.23)-(4.24), we deduce that

$$\begin{aligned}
\Psi'(t) \leq & -(1-\bar{g}(t))\left\|u_x(t)\right\|_0^2+\frac{d_2}{p}(g \diamond u)(t) \\
& +\frac{d_2}{p}\left[1-\bar{g}(t)-\left(1-\delta_1\right) L_*\right]\left\|u_x(t)\right\|_0^2-\frac{\delta_1 d_2}{p} I(t) \\
& +\frac{\varepsilon_2 R(R-1)^2}{4}\left\|u_x(t)\right\|_0^2+\frac{1}{2 \varepsilon_2}\|F(t)\|_0^2 \\
& +\frac{\varepsilon_2}{2} \bar{g}(t)\left\|u_x(t)\right\|_0^2+\frac{1}{2 \varepsilon_2}(g \diamond u)(t) \\
= & -(1-\bar{g}(t))\left\|u_x(t)\right\|_0^2 \\
& +\frac{d_2}{p}\left[1-\left(1-\delta_1\right) L_*\right]\left\|u_x(t)\right\|_0^2-\frac{\delta_1 d_2}{p} I(t) \\
& +\frac{\varepsilon_2 R(R-1)^2}{4}\left\|u_x(t)\right\|_0^2+\frac{1}{2 \varepsilon_2}\|F(t)\|_0^2 \\
& -\left(\frac{d_2}{p}-\frac{\varepsilon_2}{2}\right) \bar{g}(t)\left\|u_x(t)\right\|_0^2+\left(\frac{d_2}{p}+\frac{1}{2 \varepsilon_2}\right)(g \diamond u)(t) \\
\leq & -\left[L-\frac{d_2}{p}\left[1-\left(1-\delta_1\right) L_*\ \right]-\frac{\varepsilon_2 R(R-1)^2}{4}\right]\left\|u_x(t)\right\|_0^2 \\
& -\left(\frac{d_2}{p}-\frac{\varepsilon_2}{2}\right) \bar{g}(t)\left\|u_x(t)\right\|_0^2 \\
& -\frac{\delta_1 d_2}{p} I(t)+\frac{1}{2 \varepsilon_2}\|F(t)\|_0^2+\left(\frac{d_2}{p}+\frac{1}{2 \varepsilon_2}\right)(g \diamond u)(t) .
\end{aligned}$$

This completes the proof. □

Theorem 4.6. *Assume that* (A_1)-(A_4) *hold. Let* $I(0)>0$ *and* L_* *satisfy* (4.9). *Then, there exist positive constants* C, $\bar{\gamma}$ *such that*

$$E_1(t) \leq C e^{-\bar{\gamma} t}, \ \forall t \geq 0. \tag{4.25}$$

Proof. It follows from (4.2), (4.6) (4.21) and $\varepsilon_1 \in(0,2)$, that

$$\begin{aligned}
\mathcal{L}'(t) \leq & \tilde{F}(t)-\left(1-\frac{\varepsilon_1}{2}\right)\left\|u'(t)\right\|_0^2-\alpha\left\|u_x'(t)\right\|_0^2 \\
& -\frac{1}{2}\left[\chi_1-2 \delta\left(\frac{d_2}{p}+\frac{1}{2 \varepsilon_2}\right)\right](g \diamond u)(t)-\frac{\delta \delta_1 d_2}{p} I(t) \\
& -\delta \bar{g}(t)\left(\frac{d_2}{p}-\frac{\varepsilon_2}{2}\right)\left\|u_x(t)\right\|_0^2 \\
& -\delta\left[L-\frac{d_2}{p}\left(\ 1-\left(1-\delta_1\right) L_*\right)-\frac{\varepsilon_2}{4} R(R-1)^2\right]\left\|u_x(t)\right\|_0^2
\end{aligned} \tag{4.26}$$

$$
\begin{aligned}
\leq & \tilde{F}(t) - \frac{1}{2}\left[\chi_1 - 2\delta\left(\frac{d_2}{p} + \frac{1}{2\varepsilon_2}\right)\right](g \diamond u)(t) \\
& - \frac{\delta\delta_1 d_2}{p} I(t) - \delta\left(\frac{d_2}{p} - \frac{\varepsilon_2}{2}\right)\bar{g}(t)\left\|u_x(t)\right\|_0^2 \\
& - \delta\left[L - \frac{d_2}{p}\left(\, 1 - (1-\delta_1)L_*\right) - \frac{\varepsilon_2}{4}R(R-1)^2\right]\left\|u_x(t)\right\|_0^2,
\end{aligned}
$$

for all δ, $\varepsilon_2 > 0$, $0 < \delta_1 < 1$, where $\tilde{F}(t) = \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\|F(t)\|_0^2$.

Since $L - \frac{d_2}{p}(1 - L_*) = \frac{d_2}{p}\left[L_* - \left(1 - \frac{p}{d_2}\right) - \frac{p}{d_2}\int_0^\infty g(t)dt\right] > 0,$

$$
\lim_{\varepsilon_2 \to 0_+}\left(\frac{d_2}{p} - \frac{\varepsilon_2}{2}\right) = \frac{d_2}{p} > 0
$$

and

$$
\begin{aligned}
& \lim_{\varepsilon_2 \to 0_+,\ \delta_1 \to 0_+}\left[L - \frac{d_2}{p}\left(\, 1 - (1-\delta_1)L_*\right) - \frac{\varepsilon_2}{4}R(R-1)^2\right] \\
& = L - \frac{d_2}{p}(1 - L_*) > 0,
\end{aligned}
$$

we can choose $\varepsilon_2 > 0$, $\delta_1 \in (0,1)$ such that $\frac{d_2}{p} - \frac{\varepsilon_2}{2} > 0$, and

$$
\sigma_1 = \sigma_1(\varepsilon_2, \delta_1) = L - \frac{d_2}{p}\left(\, 1 - (1-\delta_1)L_*\right) - \frac{\varepsilon_2}{4}R(R-1)^2 > 0. \tag{4.27}
$$

Afterward, by $\chi_1 > 0$, we can choose δ such that

$$
\sigma_2 = \sigma_2(\delta) = \frac{1}{2}\left[\chi_1 - 2\delta\left(\frac{d_2}{p} + \frac{1}{2\varepsilon_2}\right)\right] > 0. \tag{4.28}
$$

Using the hypothesis (A_3), we deduce from (4.2), (4.6) and (4.21) that

$$
\begin{aligned}
\mathcal{L}'(t) \leq & - \gamma_1\Big((g \diamond u)(t) + \left\|u_x(t)\right\|_0^2 + I(t)\Big) + C_* e^{-2\gamma_0 t} \\
\leq & - \gamma\mathcal{L}(t) + C_* e^{-2\gamma_0 t},
\end{aligned} \tag{4.29}
$$

where

$$
\begin{aligned}
C_* = & \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right)\bar{C}_0^2, \\
\gamma_1 = & \min\left\{\frac{\delta\delta_1 d_2}{p}, \delta\sigma_1, \sigma_2\right\} > 0, \\
0 < & \gamma < \min\left\{\frac{\gamma_1}{\beta_2}, 2\gamma_0\right\}.
\end{aligned}
$$

By integrating (4.29), we deduce

$$\mathcal{L}(t) \leq \left(\mathcal{L}(0) + \frac{C_*}{2\gamma_0 - \gamma} \right) e^{-\gamma t}, \ \forall t \geq 0. \tag{4.30}$$

This implies (4.25) and Theorem 4.6 is proved. □

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