# RELATIVE LOGARITHMIC ORDER OF AN ENTIRE FUNCTION 

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#### Abstract

In this paper, we extend some results related to the growth rates of entire functions by introducing the relative logarithmic order $\rho_{g}^{l}(f)$ of a nonconstant entire function $f$ with respect to another nonconstant entire function $g$. Next we investigate some theorems related the behavior of $\rho_{g}^{l}(f)$. We also define the relative logarithmic proximate order of $f$ with respect to $g$ and give some theorems on it.


## 1. Introduction

Let $f$ be a nonconstant entire function. Then the maximum modulus function $M_{f}(r)$ of $f$, defined by $M_{f}(r)=\max _{|z|=r}|f(z)|$ is continuous and strictly increasing function of $r$. In such case the inverse function $M_{f}^{-1}:(|f(0)|, \infty) \rightarrow(0, \infty)$ exists and is also continuous, strictly increasing and $\lim _{s \rightarrow \infty} M_{f}^{-1}(s)=\infty$. The growth of an entire function $f$ is generally measured by its order and type.

In 1988, Luis Bernal [1] introduced the order of growth of a nonconstant entire function $f$ relative to another entire function $g$, which is defined by

$$
\begin{aligned}
\rho_{g}(f) & =\inf \left\{\mu>0: M_{f}(r)<M_{g}\left(r^{\mu}\right), \text { for all } r>r_{0}\right\} \\
& =\limsup _{r \rightarrow \infty} \frac{\log M_{g}^{-1}\left(M_{f}(r)\right)}{\log r}
\end{aligned}
$$

In general, techniques that work well for functions of finite positive order often do not work for functions of order zero. In order to make some progress with functions of order zero, in 2005, P. T. Y. Chern [2] defined the logarithmic order of an entire function $f$, given by

$$
\rho^{l}=\limsup _{r \rightarrow \infty} \frac{\log ^{+} \log ^{+} M_{f}(r)}{\log \log r},
$$

where $\log ^{+} x=\max \{\log x, 0\}$, for all $x \geq 0$.
In this paper we want to sort out the same type of limitations, occurring for the functions of relative order zero, by introducing the relative logarithmic order of $f$ with respect to $g, \rho_{g}^{l}(f)$, for two nonconstant entire functions $f$ and $g$. And then we investigate some theorems related the behavior of $\rho_{g}^{l}(f)$.

[^0]Moreover, In 1923, Valiron [7] initiated the terminology and generalized the concept of proximate order and in 1946, S. M. Shah [6] defined it in more justified form and gave a simple proof of its existence. In this paper, we also define the relative logarithmic proximate order of $f$ with respect to $g$.

## 2. Basic definitions and preliminary lemmas

In this section we state some definitions and lemmas which will be used to prove our main results.

Definition 2.1. A nonconstant entire function $f$ is said to be satisfy the property $(A)$ if and only if for each $\sigma>1$,

$$
M_{f}(r)^{2} \leq M_{f}\left(r^{\sigma}\right)
$$

exists.
For example $\exp z, \cos z$ etc satisfy the property $(A)$. But no polynomial satisfies property $(A)$. Moreover, there are some transcendental functions which do not satisfy property $(A)$.

Lemma 2.2. [1] Let $f$ be a nonconstant entire function, then $f$ satisfies the property $(A)$ if and only if for each $\sigma>1$ and positive integer $n$,

$$
M_{f}(r)^{n} \leq M_{f}\left(r^{\sigma}\right), \text { for all } r>0
$$

Lemma 2.3. [1] Let $f$ be a nonconstant entire function, $\alpha>1,0<\beta<\alpha, s>$ $1,0<\mu<\lambda$ and $n$ be a positive integer. Then
a) $M_{f}(\alpha r)>\beta M_{f}(r)$.
b) There exist, $K=K(s, f)>0$ such that

$$
f(r)^{s} \leq K M_{f}\left(r^{s}\right), \text { for all } r>0
$$

c) $\lim _{r \rightarrow \infty} \frac{M_{f}\left(r^{s}\right)}{M_{f}(r)}=\infty=\lim _{r \rightarrow \infty} \frac{M_{f}\left(r^{\lambda}\right)}{M_{f}\left(r^{\mu}\right)}$.
d) If $f$ is transcendental, then $\lim _{r \rightarrow \infty} \frac{M_{f}\left(r^{s}\right)}{r^{n} M_{f}(r)}=\infty=\lim _{r \rightarrow \infty} \frac{M_{f}\left(r^{\lambda}\right)}{r^{n} M_{f}\left(r^{\mu}\right)}$.

Lemma 2.4. [1] Suppose that $f$ and $g$ are entire functions, $f(0)=0$ and $h=g \circ f$. Then there exist $c \in(0,1)$, independent of $f$ and $g$, such that

$$
M_{h}(r)>M_{g}\left(c M_{f}\left(\frac{r}{2}\right)\right), \text { for all } r>0
$$

Lemma 2.5. [1] Let $R>0, \eta \in\left(0, \frac{3 e}{2}\right)$ and $f$ be analytic in $|z| \leq 2 e R$ with $f(0)=1$. Then on the disc $|z| \leq R$, excluding a family of discs the sum of whose radii is not greater than $4 \eta R$, it is verified that

$$
\log |f(z)|>-T(\eta) \log M_{f}(2 e R)
$$

where $T(\eta)=2+\log \left(\frac{3 e}{2 \eta}\right)$.
Lemma 2.6. [1] Let $f$ be a nonconstant entire function and $A(r)=\max \{\operatorname{Re} f(z)$ : $|z|=r\}$, then

$$
M_{f}(r)<A(145 r) .
$$

Lemma 2.7. [1] Let $f$ be a nonconstant entire function, then

$$
T(r) \leq \log ^{+} M_{f}(r) \leq\left(\frac{R+r}{R-r}\right) T(r), \text { for } 0<r<R
$$

## 3. Main Results

In this section we first define the relative logarithmic order of $f$ with respect to $g$, relative logarithmic lower order of $f$ with respect to $g$ and then establish some theorems related to these. Finally we introduce the relative logarithmic proximate order of $f$ with respect to $g$.

Definition 3.1 (Relative logarithmic order of f with respect to g ). Let $f$ and $g$ be two entire functions. The relative logarithmic order of $f$ with respect to $g$ is given by

$$
\begin{aligned}
\rho_{g}^{l}(f) & =\inf \left\{\mu>0: M_{f}(r)<M_{g}\left((\log r)^{\mu}\right), \text { for all } r>r_{0}(\mu)>0\right\} \\
& =\limsup _{r \rightarrow \infty} \frac{\log ^{+} M_{g}^{-1}\left(M_{f}(r)\right)}{\log \log r} .
\end{aligned}
$$

Definition 3.2 (Relative logarithmic lower order of f with respect to g ). Let $f$ and $g$ be two entire functions. The relative logarithmic lower order of $f$ with respect to $g$ is given by

$$
\lambda_{g}^{l}(f)=\underset{r \rightarrow \infty}{\liminf } \frac{\log ^{+} M_{g}^{-1}\left(M_{f}(r)\right)}{\log \log r}
$$

### 3.1. Some general properties on relative logarithmic order.

Theorem 3.3. Let $f, g$, $h$ be nonconstant entire functions and $L_{i}(i=1,2,3,4)$ are nonconstant linear functions, i.e. $L_{i}(z)=a_{i} z+b_{i}$, for all $z \in \mathbb{C}$, with $a_{i}, b_{i} \in \mathbb{C}, a_{i} \neq$ $0(i=1,2,3,4)$. Then
a) $\rho_{g}^{l}(f)=\limsup _{r \rightarrow \infty} \frac{\log ^{+} M_{g}^{-1}\left(M_{f}(r)\right)}{\log \log r}=\limsup _{r \rightarrow \infty} \frac{\log ^{+} M_{g}^{-1}(r)}{\log _{\log }{ }^{+} M_{f}^{-1}(r)}$,
b) If $g$ is a polynomial and $f$ is a transcendental, then $\rho_{g}^{l}(f)=\infty$,
c) If $f$ and $g$ are polynomials, then $\rho_{g}^{l}(f)=\infty$,
d) If $M_{f}(r) \leq M_{g}(r)$, then we have $\rho_{h}^{l}(f) \leq \rho_{h}^{l}(g)$,
e) If $M_{g}(r) \leq M_{h}(r)$, then we have $\rho_{g}^{l}(f) \geq \rho_{h}^{l}(f)$,
f) $\rho_{\left(L_{4} \circ g \circ L_{3}\right)}^{l}\left(L_{2} \circ f \circ L_{1}\right)=\rho_{g}^{l}(f)$.

Proof. a) This follows from the definition.
b) Let the degree of $g$ be $n$. Then $M_{g}(r) \leq K r^{n}$ and $M_{f}(r)>L r^{m}$, where $K, L$ are constant and $m>0$ be any real number, for sufficiently large $r$.

Then,

$$
\begin{aligned}
\frac{\log ^{+} M_{g}^{-1}\left(M_{f}(r)\right)}{\log \log r} & >\frac{\log ^{+} M_{g}^{-1}\left(L r^{m}\right)}{\log \log r} \\
& \geq \frac{\log ^{+}\left(\frac{1}{K}\left(L r^{m}\right)^{\frac{1}{n}}\right)}{\log \log r} \\
& =\frac{\log ^{+} \frac{L^{\frac{1}{n}}}{K}+\log ^{+} r^{\frac{m}{n}}}{\log \log r} \\
& =\frac{m}{n} \frac{\log ^{+} r}{\log \log r}+\frac{\log ^{+} \frac{L^{\frac{1}{n}}}{K}}{\log \log r},
\end{aligned}
$$

which tends to $\infty$ as $r \rightarrow \infty$.
Hence,

$$
\rho_{g}^{l}(f)=\limsup _{r \rightarrow \infty} \frac{\log ^{+} M_{g}^{-1}\left(M_{f}(r)\right)}{\log \log r}=\infty .
$$

c) Let

$$
f(z)=a_{0} z^{m}+a_{1} z^{m-1}+\ldots+a^{m}, a_{0} \neq 0
$$

and

$$
g(z)=b_{0} z^{n}+b_{1} z^{n-1}+\ldots+b^{n}, b_{0} \neq 0 .
$$

Then $M_{f}(r) \geq \frac{1}{2}\left|a_{0}\right| r^{m}$ and $M_{g}(r) \leq K r^{n}$, where $K$ is a constants, for sufficiently large $r$.

Then,

$$
\begin{aligned}
\frac{\log ^{+} M_{g}^{-1}\left(M_{f}(r)\right)}{\log \log r} & \geq \frac{\log ^{+} M_{g}^{-1}\left(\frac{1}{2}\left|a_{0}\right| r^{m}\right)}{\log \log r} \\
& \geq \frac{\log ^{+}\left(\frac{1}{K}\left(\frac{1}{2}\left|a_{0}\right| r^{m}\right)^{\frac{1}{n}}\right)}{\log \log r} \\
& =\frac{m}{n} \frac{\log ^{+} r}{\log \log r}+\frac{\log ^{+}\left(\frac{\left|a_{0}\right|^{\frac{1}{n}}}{2^{\frac{1}{n}} K}\right)}{\log \log r}
\end{aligned}
$$

which tends to $\infty$ as $r \rightarrow \infty$.
Hence,

$$
\rho_{g}^{l}(f)=\limsup _{r \rightarrow \infty} \frac{\log ^{+} M_{g}^{-1}\left(M_{f}(r)\right)}{\log \log r}=\infty .
$$

Proofs of d), e) and f) are omitted.
Remark 3.4. If $f$ is a polynomial and $g$ is a transcendental, then $\rho_{g}^{l}(f)$ may be zero or a positive finite number.

Example 3.5. Let $f(z)=z$ and $g(z)=e^{z}$.
Then, $M_{f}(r)=r$ and $M_{g}(r)=e^{r}$.
Therefore

$$
\rho_{g}^{l}(f)=\underset{r \rightarrow \infty}{\limsup } \frac{\log \log r}{\log \log r}=1 .
$$

Example 3.6. Let $f(z)=z$ and $g(z)=e^{e^{z}}$.
Then, $M_{f}(r)=r$ and $M_{g}(r)=e^{e^{r}}$.
Therefore

$$
\rho_{g}^{l}(f)=\limsup _{r \rightarrow \infty} \frac{\log \log \log r}{\log \log r}=0 .
$$

### 3.2. Relative logarithmic order of composition.

Theorem 3.7. Let $f, f_{1}, f_{2}, g$ and $m$ be nonconstant entire functions and $h=g \circ f$, then
a) $\rho_{g \circ f_{2}}^{l}\left(g \circ f_{1}\right)=\rho_{f_{2}}^{l}\left(f_{1}\right)$,
b) $\max \left\{\rho_{m}^{l}(f), \rho_{m}^{l}(g)\right\} \leq \rho_{m}^{l}(h)$,
c) If $f$ is a polynomial, then $\rho_{m}^{l}(h)=\rho_{m}^{l}(g)$ and $\rho_{g}^{l}(h)=\infty$.

Proof. a) Let $h_{i}=g \circ f_{i},(i=1,2)$. Then $h_{i}$ is a nonconstant entire function.
We can suppose that $f_{i}(0)=0$, if not, we take $f_{i}^{*}(z)=f_{i}(z)-f_{i}(0)$ and $g_{i}^{*}(z)=$ $g\left(z+f_{i}(0)\right)$ and we would have $h_{i}=g_{i}^{*} \circ f_{i}^{*}$, and by the first Theorem [f) part ], we $\operatorname{get} \rho_{f_{2}^{*}}^{l}\left(f_{1}^{*}\right)=\rho_{f_{2}}^{l}\left(f_{1}\right)$.

So, without loss of generality we take $f_{i}(0)=0$.
We have by Lemma 2.4

$$
M_{h_{i}}(r) \geq M_{g}\left(c M_{f_{i}}\left(\frac{r}{2}\right)\right), \text { for all } r>0, i=1,2 .
$$

Again using Lemma 2.3we have

$$
\begin{aligned}
M_{f_{i}}\left(\frac{1}{d} \cdot \frac{d r}{2}\right) & >\frac{1}{c} \cdot M_{f_{i}}\left(\frac{d r}{2}\right) \\
& \Rightarrow M_{f_{i}}\left(\frac{r}{2}\right)>\frac{1}{c} M_{f_{i}}\left(\frac{d r}{2}\right) \text { for all } d \in(0, c), \text { since } M_{h_{i}} \leq M_{g} \circ M_{f_{i}} .
\end{aligned}
$$

Then

$$
\begin{equation*}
M_{h_{i}}(r)>M_{g}\left(M_{f_{i}}\left(\frac{d r}{2}\right)\right) \geq M_{h_{i}}\left(\frac{d r}{2}\right), \text { for } i=1,2 . \tag{1}
\end{equation*}
$$

Again from (1)

$$
\begin{aligned}
M_{h_{1}}(r) & >M_{g}\left(M_{f_{1}}\left(\frac{d r}{2}\right)\right) \\
& \Rightarrow M_{h_{2}}^{-1}\left(M_{h_{1}}(r)\right)>M_{h_{2}}^{-1}\left(M_{g}\left(M_{f_{1}}\left(\frac{d r}{2}\right)\right)\right)
\end{aligned}
$$

Again since, $M_{h_{2}}^{-1} \circ M_{g}(t) \geq M_{f_{2}}^{-1}(t)$,

$$
\begin{equation*}
\left.M_{h_{2}}^{-1}\left(M_{h_{1}}(r)\right)>M_{h_{2}}^{-1}\left(M_{g}\left(M_{f_{1}}\left(\frac{d r}{2}\right)\right)\right)\right)>M_{f_{2}}^{-1}\left(M_{f_{1}}\left(\frac{d r}{2}\right)\right) . \tag{2}
\end{equation*}
$$

In (1), for $i=2$, we put $M_{h_{2}}(r)=t$. i.e., $r=M_{h_{2}}^{-1}(t)$ and we get

$$
\begin{aligned}
t & >M_{g}\left(M_{f_{2}}\left(\frac{d}{2} M_{h_{2}}^{-1}(t)\right)\right) \\
M_{f_{2}}^{-1}\left(M_{g}^{-1}(t)\right) & >\frac{d}{2} M_{h_{2}}^{-1}(t) \Rightarrow M_{h_{2}}^{-1}(t)<\frac{2}{d} M_{f_{2}}^{-1}\left(M_{g}^{-1}(t)\right) .
\end{aligned}
$$

Putting $t=M_{h_{1}}(r)$, we have

$$
\begin{aligned}
M_{h_{2}}^{-1}\left(M_{h_{1}}(r)\right) & <\frac{2}{d} M_{f_{2}}^{-1}\left(M_{g}^{-1}\left(M_{h_{1}}(r)\right)\right) \\
& \leq \frac{2}{d} M_{f_{2}}^{-1}\left(M_{f_{1}}(r)\right) .
\end{aligned}
$$

Combining (2) and (3) we have,

$$
M_{f_{2}}^{-1}\left(M_{f_{1}}\left(\frac{d r}{2}\right)\right)<M_{h_{2}}^{-1}\left(M_{h_{1}}(r)\right)<\frac{2}{d} M_{f_{2}}^{-1}\left(M_{f_{1}}(r)\right) .
$$

Taking logarithm and dividing by $\log \log r$ and then taking $\lim \sup$ as $r \rightarrow \infty$, we get

$$
\rho_{g \circ f_{2}}^{l}\left(g \circ f_{1}\right)=\rho_{f_{2}}^{l}\left(f_{1}\right) .
$$

b) As in part (a), we can assume that $f(0)=0$.

Since $f$ and $g$ are nonconstant, there exist $\alpha>0$ such that $M_{f}(r)>\alpha r$ and $M_{g}(r)>\alpha r$.

Applying the Lemma $2.4 c \in(0,1)$ such that

$$
\begin{aligned}
M_{h}(r) & \geq M_{g}\left(c M_{f}\left(\frac{r}{2}\right)\right)>\alpha . c . M_{f}\left(\frac{r}{2}\right)>M_{f}\left(r^{\sigma}\right), \text { for sufficiently large } r, \\
& \Rightarrow M_{m}^{-1}\left(M_{h}(r)\right)>M_{m}^{-1}\left(M_{f}\left(r^{\sigma}\right)\right),
\end{aligned}
$$

and also

$$
M_{h}(r)>M_{g}\left(c M_{f}\left(\frac{r}{2}\right)\right)>M_{g}\left(c \alpha \frac{r}{2}\right)>M_{g}\left(r^{\sigma}\right), \text { for sufficiently large } r \text {. }
$$

Taking logarithm and dividing by $\log \log r$ and using (3), we get

$$
\begin{aligned}
\frac{\log ^{+} M_{m}^{-1}\left(M_{h}(r)\right)}{\log \log r} & >\frac{\log ^{+} M_{m}^{-1}\left(M_{f}\left(r^{\sigma}\right)\right)}{\log \log r} \\
& =\frac{\log ^{+} M_{m}^{-1}\left(M_{f}(s)\right)}{\log \log s^{\frac{1}{\sigma}}},\left[\text { taking } r^{\sigma}=s\right] \\
& =\frac{\log ^{+} M_{m}^{-1}\left(M_{f}(s)\right)}{\log \left(\frac{1}{\sigma} \log s\right)} \\
& =\frac{\log ^{+} M_{m}^{-1}\left(M_{f}(s)\right)}{\log \frac{1}{\sigma}+\log \log s} \\
& =\frac{\frac{\log +M_{m}^{-1}\left(M_{f}(s)\right)}{\log \log s}}{\frac{\log \frac{1}{\sigma}}{\log \log s}+1}
\end{aligned}
$$

Now taking limsup as $r \rightarrow \infty$, we get

$$
\begin{aligned}
\rho_{m}^{l}(h) & \geq \limsup _{s \rightarrow \infty} \frac{\log ^{+} M_{m}^{-1}\left(M_{f}(s)\right)}{\log \log s},[\text { since } s \rightarrow \infty \text { as } r=\infty] \\
& \Rightarrow \rho_{m}^{l}(h) \geq \rho_{m}^{l}(f) .
\end{aligned}
$$

Similarly from (3.7), we get

$$
\rho_{m}^{l}(h) \geq \rho_{m}^{l}(g) .
$$

From the above two results (b) follows.
c) Let $f$ be a polynomial of degree $n \geq 1$, then there exist $\alpha>0, \beta>0$ such that $\alpha r^{n}<M_{f}(r)<\beta r^{n}$.

So, using Lemma 2.4

$$
\begin{aligned}
M_{g}\left(\gamma r^{n}\right) & <M_{g}\left(c M_{f}\left(\frac{r}{2}\right)\right) \leq M_{h}(r) \leq M_{g}\left(M_{f}(r)\right)<M_{g}\left(\beta r^{n}\right) \\
& \Rightarrow M_{m}^{-1}\left(M_{g}\left(\gamma r^{n}\right)\right)<M_{m}^{-1}\left(M_{h}(r)\right)<M_{m}^{-1}\left(M_{g}\left(\beta r^{n}\right)\right)
\end{aligned}
$$

where $\gamma=c\left(\frac{\alpha}{2}\right)^{n}$.
Taking logarithms and dividing by $\log \log r$, we get

$$
\begin{aligned}
& \frac{\log ^{+} M_{m}^{-1}\left(M_{g}\left(\gamma r^{n}\right)\right)}{\log \log r}<\frac{\log ^{+} M_{m}^{-1}\left(M_{h}(r)\right)}{\log \log r}<\frac{\log ^{+} M_{m}^{-1}\left(M_{g}\left(\beta r^{n}\right)\right)}{\log \log r} \\
& \Rightarrow \frac{\log ^{+} M_{m}^{-1}\left(M_{g}(s)\right)}{\log \log \left(\frac{s}{\gamma}\right)^{\frac{1}{n}}}<\frac{\log ^{+} M_{m}^{-1}\left(M_{h}(r)\right)}{\log \log r} \\
& \quad<\frac{\log ^{+} M_{m}^{-1}\left(M_{g}(t)\right)}{\log \log \left(\frac{t}{\beta}\right)^{\frac{1}{n}}}\left[\text { taking } \gamma r^{n}=s \text { and } \beta r^{n}=t\right] \\
& \Rightarrow \frac{\log ^{+} M_{m}^{-1}\left(M_{g}(s)\right)}{\log \frac{1}{n}+\log \log \left(\frac{s}{\gamma}\right)}<\frac{\log ^{+} M_{m}^{-1}\left(M_{h}(r)\right)}{\log \log r}<\frac{\log ^{+} M_{m}^{-1}\left(M_{g}(t)\right)}{\log \frac{1}{n}+\log \log \left(\frac{t}{\beta}\right)} \\
& \Rightarrow \quad \frac{\frac{\log +M_{m}^{-1}\left(M_{g}(s)\right)}{\log \log s}}{\frac{\log \frac{1}{n}}{\log \log s}+\frac{\log \log \left(\frac{s}{\gamma}\right)}{\log \log s}}<\frac{\log ^{+} M_{m}^{-1}\left(M_{h}(r)\right)}{\log \log r}<\frac{\frac{\log M_{m}^{-1}\left(M_{g}(t)\right)}{\log \log s}}{\frac{\log \frac{1}{n}}{\log \log t}+\frac{\log \log \left(\frac{t}{f}\right)}{\log \log t}}
\end{aligned}
$$

Now taking limsup as $r \rightarrow \infty$, we get

$$
\rho_{m}^{l}(h)=\rho_{m}^{l}(g)
$$

Again from (4), we get

$$
\begin{aligned}
M_{g}^{-1}\left(M_{h}(r)\right) & >M_{g}^{-1}\left(M_{g}\left(\gamma r^{n}\right)\right)=\gamma r^{n} \\
& \Rightarrow \frac{\log ^{+} M_{g}^{-1}\left(M_{h}(r)\right)}{\log \log r}>\frac{\log ^{+} \gamma+n \log ^{+} r}{\log \log r}
\end{aligned}
$$

which tends to $\infty$ as $r \rightarrow \infty$.
Hence,

$$
\rho_{g}^{l}(h)=\limsup _{r \rightarrow \infty} \frac{\log ^{+} M_{g}^{-1}\left(M_{h}(r)\right)}{\log \log r}=\infty .
$$

### 3.3. Relative logarithmic order of sum and product.

Theorem 3.8. Let $f, g, f_{1}, f_{2}$ be nonconstant entire functions and $P$ be a polynomial not identically zero. Then
a) $\rho_{g}^{l}\left(f_{1}+f_{2}\right) \leq \max \left\{\rho_{g}^{l}\left(f_{1}\right), \rho_{g}^{l}\left(f_{2}\right)\right\}$, giving equality if $\rho_{g}^{l}\left(f_{1}\right) \neq \rho_{g}^{l}\left(f_{2}\right)$,
b) If $f$ is transcendent, then $\rho_{g}^{l}(P f)=\rho_{g}^{l}(f)$, and if $g$ is transcendent, then $\rho_{P g}^{l}(f)=$ $\rho_{g}^{l}(f)$,
c) $\rho_{g}^{l}(f)=\rho_{g}^{l}\left(f^{n}\right)$, where $n$ is a positive integer.
d) if $g$ satisfies property $(A)$, then $\rho_{g}^{l}\left(f_{1} f_{2}\right) \leq \max \left\{\rho_{g}^{l}\left(f_{1}\right), \rho_{g}^{l}\left(f_{2}\right)\right\}$, giving equality if $\rho_{g}^{l}\left(f_{1}\right) \neq \rho_{g}^{l}\left(f_{2}\right)$.

The same result is true for $\frac{f_{1}}{f_{2}}$, assuming it is an entire function.
Proof. a) Let $h=f_{1}+f_{2}, \rho^{l}=\rho_{g}^{l}(h), \rho_{i}^{l}=\rho_{g}^{l}\left(f_{i}\right)$ for $i=1,2$.
If $h$ is constant, then it is trivial.
Suppose $h$ is not a constant and $\rho_{1}^{l} \leq \rho_{2}^{l}$.
Given $\varepsilon>0$,

$$
M_{f_{1}}(r) \leq M_{g}\left((\log r)^{\rho_{1}^{\rho}+\varepsilon}\right) \leq M_{g}\left((\log r)^{\rho_{2}^{\rho}+\varepsilon}\right)
$$

and

$$
M_{f_{2}}(r) \leq M_{g}\left((\log r)^{\rho_{2}^{\rho}+\varepsilon}\right),
$$

for $r>r_{0}$.
Therefore,

$$
M_{h}(r) \leq M_{f_{1}}(r)+M_{f_{2}}(r) \leq 2 M_{g}\left((\log r)^{\rho_{2}^{l}+\varepsilon}\right)<M_{g}\left(3(\log r)^{\rho_{2}^{l}+\varepsilon}\right) .
$$

Taking logarithm and dividing by $\log \log r$, we get

$$
\begin{aligned}
\frac{\log ^{+} M_{g}^{-1} M_{h}(r)}{\log \log r} & <\frac{\log ^{+} 3+\left(\rho_{2}^{l}+\varepsilon\right) \log ^{+} \log r}{\log \log r} \\
& \Rightarrow \limsup _{r \rightarrow \infty} \frac{\log ^{+} M_{g}^{-1} M_{h}(r)}{\log \log r} \\
& <\limsup _{r \rightarrow \infty} \frac{\log ^{+} 3+\left(\rho_{2}^{l}+\varepsilon\right) \log ^{+} \log r}{\log \log r} \\
& =\rho_{2}^{l}+\varepsilon \\
& \Rightarrow \rho_{g}^{l}(h) \leq \rho_{2}^{l}+\varepsilon, \text { for each } \varepsilon>0,
\end{aligned}
$$

and consequently,

$$
\rho^{l} \leq \rho_{2}^{l}=\max \left\{\rho_{1}^{l}, \rho_{2}^{l}\right\} .
$$

Now suppose that, $\rho_{1}^{l}<\rho_{2}^{l}$ and take $\lambda \in\left(\rho_{1}^{l}, \rho_{2}^{l}\right)$ and $\mu \in\left(\rho_{1}^{l}, \lambda\right)$.
Then, $M_{f_{1}}(r)<M_{g}\left((\log r)^{\mu}\right)$ and there is a sequence $\left\{r_{n}\right\} \rightarrow \infty$ with $M_{g}\left(\left(\log r_{n}\right)^{\lambda}\right)<$ $M_{f_{2}}(r)$, for all $n$.

Again by Lemma 2.3¿2 $M_{g}\left((\log r)^{\mu}\right)$.
Therefore
$2 M_{f_{1}}\left(r_{n}\right)<2 M_{g}\left(\left(\log r_{n}\right)^{\mu}\right)<M_{g}\left(\left(\log r_{n}\right)^{\lambda}\right)<M_{f_{2}}\left(r_{n}\right)$ for sufficiently large $n$.
Which implies

$$
\begin{aligned}
M_{h}\left(r_{n}\right) & \geq M_{f_{2}}\left(r_{n}\right)-M_{f_{1}}\left(r_{n}\right) \\
& >\frac{1}{2} M_{f_{2}}\left(r_{n}\right) \\
& >\frac{1}{2} M_{g}\left(\left(\log r_{n}\right)^{\lambda}\right) \\
& \left.>M_{g}\left(\frac{1}{3}\left(\log r_{n}\right)^{\lambda}\right), \text { for } n \text { sufficiently large } n, \text { by Lemma } 2.3 a\right) .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\rho^{l} & \geq \limsup _{r \rightarrow \infty} \frac{\log ^{+} M_{g}^{-1} M_{h}\left(r_{n}\right)}{\log \log r_{n}} \\
& >\limsup _{r \rightarrow \infty} \frac{\log ^{+} \frac{1}{3}+\lambda \log { }^{+} \log r_{n}}{\log \log r_{n}}=\lambda, \text { for each } \lambda \in\left(\rho_{1}^{l}, \rho_{2}^{l}\right) .
\end{aligned}
$$

So, $\rho^{l} \geq \rho_{2}^{l}=\max \left\{\rho_{1}^{l}, \rho_{2}^{l}\right\}$.
Hence

$$
\rho^{l}=\max \left\{\rho_{1}^{l}, \rho_{2}^{l}\right\} .
$$

b) Since $P(z)$ is a polynomial there exists a real number $\alpha>0$ and a positive integer $n$ such that

$$
2 \alpha<|P(z)|<r^{n} \quad(|z|=r)
$$

Since $f$ is transcendental, $h=P f$ and $s>1$, then

$$
\begin{aligned}
M_{f}(\alpha r) & \left.<2 \alpha M_{f}(r), \text { using Lemma } 2.3 a\right) \\
& <|P(z)| M_{f}(r), \text { on }|z|=r \\
& =M_{h}(r) \\
& <r^{n} M_{f}(r) \\
& \left.<M_{f}\left(r^{s}\right), \text { using Lemma } 2.3 d\right), \text { for sufficiently large } r .
\end{aligned}
$$

Therefore

$$
M_{g}^{-1} M_{f}(\alpha r)<M_{g}^{-1} M_{h}(r)<M_{g}^{-1} M_{f}\left(r^{s}\right)
$$

$\Rightarrow \frac{\log ^{+} M_{g}^{-1} M_{f}(\alpha r)}{\log \log (\alpha r)} \cdot \frac{\log ^{+} \log (\alpha r)}{\log \log (r)}<\frac{\log ^{+} M_{g}^{-1} M_{h}(r)}{\log \log r}<\frac{\log ^{+} M_{g}^{-1} M_{f}\left(r^{s}\right)}{\log \log r^{s}} \cdot \frac{\log \log r^{s}}{\log \log r}$
Taking lim sup as $r \rightarrow \infty$, we have

$$
\begin{aligned}
\rho_{g}^{l}(f) \cdot 1 & \leq \rho_{g}^{l}(h) \leq \rho_{g}^{l}(f) \cdot 1 \\
& \Rightarrow \rho_{g}^{l}(h)=\rho_{g}^{l}(f)
\end{aligned}
$$

c) It is obvious that $\rho_{g}^{l}\left(f^{n}\right) \geq \rho_{g}^{l}(f)$.

Let $M_{f^{n}}(r)=\max \left\{\left|f^{n}(z)\right|:|z|=r\right\}$.
Therefore

$$
\begin{aligned}
M_{f^{n}}(r) & \left.\left.\leq K M_{f}\left(r^{n}\right)<M_{f}\left((K+1) r^{n}\right), \text { by Lemma } 2.3 a\right), 2.3 b\right) \\
& \Rightarrow M_{g}^{-1} M_{f^{n}}(r)<M_{g}^{-1} M_{f}\left((K+1) r^{n}\right) \\
& \Rightarrow \frac{\log ^{+} M_{g}^{-1} M_{f^{n}}(r)}{\log \log r}<\frac{\log ^{+} M_{g}^{-1} M_{f}\left((K+1) r^{n}\right)}{\log \log r}
\end{aligned}
$$

Taking limsup as $r \rightarrow \infty$, we get

$$
\rho_{g}^{l}\left(f^{n}\right) \leq \rho_{g}^{l}(f) .
$$

Therefore

$$
\rho_{g}^{l}\left(f^{n}\right)=\rho_{g}^{l}(f) .
$$

d) Let us assume $f_{1}, f_{2}$ be transcendental, otherwise it would be trivial.

Denote $h=f_{1} f_{2}, \rho^{l}=\rho_{g}^{l}(h), \rho_{i}^{l}=\rho_{g}^{l}\left(f_{i}\right),(i=1,2)$.
First we assume $\rho_{1} \leq \rho_{2}<\infty$ (If $\rho_{2}=\infty$, it is trivial)
Now given $\varepsilon>0$,

$$
M_{f_{i}}(r)<M_{g}\left((\log r)^{\rho_{2}+\frac{\varepsilon}{2}}\right), i=1,2 .
$$

Then

$$
M_{h}(r) \leq M_{f_{1}}(r) M_{f_{2}}(r)<\left(M_{g}\left((\log r)^{\rho_{2}+\frac{\varepsilon}{2}}\right)\right)^{2}<M_{g}\left((\log r)^{\rho_{2}+\varepsilon}\right),
$$

applying Property $(A)$, with $\sigma=\frac{\rho_{2}^{l}+\varepsilon}{\rho_{2}^{L}+\frac{\varepsilon}{2}}$.
Then

$$
\rho^{l} \leq \rho_{2}^{l}=\max \left\{\rho_{1}^{l}, \rho_{2}^{l}\right\} .
$$

Next suppose that, $\rho_{1}^{l}<\rho_{2}^{l}$.
From part b) we have, the product of $f$ by a factor $\frac{c}{z^{n}}$ does not alter its order, so we can assume without loss of generality that $f_{1}(0)=1$.

Take $\lambda, \mu$ with $\rho_{1}^{l}<\mu<\lambda<\rho_{2}^{l}$.
Then there is a succession $R_{n} \rightarrow \infty$ such that

$$
M_{f_{2}}\left(R_{n}\right)>M_{g}\left(\left(\log R_{n}\right)^{\lambda}\right), \text { for all } n \text { and } M_{f_{1}}(r)<M_{g}\left((\log r)^{\mu}\right) .
$$

Let us apply the Lemma $2.5 \frac{1}{16}$, we get

$$
\log \left|f_{1}(z)\right|>-(2+\log (24 e)) \log M_{f_{1}}\left(4 e R_{n}\right)
$$

on the disc $|z| \leq 2 R_{n}$, excluding a family of discs, the sum of whose radii exceeds $\frac{R_{n}}{2}$.
Therefore there exists $r_{n} \in\left(R_{n}, 2 R_{n}\right)$ such that $|z|=r_{n}$ does not intersect any of the excluded discs, then

$$
\log \left|f_{1}(z)\right|>-7 \log M_{f_{1}}\left(4 e R_{n}\right) \text { in }|z|=r_{n} .
$$

Also

$$
M_{f_{2}}\left(r_{n}\right)>M_{f_{2}}\left(R_{n}\right)>M_{g}\left(\left(\log R_{n}\right)^{\lambda}\right)>M_{g}\left(\left(\log \frac{r_{n}}{2}\right)^{\lambda}\right)
$$

If $z_{r}$ is a point in $|z|=r$, with $M_{f_{2}}(r)=\left|f_{2}\left(z_{r}\right)\right|$, we have

$$
M_{h}(r) \geq\left|f_{1}\left(z_{r}\right)\right|\left|f_{2}\left(z_{r}\right)\right|=\left|f_{1}\left(z_{r}\right)\right| M_{f_{2}}(r) .
$$

Therefore

$$
\begin{aligned}
M_{h}(r) & >\left(M_{f_{1}}\left(4 e R_{n}\right)\right)^{-7} M_{g}\left(\left(\log \frac{r_{n}}{2}\right)^{\lambda}\right) \\
& >\left(M_{g}\left(\left(\log 4 e R_{n}\right)^{\mu}\right)\right)^{-7} M_{g}\left(\left(\log \frac{r_{n}}{2}\right)^{\lambda}\right) \\
& >\left(M_{g}\left(\left(\log 4 e r_{n}\right)^{\mu}\right)\right)^{-7} M_{g}\left(\left(\log \frac{r_{n}}{2}\right)^{\lambda}\right), \text { for sufficiently large } n .
\end{aligned}
$$

Take $\nu \in(\mu, \lambda), \sigma=\frac{v}{\mu}>1$, we obtain

$$
\begin{aligned}
M_{h}\left(r_{n}\right) & >M_{g}\left(\left(\log 4 e r_{n}\right)^{\nu}\right)\left(M_{g}\left(\left(\log 4 e r_{n}\right)^{\mu}\right)\right)^{-7} \\
& >\left(M_{g}\left(\left(\log 4 e r_{n}\right)^{\mu}\right)\right)^{8}\left(M_{g}\left(\left(\log 4 e r_{n}\right)^{\mu}\right)\right)^{-7} \\
& =M_{g}\left(\left(\log 4 e r_{n}\right)^{\mu}\right), \text { applying Lemma } 2.2 \text { for } n=8 \text { and } r=\left(\log 4 e r_{n}\right)^{\mu .} \\
& >M_{g}\left(\left(\log r_{n}\right)^{\mu}\right), \text { for sufficiently large } n .
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\rho^{l} & \geq \mu, \text { for each } \mu<\rho_{2}^{l} \\
& \Rightarrow \rho^{l} \geq \rho_{2}^{l}
\end{aligned}
$$

Hence

$$
\rho^{l}=\rho_{2}^{l} .
$$

For the last part of d), let $h=\frac{f_{1}}{f_{2}}$, i.e. $f_{1}=h f_{2}$.
We keep the same notations and without loss of generality let us suppose $\rho_{1}^{l} \leq \rho_{2}^{l}$.
If possible let, $\rho^{l}>\rho_{2}^{l}=\max \left\{\rho_{1}^{l}, \rho_{2}^{l}\right\}$. Then from the previous part equality occurs, i.e. $\rho_{1}^{l}=\max \left\{\rho^{l}, \rho_{2}^{l}\right\}=\rho^{l}$. Therefore $\rho_{1}^{l}>\rho_{2}^{l}$ and we came to a contradiction.

Therefore,

$$
\rho^{l} \leq \rho_{2}^{l} .
$$

Next we suppose that $\rho_{1}^{l}<\rho_{2}^{l}$. We are to show in this case equality holds. If possible assume that $\rho^{l}<\rho_{2}^{l}$, then $\max \left\{\rho^{l}, \rho_{2}^{l}\right\}=\rho_{1}^{l}$. From the previous part, then $\rho_{2}^{l}=\rho_{1}^{l}$ and we come back to a contradiction again.

So,

$$
\rho^{l}=\rho_{2}^{l} .
$$

### 3.4. Relative logarithmic order of derivative.

Theorem 3.9. Let $f$ and $g$ be both transcendental entire functions. Then

$$
\rho_{g}^{l}(f)=\rho_{g}^{l}\left(f^{\prime}\right)=\rho_{g^{\prime}}^{l}(f)=\rho_{g^{\prime}}^{l}\left(f^{\prime}\right)
$$

Proof. Without loss of generality we can assume that $f(0)=0$.
Then

$$
f(z)=\int_{0}^{z} f^{\prime}(t) d t
$$

where we have taken the integral over the segment that joins the origin with $z$. Then

$$
M_{f}(r) \leq r M_{f^{\prime}}(r)
$$

Using Cauchy‘s formula, we get

$$
f^{\prime}(z)=\frac{1}{2 \pi i} \int_{C} \frac{f(t)}{(t-z)^{2}} d t
$$

where $C=\{t:|t-z|=r\}$; then

$$
M_{f^{\prime}}(r) \leq \frac{1}{2 \pi} \frac{M_{f}(r)}{r^{2}} \cdot 2 \pi r=\frac{M_{f}(r)}{r} \leq \frac{M_{f}(2 r)}{r} .
$$

Summarizing we get,

$$
\frac{M_{f}(r)}{r} \leq M_{f^{\prime}}(r) \leq \frac{M_{f}(2 r)}{r}, \text { for each } r>0
$$

Next let $\sigma \in(0,1)$, then from Lemma $2.3 d)$ and taking $\lambda=1, \mu=\sigma$, we get

$$
\begin{aligned}
\lim _{r \rightarrow \infty} \frac{M_{f}(r)}{r M_{f}\left(r^{\sigma}\right)} & =\infty \\
& \Rightarrow M_{f}(r)>r M_{f}\left(r^{\sigma}\right), \text { for sufficiently large } r
\end{aligned}
$$

Therefore

$$
M_{f}\left(r^{\sigma}\right) \leq \frac{M_{f}(r)}{r} \leq M_{f^{\prime}}(r) \leq \frac{M_{f}(2 r)}{r} \leq M_{f}(2 r)
$$

$$
\begin{aligned}
& \Rightarrow \quad M_{f}\left(r^{\sigma}\right) \leq M_{f^{\prime}}(r) \leq M_{f}(2 r) \\
& \Rightarrow \quad M_{g}^{-1} M_{f}\left(r^{\sigma}\right) \leq M_{g}^{-1} M_{f^{\prime}}(r) \leq M_{g}^{-1} M_{f}(2 r) \\
& \Rightarrow \quad \frac{\log ^{+} M_{g}^{-1} M_{f}\left(r^{\sigma}\right)}{\log \log r} \leq \frac{\log ^{+} M_{g}^{-1} M_{f^{\prime}}(r)}{\log \log r} \leq \frac{\log ^{+} M_{g}^{-1} M_{f}(2 r)}{\log \log r} \\
& \Rightarrow \quad \frac{\log ^{+} M_{g}^{-1} M_{f}\left(r^{\sigma}\right)}{\log \log r^{\sigma}} \cdot \frac{\log \log r^{\sigma}}{\log \log r} \leq \frac{\log ^{+} M_{g}^{-1} M_{f^{\prime}}(r)}{\log \log r} \leq \frac{\log ^{+} M_{g}^{-1} M_{f}(2 r)}{\log \log 2 r} \cdot \frac{\log \log 2 r}{\log \log r},
\end{aligned}
$$

taking limsup as $r \rightarrow \infty$, we get

$$
\begin{aligned}
\rho_{g}^{l}(f) \cdot 1 & \leq \rho_{g}^{l}\left(f^{\prime}\right) \leq \rho_{g}^{l}(f) \cdot 1 \\
& \Rightarrow \rho_{g}^{l}(f)=\rho_{g}^{l}\left(f^{\prime}\right)
\end{aligned}
$$

Again from (4)

$$
\begin{gathered}
\frac{\log \log ^{+} M_{g}^{-1} M_{f}\left(r^{\sigma}\right)}{\log r} \leq \frac{\log \log ^{+} M_{g}^{-1} M_{f^{\prime}}(r)}{\log r} \leq \frac{\log \log ^{+} M_{g}^{-1} M_{f}(2 r)}{\log r} \\
\Rightarrow \frac{\log \log ^{+} M_{g}^{-1}(s)}{\log ^{+}\left(\frac{1}{\sigma} M_{f}^{-1}(s)\right)} \leq \frac{\log \log ^{+} M_{g}^{-1}(s)}{\log ^{+} M_{f^{\prime}}^{-1}(s)} \leq \frac{\log \log ^{+} M_{g}^{-1}(s)}{\log ^{+}\left(\frac{1}{2} M_{f}^{-1}(s)\right)},
\end{gathered}
$$

taking liminf as $s \rightarrow \infty$, we get

$$
\begin{aligned}
\liminf _{s \rightarrow \infty} \frac{\log \log ^{+} M_{g}^{-1}(s)}{\log ^{+} M_{f}^{-1}(s)} & \leq \liminf _{s \rightarrow \infty} \frac{\log \log ^{+} M_{g}^{-1}(s)}{\log ^{+} M_{f^{\prime}}^{-1}(s)} \leq \liminf _{s \rightarrow \infty} \frac{\log \log ^{+} M_{g}^{-1}(s)}{\log ^{+} M_{f}^{-1}(s)} \\
& \Rightarrow \liminf _{s \rightarrow \infty} \frac{\log \log ^{+} M_{g}^{-1}(s)}{\log ^{+} M_{f}^{-1}(s)}=\liminf _{s \rightarrow \infty} \frac{\log _{\log ^{+} M_{g}^{-1}(s)}^{\log ^{+} M_{f^{\prime}}^{-1}(s)}}{}
\end{aligned}
$$

Interchanging the role of $f$ and $g$, we get

$$
\begin{aligned}
\liminf _{s \rightarrow \infty} \frac{\log \log ^{+} M_{f}^{-1}(s)}{\log ^{+} M_{g}^{-1}(s)} & =\liminf _{s \rightarrow \infty} \frac{\log \log ^{+} M_{f}^{-1}(s)}{\log ^{+} M_{g^{\prime}}^{-1}(s)} \\
& \Rightarrow \frac{1}{\limsup _{s \rightarrow \infty} \frac{\log ^{+} M_{g}^{-1}(s)}{\log _{\log }+M_{f}^{-1}(s)}}=\frac{1}{\limsup _{s \rightarrow \infty} \frac{\log ^{+} M_{f^{\prime}}^{-1}(s)}{\log ^{\log +} M_{g}^{-1}(s)}} \\
& \Rightarrow \frac{1}{\rho_{g}^{l}(f)}=\frac{1}{\rho_{g^{\prime}}^{l}(f)} \\
& \Rightarrow \rho_{g}^{l}(f)=\rho_{g^{\prime}}^{l}(f) .
\end{aligned}
$$

Consequently from (4) and (4), we get

$$
\rho_{g}^{l}(f)=\rho_{g}^{l}\left(f^{\prime}\right)=\rho_{g^{\prime}}^{l}(f)=\rho_{g^{\prime}}^{l}\left(f^{\prime}\right)
$$

Note that, it is trivial when either $f$ and $g$ both are polynomials, or $f$ is transcendent and $g$ is polynomial. But the theorem does not hold for $f$ is polynomial and $g$ is transcendental, as shown in the following example.

Example 3.10. let $f(z)=z, g(z)=\exp z$.
Then $f^{\prime}(z)=1, M_{f}(r)=r, M_{f^{\prime}}(r)=1$ and $M_{g}(r)=\exp r$.

Therefore

$$
\begin{aligned}
\rho_{g}^{l}(f) & =\frac{\log ^{+} \log r}{\log \log r} \\
& =1
\end{aligned}
$$

whereas,

$$
\begin{aligned}
\rho_{g}^{l}\left(f^{\prime}\right) & =\frac{\log ^{+} \log 1}{\log \log r} \\
& =0
\end{aligned}
$$

### 3.5. Relative logarithmic order of real and imaginary parts.

Theorem 3.11. Let $f$ and $g$ are nonconstant entire functions.
Let

$$
\begin{aligned}
& A(r)=\max \{\operatorname{Re} f(z):|z|=r\}, \\
& B(r)=\max \{\operatorname{Im} f(z):|z|=r\}, \\
& C(r)=\max \{\operatorname{Re} g(z):|z|=r\}, \\
& D(r)=\max \{\operatorname{Im} g(z):|z|=r\} .
\end{aligned}
$$

Then

$$
\begin{aligned}
\rho_{g}^{l}(f) & =\inf \left\{\mu>0: M(r)<N\left((\log r)^{\mu}\right)\right\} \\
& =\limsup _{r \rightarrow \infty} \frac{\log N^{-1}(M(r))}{\log \log r}
\end{aligned}
$$

where $M$ is any of the functions $A, B \circ F$ and $N$ is any of the functions $C, D \circ G$.
Proof. It is clear that $A, B, C$ and $D$ are continuous strictly increasing functions of $r$, then $A^{-1}, B^{-1}, C^{-1}$ and $D^{-1}$ exist. From Lemma 2.5 we obtain the existence of a constant $\alpha>0$ with

$$
M(r) \leq F(r) \leq M(\alpha r)
$$

and

$$
N(r) \leq G(r) \leq N(\alpha r) .
$$

Let $\rho^{l}=\rho_{g}^{l}(f)$ and $\beta=\inf \left\{\mu>0: M(r)<N\left((\log r)^{\mu}\right)\right\}$.
We first prove that $\beta \leq \rho^{l}$.
If $\rho^{l}=\infty$, it is trivial.
So assume that $\rho^{l}$ is finite, choose $\lambda, \mu$ with $\rho^{l}<\lambda<\mu<\infty$.
Then $M_{f}(r)<M_{g}\left((\log r)^{\lambda}\right)$ and
$M(r) \leq M_{f}(r)<M_{g}\left((\log r)^{\lambda}\right)<N\left((\log \alpha r)^{\lambda}\right)<N\left((\log r)^{\mu}\right)$, for sufficiently large $r$.
Which implies

$$
\begin{aligned}
\beta & \leq \mu, \text { for all } \mu>\rho^{l} . \\
& \Rightarrow \beta \leq \rho^{l}
\end{aligned}
$$

Finally let us prove, $\beta \geq \rho^{l}$.
If $\rho^{l}=0$, the case is trivial.
So let $\rho^{l}>0$, choose $\lambda, \mu$ such that $0<\mu<\lambda<\rho^{l}$.
Then there is a sequence $\left\{r_{n}\right\} \rightarrow \infty$ such that

$$
M_{f}\left(r_{n}\right)>M_{g}\left((\log r)^{\lambda}\right), \text { for all } n
$$

Therefore

$$
M\left(\alpha r_{n}\right)>M_{f}\left(r_{n}\right)>M_{g}\left(\left(\log r_{n}\right)^{\lambda}\right)>M_{g}\left(\left(\log \alpha r_{n}\right)^{\mu}\right)>N\left(\left(\log \alpha r_{n}\right)^{\mu}\right)
$$

for sufficiently large $n$.
Which implies

$$
\begin{aligned}
\beta & \geq \mu \text { for each } \mu<\rho^{l} . \\
& \Rightarrow \beta \geq \rho^{l} .
\end{aligned}
$$

Therefore we have $\beta=\rho^{l}$.

### 3.6. Relative logarithmic order of Nevanlinna's characteristic function.

Theorem 3.12. Let $f$ and $g$ are nonconstant entire functions. Then

$$
\begin{aligned}
\rho_{g}^{l}(f) & =\inf \left\{\mu>0: T_{f}(r)<T_{g}\left((\log r)^{\mu}\right)\right\} \\
& =\limsup _{r \rightarrow \infty} \frac{\log ^{+} T_{g}^{-1}\left(T_{f}(r)\right)}{\log \log r}
\end{aligned}
$$

Proof. Let $\rho^{l}=\rho_{g}^{l}(f)$ and $\alpha=\inf \left\{\mu>0: T_{f}(r)<T_{g}\left((\log r)^{\mu}\right)\right\}$
Let us prove that $\alpha \leq \rho^{l}$.
If $\rho^{l}=\infty$, the case is trivial.
So, we take $\rho^{l}$ be finite and let's take $\gamma, \delta, \lambda, \mu$ such that $\rho^{l}<\gamma<\delta<\lambda<\mu<\infty$.
Now for sufficiently large $r$, it is clear that

$$
\frac{\gamma}{\delta}<\frac{(\log r)^{\mu}-(\log r)^{\lambda}}{(\log r)^{\mu}+(\log r)^{\lambda}}
$$

By Lemma 2.3

$$
M_{g}\left(r^{\gamma}\right)^{s}=M_{g}\left(r^{\gamma}\right)^{\frac{\delta}{\gamma}} \leq K M_{g}\left(r^{\delta}\right)<M_{g}\left(r^{\lambda}\right) .
$$

Hence

$$
\begin{aligned}
M_{g}\left((\log r)^{\gamma}\right)^{s} & =M_{g}\left((\log r)^{\gamma}\right)^{\frac{\delta}{\gamma}} \\
& \leq K M_{g}\left((\log r)^{\delta}\right), \text { for sufficiently large } r \\
& <M_{g}\left((\log r)^{\lambda}\right) .
\end{aligned}
$$

Therefore,

$$
\frac{\delta}{\gamma} \log M_{g}\left((\log r)^{\gamma}\right)<\log M_{g}\left((\log r)^{\lambda}\right) .
$$

Which implies

$$
\begin{aligned}
\log ^{+} M_{g}\left((\log r)^{\gamma}\right) & <\frac{\gamma}{\delta} \log ^{+} M_{g}\left((\log r)^{\lambda}\right) \\
& <\frac{(\log r)^{\mu}-(\log r)^{\lambda}}{(\log r)^{\mu}+(\log r)^{\lambda}} \log ^{+} M_{g}\left((\log r)^{\lambda}\right) \\
& \leq T_{g}\left((\log r)^{\mu}\right) .
\end{aligned}
$$

Again from Lemma 2.7

$$
\begin{aligned}
T_{f}(r) & \leq \log ^{+} M_{f}(r)<\log ^{+} M_{g}\left((\log r)^{\lambda}\right) \\
& \Rightarrow T_{f}(r)<T_{g}\left((\log r)^{\mu}\right) \\
& \Rightarrow \mu \geq \alpha, \text { for all } \mu>\rho^{l} \\
& \Rightarrow \rho^{l} \geq \alpha
\end{aligned}
$$

Next we prove, $\alpha \geq \rho^{l}$.
If $\rho^{l}=0$, the case is trivial.
So let $\rho^{l}>0$, and take $\gamma, \delta, \mu$ with $0<\mu<\lambda<\gamma<\rho^{l}$.
Then there exist $\left\{r_{n}\right\} \rightarrow \infty$ such that

$$
M_{f}\left(r_{n}\right)>M_{g}\left(\left(\log r_{n}\right)^{\gamma}\right), \text { for all } n .
$$

Let $c \in\left(\frac{\lambda}{\gamma}, 1\right)$ and $d>\frac{1+c}{1-c}$.
Then

$$
\begin{aligned}
T_{f}\left(d r_{n}\right) & >\frac{d r_{n}-r_{n}}{d r_{n}+r_{n}} \log ^{+} M_{f}\left(r_{n}\right) \\
& =\frac{d-1}{d+1} \log ^{+} M_{f}\left(r_{n}\right) \\
& >c \log ^{+} M_{f}\left(r_{n}\right) \\
& >\log ^{+} M_{g}\left(\left(\log r_{n}\right)^{\gamma}\right)^{c} \\
& \left.>\log ^{+} \frac{M_{g}\left(\left(\log r_{n}\right)^{\gamma c}\right)}{K}, \text { using Lemma } 2.3 b\right) \text { for } c<1 \\
& >\log ^{+} M_{g}\left(\left(\log r_{n}\right)^{\lambda}\right), \text { as } c>\frac{\lambda}{\gamma} \\
& \geq \log ^{+} M_{g}\left(\left(d \log r_{n}\right)^{\mu}\right), \text { for sufficiently large } n \\
& \geq T_{g}\left(\left(d \log r_{n}\right)^{\mu}\right) .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
T_{f}\left(d r_{n}\right) & >T_{g}\left(\left(d \log r_{n}\right)^{\mu}\right), \text { for sufficiently large } n \\
& \Rightarrow \alpha \geq \mu, \text { for all } \mu<\rho^{l} \\
& \Rightarrow \alpha \geq \rho^{l} .
\end{aligned}
$$

Hence,

$$
\rho^{l}=\alpha=\left\{\mu>0: T_{f}(r)<T_{g}\left((\log r)^{\mu}\right)\right\} .
$$

### 3.7. Relative logarithmic proximate order.

Definition 3.13 (Relative logarithmic proximate order of f with respect to g ). Let $f$ and $g$ be two entire functions with finite logarithmic order of growth of $f$ relative to $g$ (i.e. $\rho_{g}^{l}(f)$ is finite). A non-negative real valued continuous function $\rho_{g}^{l}(f)(r)$, defined in $(0,+\infty)$, is said to be a logarithmic proximate order of growth of $f$ relative to $g$ if the following properties holds:
i) $\quad \rho_{g}^{l}(f)(r)$ is differentiable for $r>r_{0}$ except at isolated points at which $\left[\rho_{g}^{l}(f)\right]^{\prime}$ $(r-0)$ and $\left[\rho_{g}^{l}(f)\right]^{\prime}(r+0)$ exist,
ii) $\lim _{r \rightarrow \infty} \rho_{g}^{l}(f)(r)=\rho_{g}^{l}(f)$,
iii) $\quad \lim _{r \rightarrow \infty} r \cdot\left[\rho_{g}^{l}(f)\right]^{\prime}(r) \cdot \log \log r=0$,
iv) $\quad \limsup _{r \rightarrow \infty} \frac{M_{g}^{-1}\left(M_{f}(r)\right)}{(\log r)^{\rho_{g}^{g}(f)(r)}}=1$.

Theorem 3.14. Let $f$ and $g$ be two entire functions with finite logarithmic order of $f$ with respect to $g$. Then there exist a logarithmic proximate order $\rho_{g}^{l}(f)(r)$.

The proof of this theorem is omitted because it can be carried out in the same line of S. M. Shah [6].
3.8. Future aspects. Keeping in mind the results already established, one may explore the analogous theorems using more generalized order such as iterated order [5], $(p, q)$-order [4], $\phi$-order [3] etc.

## References

[1] L. Bernal, Orden relative de crecimiento de funciones enteras, Collect. Math. 39 (1988), 209229.
[2] P. T. Y. Chern, On meromorphic functions with finite logarithmic order, Amer. Math. Soc. 358 (2) (2005), 473-489.
[3] I. Chyzhykov, J. Heittokangas and J. Rättyä:, Finiteness of $\phi$-order of solutions of linear differential equations in the unit disc., J. Anal. Math. 109 (2009), 163-198.
[4] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, On the ( $p, q$ )-order and lower $(p, q)$-order of an entire function, Journal für die Reine und Angewandte Mathematik 282 (1976), 53-67.
[5] D. Sato, On the rate of growth of entire functions of fast growth, Bull. Amer. Math. Soc. 69 (1963), 411-414.
[6] S. M. Shah, On proximate orders of integral functions, Bull. Amer. Math. Soc. 52 (1946), 326328.
[7] G. Valiron, Lectures on the general theory of integral functions, Toulouse, 1923.

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