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EMBEDDING THEOREMS ON THE FRACTIONAL ORLICZ-SOBOLEV SPACES

Tacksun Jung[†] and Q-Heung Choi[∗,‡]

Abstract. In this paper we deal with the embedding inclusions on the fractional Orlicz-Sobolev spaces which are crucial roles for studying the theories of the partial differential equations. We get some properties and theories of the embedding inclusions on the fractional Orlicz-Sobolev spaces.

1. Introduction and preliminary

Let Ω be a bounded domain of R^N with smooth boundary $\partial\Omega$, $s \in (0,1)$ and $p : \Omega \times \Omega \to (1,\infty)$ be a continuous function. The fractional Sobolev spaces with variable exponent $p(x,y)$ are defined as:

$$W^{s,p(x,y)}(\Omega) = \{u \in L^{p(x,y)}(\Omega) | \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} dxdy < \infty, \text{ for some } \lambda > 0\}$$

endowed with the norm

$$\|u\|_{s,p(x,y)} = \inf\{\lambda > 0 | \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} dxdy \le 1\}.$$

In this paper, we are trying to relax the growth condition on $W^{s,p(x,y)}$ and deal with more generalized spaces on the growth condition than the fractional Sobolev spaces. When we are trying to relax the growth conditions, we can not formulate with the fractional Lebesgue spaces and the fractional Sobolev spaces $W^{s,p}$. We adopt the fractional Orlicz spaces with variable exponent and the fractional Orlicz-Sobolev spaces with variable exponent as the adequate function spaces. We refer the readers to [4, 9] and the references therein for the theory of Orlicz and Orlicz-Sobolev spaces. We also refer the readers to [2, 10] for some results about the fractional Orlicz-Sobolev spaces

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∗ Corresponding author.

and the fractional N-Laplacian operator. In [3], the authors provide the connection between the fractional order theories and the Orlicz-Sobolev ones, and define the fractional order Orlicz-Sobolev space associated to a Young function and a fractional parameter.

In this paper, we investigate some properties and theories for the fractional Orlicz space, the fractional Orlicz-Sobolev space and the embedding inclusions on the fractional Orlicz-Sobolev spaces.

In last years, the fractional Sobolev space and the corresponding fractional Laplace operators with variable exponent of elliptic type have been interested and researched by some mathematicians for pure mathematical research and concrete real-world applications (cf, [9], [7], [9], [12]). These problems arise in applications of natural science, for example, nonlinear elasticity theory, electro rheological fluids, non-Newtonian fluid theory in a porous medium and image processing (cf. [3], [10], [13]).

To state main results we need some notations.

Let h be an odd and increasing homeomorphism from R onto R and let H be the function defined by

$$H(x)=\int_0^x h(t)dt \qquad \text{for all } \; x\in R.$$

Then H is a Young function and also a N-function (We call that H is a Young function if $H(0)=0$, $\lim_{x\to+\infty}H(x)=+\infty$ and H is convex, and we call that H is a N-function if H satisfies that $H(x)=0$ if and only if $x=0$, $\lim_{x\to 0}\frac{H(x)}{x}=0$, $\lim_{x\to\infty}\frac{H(x)}{x}=+\infty$). Let H^* be the function defined by

$$H^*(x)=\int_0^x h^{-1}(t)dt \qquad \text{for all } \; x\in R.$$

The function H^* is called the complementary function of H and satisfies

$$H^*(x)=\sup\{yx-H(y)|\; y\geq 0\} \qquad \text{for all } \; x\geq 0.$$

Then H^* satisfies that

$$\lim_{x\to 0}\frac{H^*(x)}{x}=0, \qquad \lim_{x\to\infty}\frac{H^*(x)}{x}=+\infty,$$

i.e., H^* is a N-function. Moreover, by Young's inequality,

$$xy\leq H(x)+H^*(y), \qquad \text{for all } \; x,\; y\geq 0. \tag{1.3}$$

The Orlicz space $L_H(\Omega)$ defined by N-function H is the space defined by

$$L_H(\Omega)=\{u|\; u:\Omega\to R \text{ is a measurable function with}$$

$$\|u\|_{L_H}=\sup\{\int_\Omega uvdx|\; \int_\Omega H^*(|v|)dx\leq 1\}<\infty\}.$$

Then $L_H(\Omega)$ is a Banach space with a norm $\|u\|_{L_H}$. We note that the norm $\|u\|_{L_H}$ is equivalent to the Luxemburg norm

$$\|u\|_H=\inf\{\lambda>0|\; \int_\Omega H(|\frac{u(x)}{\lambda}|)\leq 1\}.$$

In the Orlicz space $L_H(\Omega)$, *Hölder* inequality is valid (see [11]): for all $u\in L_H(\Omega)$, $v\in L_{H^*}(\Omega)$, we have

$$\int_\Omega |uv|dx\leq 2\|u\|_{L_H}\|v\|_{L_{H^*}}.$$

In [2], the Orlicz-Sobolev space $W^1L_H(\Omega)$ is defined by

$$W^1L_H(\Omega) = \{u \in L_H(\Omega)|\ \frac{\partial u}{\partial x_i} \in L_H(\Omega),\ i = 1, \dots, N\}$$

endowed with the norm

$$\|u\|_{1,H} = \|u\|_H + \|\nabla u\|_H.$$

Then $W^1L_H(\Omega)$ is a reflexive Banach space. The Orlicz-Sobolev space $W_0^1L_H(\Omega)$ is defined by the closure of $C_0^\infty(\Omega)$ in $W^1L_H(\Omega)$. The space $W^1L_H(\Omega)$ is also a reflexive Banach space. By Lemma 5.7 in [5], the norm $\|\nabla u\|_H$ is an equivalent to the norm $\|u\|_{1,H}$ in $W_0^1L_H(\Omega)$. For any given $0 < s < 1$ and H a N-function, the fractional Orlicz-Sobolev space $W^sL_H(\Omega)$ is the space defined by

$$W^sL_H(\Omega) = \{u \in L_H(\Omega) :\ \int_\Omega\int_\Omega H(\frac{|u(x) - u(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N} < \infty\}$$

endowed with the norm

$$\|u\|_{s,H} = \|u\|_H + [u]_{s,H},$$

where $[u]_{s,H}$ is the Gagliardo semi-norm defined by

$$[u]_{s,H} = \inf\{\lambda > 0|\ \int_\Omega\int_\Omega H(\frac{|u(x) - u(y)|}{\lambda|x-y|^s})\frac{dxdy}{|x-y|^N} \leq 1\}.$$

By [2], for any $0 < s < 1$ and H a Young function such that H and H^* satisfy that

$$H(2t) \leq C_1H(t) \text{ and } H^*(2t) \leq C_2H^*(t), \qquad \forall t \geq 0,\ C_1,\ C_2 > 0,$$

$W^sL_H(R^N)$ is a reflexive and separable Banach space. Furthermore $C_0^\infty(R^N)$ is dense in $W^sL_H(R^N)$ in the norm $\|\cdot\|_{s,H}$. If $h(t) = |t|^{r(x,y)-2}t$, where $r(\cdot)$ is a continuous function on $\bar{\Omega}\times\bar{\Omega}$, $(-\Delta)^s_{r(\cdot)}u_n$ is the fractional $r(\cdot)-$Laplacian operator with variable exponent defined by

$$(-\Delta)^s_{r(\cdot)}u(x) = \text{P.V.}\int_\Omega \frac{|u(x)-u(y)|^{r(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sr(x,y)}}\frac{u(x)-u(y)}{|u(x)-u(y)|}dy, \qquad x \in \Omega.$$

Let $W_0^sL_H(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|u\|_{s,H}$ such that

$$W_0^sL_H(\Omega) = \{u \in W^sL_H(\Omega)|\ u = 0 \qquad \text{a.e, in } \ R^N\backslash\Omega\}.$$

Let us set

$$h_0 = \inf_{t>0}\frac{th(t)}{H(t)} \qquad h^0 = \sup_{t>0}\frac{th(t)}{H(t)}.$$

We assume that

$$1 < h_0 \leq \frac{th(t)}{H(t)} \leq h^0 < \infty \qquad \forall t \geq 0. \tag{1.5}$$

By Proposition 2.3 of [8], it implies that each H satisfies the Δ_2-condition, i.e., there exists a constant $C > 0$ such that

$$H(2t) \leq CH(t), \qquad t \geq 0.$$

We also assume that H is a function such that

$$H : t \in [0,\infty) \mapsto H(\sqrt{t}) \text{ is convex.} \tag{1.6}$$

2. Main results

Let $0 < s < 1$, h, H, H^* be functions, $L_H(\Omega)$ be the Orlicz space, $W^s L_H(\Omega)$ be the fractional Orlicz-Sobolev space and $W_0^s L_H(\Omega)$ be the space defined in Section 1.

Let $W^{s,r(x,y)}(\Omega)$ be the fractional Sobolev space with variable exponent $r(x,y)$ defined in Section 1. Let us define the functional $\Lambda : W^s L_H(\Omega) \to R$ by

$$\Lambda(u) = \int_\Omega \int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}.$$

Then the functional Λ is of class $C^1(W^s L_H(\Omega), R)$ and

$$\begin{aligned} < \Lambda'(u), v > &= \int_\Omega \int_\Omega h(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{v(x)-v(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N} \\ &= < (-\Delta)_h^s u, v >, \end{aligned} \quad (2.1)$$

which is proved in Proposition 3.3 in [11].

Lemma 2.1. [11] *(Generalized Poincaré inequality on the Orlicz-Sobolev space) Let Ω be a bounded open subset of R^N, $0 < s < 1$ and H be a Young function. Then there exists a positive constant $C > 0$ such that*

$$\|u\|_H \leq [u]_{s,H}, \qquad \forall u \in W_0^s L_H(\Omega). \quad (2.2)$$

That is, the embedding

$$W_0^s L_H(\Omega) \hookrightarrow L_H(\Omega)$$

is continuous and compact. Furthermore $[u]_{s,H}$ is a norm of $W_0^s L_H(\Omega)$ equivalent to $\|\cdot\|_{s,H}$.

Lemma 2.2. [1] *Let $u \in W^s L_H(\Omega)$. Then*

$$\|u\|_{s,h_0}^{h_0} \leq \Lambda(u) \leq \|u\|_{s,h^0}^{h^0}, \qquad \text{if } \|u\|_{s,H} > 1, \quad (2.3)$$

$$\|u\|_{s,h^0}^{h^0} \leq \Lambda(u) \leq \|u\|_{s,h_0}^{h_0}, \qquad \text{if } \|u\|_{s,H} < 1.$$

It follows that the embedding

$$W^s L_H(\Omega) \hookrightarrow W^{s,h_0}(\Omega)$$

is continuous.

Proof. The proof is given by (1.4) and Theorem 3.11 of [1]. □

Theorem 2.3. [11] *Let Ω be a bounded open subset of R^N, $0 < s < 1$ and H be a N-function. Let $1 \leq r(x) < h_0^* = \frac{Nh_0}{N-sh_0}$, $N > sh_0$. Then the embedding*

$$W^{s,h_0} \hookrightarrow L^{r(x)}$$

is continuous and compact for all $1 \leq r(x) < h_0^*$. Moreover the embedding

$$W^s L_H(\Omega) \hookrightarrow L^{r(x)}$$

is continuous and compact for all $1 \leq r(x) < h_0^*$. Furthermore there exists a positive constant C such that

$$\|u\|_{L^{r(x)}} \leq C[u]_{s,H} \quad (2.4)$$

Proof. Since the embedding $W^{s,h_0} \hookrightarrow L^{r(x)}$ is continuous and compact for all $1 \le r(x) < h_0^*$ and by Lemma 2.2, the embedding $W^sL_H(\Omega) \hookrightarrow W^{s,h_0}$ is continuous, it follows that for all $1 \le r(x) < h_0^*$, the embedding $W^sL_H(\Omega) \hookrightarrow L^{r(x)}$ is continuous and compact. □

THEOREM 2.4. *Let Ω be a bounded open subset of R^N, $0 < s_1 < s < s_2 < 1$ and H be a N-function. Then the embeddings*

$$W_0^{s_2}L_H(\Omega) \hookrightarrow W_0^{s}L_H(\Omega) \hookrightarrow W_0^{s_1}L_H(\Omega)$$

are continuous.

Proof. For any $u \in W_0^{s_2}L_H(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega \int_{\Omega\cap\{|x-y|\ge 1\}} H(\frac{|u(x)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}\\
&\le \int_\Omega \int_{\Omega\cap\{|x-y|\ge 1\}} H(\frac{|u(x)|}{|x-y|^{s_1}})\frac{dxdy}{|x-y|^N}\\
&\le \int_\Omega \int_{\Omega\cap\{|z|\ge 1\}} H(\frac{|u(x)|}{|z|^{s_1}})\frac{dxdz}{|z|^N}\\
&\le C\int_\Omega H(|u(x)|)dx.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
&\int_\Omega \int_{\Omega\cap\{|x-y|\le 1\}} H(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}\\
&\le \int_\Omega \int_{\Omega\cap\{|x-y|\le 1\}} H(\frac{|u(x)-u(y)|}{|x-y|^{s_2}})\frac{dxdy}{|x-y|^N}\\
&\le \int_\Omega \int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^{s_2}})\frac{dxdy}{|x-y|^N}
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\int_\Omega \int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}\\
&\le \int_\Omega \int_{\Omega\cap\{|x-y|\ge 1\}} H(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}\\
&\quad + \int_\Omega \int_{\Omega\cap\{|x-y|\le 1\}} H(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}\\
&\le C\int_\Omega H(|u(x)|)dx + \int_\Omega \int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}\\
&\le C\Big(\int_\Omega H(|u(x)|)dx + \int_\Omega \int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^{s_2}})\frac{dxdy}{|x-y|^N}\Big). \qquad (2.5)
\end{aligned}$$

It follows from this inequality that we can easily verify that the embedding $W_0^{s_2}L_H(\Omega) \hookrightarrow W_0^{s}L_H(\Omega)$ is continuous. Similarly, for any $u \in W_0^s L_H(\Omega)$, we have

$$\int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} H(\frac{|u(x)|}{|x-y|^{s_1}})\frac{dxdy}{|x-y|^N}$$
$$\leq \int_\Omega \int_{\Omega\cap\{|z|\geq 1\}} H(\frac{|u(x)|}{|z|^{s_1}})\frac{dxdz}{|z|^N}$$
$$\leq D\int_\Omega H(|u(x)|)dx.$$

Moreover we have

$$\int_\Omega \int_{\Omega\cap\{|x-y|\leq 1\}} H(\frac{|u(x)-u(y)|}{|x-y|^{s_1}})\frac{dxdy}{|x-y|^N}$$
$$\leq \int_\Omega \int_{\Omega\cap\{|x-y|\leq 1\}} H(\frac{|u(x)-u(y)|}{|x-y|^{s}})\frac{dxdy}{|x-y|^N}$$
$$\leq \int_\Omega \int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^{s}})\frac{dxdy}{|x-y|^N}.$$

Thus we have

$$\int_\Omega \int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^{s_1}})\frac{dxdy}{|x-y|^N}$$
$$\leq \int_\Omega \int_{\Omega\cap\{|x-y|\geq 1\}} H(\frac{|u(x)-u(y)|}{|x-y|^{s_1}})\frac{dxdy}{|x-y|^N}$$
$$+\int_\Omega \int_{\Omega\cap\{|x-y|\leq 1\}} H(\frac{|u(x)-u(y)|}{|x-y|^{s_1}})\frac{dxdy}{|x-y|^N}$$
$$\leq D\int_\Omega H(|u(x)|)dx + \int_\Omega \int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^{s}})\frac{dxdy}{|x-y|^N}$$
$$\leq D(\int_\Omega H(|u(x)|)dx + \int_\Omega \int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^{s}})\frac{dxdy}{|x-y|^N}).$$

It follows that the embedding $W_0^s L_H(\Omega) \hookrightarrow W_0^{s_1}L_H(\Omega)$ is continuous. Thus the proof of the lemma is complete. □

THEOREM 2.5. *Assume that (1.5) and (1.6) hold, that the sequence $\{u_n\}$ converges weakly to u in $W_0^s L_H(\Omega)$ and*

$$\lim_{n\to+\infty}\sup < \Lambda'(u_n), u_n - u > \leq 0.$$

Then $\{u_n\}$ converges strongly to u in $W^s L_H(\Omega)$.

Proof. Since the sequence $\{u_n\}$ converges weakly to u in $W_0^s L_H(\Omega)$ and $\lim_{n\to+\infty}\sup < \Lambda'(u_n), u_n - u > \leq 0$, we have

$$\lim_{n\to+\infty}\int_\Omega \int_\Omega h(\frac{|u_n(x)-u_n(y)|}{|x-y|^s})\frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|}\frac{u_n(x)-u_n(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N}$$
$$\leq \int_\Omega \int_\Omega h(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{u(x)-u(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N}.$$

Thus the sequence

$$\left\{\int_\Omega\int_\Omega h(\frac{|u_n(x)-u_n(y)|}{|x-y|^s})\frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|}\frac{u_n(x)-u_n(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N}\right\}$$

is bounded and converges to

$$\int_\Omega\int_\Omega h(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{u(x)-u(y)}{|u(x)-u(y)|}\frac{u(x)-u(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N}.$$

By (1.5), we have

$$\begin{aligned}&\lim_{n\to+\infty} h_0\int_\Omega\int_\Omega H(\frac{|u_n(x)-u_n(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}\\ &\leq \lim_{n\to+\infty}\int_\Omega\int_\Omega h(\frac{|u_n(x)-u_n(y)|}{|x-y|^s})\frac{u_n(x)-u_n(y)}{|u_n(x)-u_n(y)|}\frac{u_n(x)-u_n(y)}{|x-y|^s}\frac{dxdy}{|x-y|^N}.\end{aligned}$$

Thus the sequence

$$\left\{\int_\Omega\int_\Omega H(\frac{|u_n(x)-u_n(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}\right\}$$

is bounded and converges to

$$\int_\Omega\int_\Omega H(\frac{|u(x)-u(y)|}{|x-y|^s})\frac{dxdy}{|x-y|^N}.$$

Thus the sequence $\{u_n\}$ is bounded and converges weakly to u in $L_H(\Omega)$. Since the embedding $W_0^sL_H(\Omega)\hookrightarrow L_H(\Omega)$ is continuous and compact, $\{u_n\}$ converges strongly to u in $W_0^sL_H(\Omega)$. □

THEOREM 2.6. *If u, $u_n\in W^sL_H(\Omega)$, $n=1,2,\ldots$, then the following statement are equivalent to each other*
(i) $\lim_{n\to\infty}\|u_n-u\|_{s,H}=0$, $i=1,2$,
(ii) $\lim_{n\to\infty}\int_\Omega H(u_n(x)-u(x))dx=0$ *and* $\lim_{n\to\infty}[u_n-u]_{s,H}=0$,
(iii) $u_n\to u$ *in measure in* $W^sL_H(\Omega)$ *and* $\lim_{n\to\infty}\int_\Omega H(u_n(x))dx=\int_\Omega H(u(x))dx$.

Proof. By the definition of $\|\cdot\|_{s,H}$, (i)⇔(ii) holds. We shall show that (i) implies (iii). We assume that (i) holds. Then

$$\begin{aligned}\int_\Omega[H(u_n)-H(x)]dx &\leq \int_\Omega h(u+\theta(u_n-u))(u_n-u)dx\\ &\leq \|h(u+\theta(u_n-u))\|_{H^*}\|u_n-u\|_H\\ &\leq \|h(u+\theta(u_n-u))\|_{H^*}\|u_n-u\|_{s,H}\to 0\end{aligned}$$

for $0<\theta<1$. It follows that (iii) holds. Assume that (iii) holds. Since

$$\lim_{n\to\infty}\int_\Omega H(u_n(x))dx=\int_\Omega H(u(x))dx,$$

u_n converges weakly to u in $L_H(\Omega)$. By Lemma 2.1, since $u_n\to u$ in measure in $W^sL_H(\Omega)$ and the embedding $W^sL_H(\Omega)\hookrightarrow L_H(\Omega)$ is continuous and compact, $u_n\to u$ strongly □

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Tacksun Jung
Department of Mathematics, Kunsan National University,
Kunsan 573-701, Korea
E-mail: tsjung@kunsan.ac.kr

Q-Heung Choi
Department of Mathematics Education, Inha University,
Incheon 402-751, Korea
E-mail: qheung@inha.ac.kr