

Some Congruences for Andrews' Partition Function $\overline{\mathcal{EO}}(n)$

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ABSTRACT. Recently, Andrews introduced partition functions $\mathcal{EO}(n)$ and $\overline{\mathcal{EO}}(n)$ where the function $\mathcal{EO}(n)$ denotes the number of partitions of n in which every even part is less than each odd part and the function $\overline{\mathcal{EO}}(n)$ denotes the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. In this paper we obtain some congruences modulo 2, 4, 10 and 20 for the partition function $\overline{\mathcal{EO}}(n)$. We give a simple proof of the first Ramanujan-type congruences $\overline{\mathcal{EO}}(10n + 8) \equiv 0 \pmod{5}$ given by Andrews.

1. Introduction

A partition of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ such that $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$. Let $p(n)$ be the number of partitions of n . For example $p(5) = 7$. The seven partitions of 5 are 5, 4 + 1, 3 + 2, 3 + 1 + 1, 2 + 2 + 1, 2 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}},$$

where throughout this paper, for any complex numbers a and $|q| < 1$ we define

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad (a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k).$$

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Received March 14 2019; revised October 7, 2020; accepted October 8, 2020.

2020 Mathematics Subject Classification: 11P84, 11P83, 05A17.

Key words and phrases: partitions, partitions with even parts below odd parts, congruences.

This work was supported by DST/INSPIRE Fellowship, IF130961, Government of India, Department Of Science & Technology, Technology Bhawan, New Delhi-110016.

Almost a century back Ramanujan established the following identity [7],

$$(1.1) \quad \sum_{n=0}^{\infty} p(5n+4)q^n = 5 \frac{(q^5; q^5)_{\infty}^5}{(q; q)_{\infty}^6},$$

which in fact implies Ramanujan's congruences for $p(n)$ modulo 5,

$$(1.2) \quad p(5n+4) \equiv 0 \pmod{5}.$$

Recently, Andrews [2] introduced the partition function $\mathcal{EO}(n)$ which counts the number of partitions of n in which every even part is less than each odd part. For example, $\mathcal{EO}(6) = 7$. The seven partitions of 6 it enumerates are 6, 5 + 1, 4 + 2, 3 + 3, 3 + 1 + 1 + 1, 2 + 2 + 2, 1 + 1 + 1 + 1 + 1 + 1. In [2], Andrews shows that the generating function for $\mathcal{EO}(n)$ is

$$(1.3) \quad \sum_{n=0}^{\infty} \mathcal{EO}(n)q^n := \frac{1}{(1-q)(q^2; q^2)_{\infty}}.$$

Andrews [2], also defined the partition function $\overline{\mathcal{EO}}(n)$ which counts the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times. For example, $\overline{\mathcal{EO}}(6) = 4$. The four partitions of 6 it enumerates are 6, 3 + 3, 2 + 2 + 2, 1 + 1 + 1 + 1 + 1 + 1. In [2], Andrews shows that the generating function for $\overline{\mathcal{EO}}(n)$ is

$$(1.4) \quad \sum_{n=0}^{\infty} \overline{\mathcal{EO}}(n)q^n = \frac{(q^4; q^4)_{\infty}^3}{(q^2; q^2)_{\infty}^2}.$$

In Section 3 of this paper, we prove some congruences modulo 2 and 4 for the partition function $\overline{\mathcal{EO}}(n)$. In Section 4, we give a simple proof of Andrews' congruences

$$\overline{\mathcal{EO}}(10n+8) \equiv 0 \pmod{5},$$

and we prove some interesting congruences modulo 10 and 20. In the Section 5, we consider

$$(1.5) \quad \sum_{n=0}^{\infty} \mathcal{EO}_e(n)q^n := \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^2},$$

where the function $\mathcal{EO}_e(n)$ counts the elements in the set of partitions which are enumerated by $\overline{\mathcal{EO}}(n)$ together with the partitions enumerated by $\mathcal{EO}(n)$ where all parts are odd and number of parts is even, i.e. $\mathcal{EO}_e(n)$ denotes the number of partitions enumerated by $\mathcal{EO}(n)$ in which only the largest even part appears an odd number of times except when parts are odd and number of parts is even. For example, $\mathcal{EO}_e(6) = 6$. The six partitions of 6 it enumerates are 6, 3 + 3, 2 + 2 + 2, 1 + 1 + 1 + 1 + 1 + 1 (which are counted by $\overline{\mathcal{EO}}(n)$) and 5 + 1 and 3 + 1 + 1 + 1

(only counted by $\mathcal{EO}(n)$ in which all parts are odd and the number of parts is even). We prove some arithmetic properties modulo 2 satisfied by $\mathcal{EO}_e(n)$. All of the proofs will follow from elementary generating function considerations and q -series manipulations. The paper concludes with a conjecture on $\overline{\mathcal{EO}}(n)$.

2. Preliminaries

We require the following definitions and lemmas to prove the main results in the next three sections. For $|ab| < 1$, Ramanujan's general theta function $f(a, b)$ is defined as

$$(2.1) \quad f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.$$

Using Jacobi's triple product identity [1, Theorem 2.8], (2.1) takes the shape

$$(2.2) \quad f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}.$$

The special cases of $f(a, b)$ are

$$(2.3) \quad \phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty} = \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2},$$

$$(2.4) \quad \psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}} = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}},$$

$$(2.5) \quad \phi(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_{\infty}^2}{(q^2; q^2)_{\infty}}.$$

Lemma 2.1.(Hirschhorn [6, p. 14, Eqn. 1.9.4]) *We have the following 2-dissection of $\phi(q)$,*

$$(2.6) \quad \phi(q) = \phi(q^4) + 2q\psi(q^8).$$

Lemma 2.2.(Hirschhorn [5] or Hirschhorn [6, p. 36, Eqn. 3.6.4]) *We have,*

$$(2.7) \quad \begin{aligned} (q; q)_{\infty}^3 &= \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{(n^2+n)/2} \\ &\equiv f(-q^{10}, -q^{15}) - 3qf(-q^5, -q^{20}) \pmod{5}. \end{aligned}$$

Lemma 2.3.(Hirschhorn [6, p. 105, Eqn. 10.7.6]) *We have the following beautiful identity due to Ramanujan,*

$$(2.8) \quad \frac{(q; q)_\infty^2 (q^4; q^4)_\infty^2}{(q^2; q^2)_\infty} = \sum_{n=-\infty}^{\infty} (3n+1)q^{3n^2+2n}.$$

From the Binomial Theorem, for any positive integer, k ,

$$(2.9) \quad (q^k; q^k)_\infty^5 \equiv (q^{5k}; q^{5k})_\infty \pmod{5}.$$

3. Congruences Modulo 2 and 4 for $\overline{\mathcal{E}\mathcal{O}}(n)$

In this section we prove some congruences modulo 2 and 4 satisfied by $\overline{\mathcal{E}\mathcal{O}}(n)$. We require the following generating functions to prove congruences for $\overline{\mathcal{E}\mathcal{O}}(n)$.

Theorem 3.1. *We have,*

$$(3.1) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(4n)q^n = \frac{(q^4; q^4)_\infty^5}{(q; q)_\infty^2 (q^8; q^8)_\infty^2},$$

$$(3.2) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(4n+2)q^n = 2 \frac{(q^2; q^2)_\infty^2 (q^8; q^8)_\infty^2}{(q; q)_\infty^2 (q^4; q^4)_\infty},$$

$$(3.3) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n)q^n = \frac{(q^2; q^2)_\infty^5 (q^4; q^4)_\infty^3}{(q; q)_\infty^5 (q^8; q^8)_\infty^2},$$

$$(3.4) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+2)q^n = 2 \frac{(q^4; q^4)_\infty^7}{(q; q)_\infty^3 (q^2; q^2)_\infty (q^8; q^8)_\infty^2},$$

$$(3.5) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+4)q^n = 2 \frac{(q^2; q^2)_\infty^7 (q^8; q^8)_\infty^2}{(q; q)_\infty^5 (q^4; q^4)_\infty^3},$$

$$(3.6) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+6)q^n = 4 \frac{(q^2; q^2)_\infty (q^4; q^4)_\infty (q^8; q^8)_\infty^2}{(q; q)_\infty^3}.$$

Proof. From (1.4), we have

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(n)q^n = \frac{(q^4; q^4)_\infty^3}{(q^2; q^2)_\infty^2},$$

since there are no terms on the right in which the power of q is odd, we have

$$\overline{\mathcal{E}\mathcal{O}}(2n+1) = 0,$$

thus by using (2.6), we obtain

$$(3.7) \quad \begin{aligned} \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(2n)q^n &= \frac{(q^2; q^2)_{\infty}^3}{(q; q)_{\infty}^2} = (q^2; q^2)_{\infty}^3 \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^5} \phi(q) \\ &= \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^2} (\phi(q^4) + 2q\psi(q^8)). \end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(4n)q^n = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \phi(q^2) = \frac{(q^4; q^4)_{\infty}^5}{(q; q)_{\infty}^2 (q^8; q^8)_{\infty}^2}$$

and

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(4n+2)q^n = 2 \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \psi(q^4) = 2 \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2},$$

which is our (3.1) and (3.2). We have

$$(3.8) \quad \begin{aligned} \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(4n)q^n &= \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \phi(q^2) \\ &= (q^2; q^2)_{\infty}^2 \phi(q^2) \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^5} \phi(q) \\ &= \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^3} \phi(q^2) \phi(q) \\ &= \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^3} \phi(q^2) (\phi(q^4) + 2q\psi(q^8)). \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n)q^n &= \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^3} \phi(q) \phi(q^2) \\ &= \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^3} \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2} \frac{(q^4; q^4)_{\infty}^5}{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2} \\ &= \frac{(q^2; q^2)_{\infty}^5 (q^4; q^4)_{\infty}^3}{(q; q)_{\infty}^5 (q^8; q^8)_{\infty}^2} \end{aligned}$$

and

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+4)q^n = 2 \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^3} \phi(q) \psi(q^4) = 2 \frac{(q^2; q^2)_{\infty}^7 (q^8; q^8)_{\infty}^2}{(q; q)_{\infty}^5 (q^4; q^4)_{\infty}^3},$$

which is our (3.3) and (3.5). We have

$$\begin{aligned}
 \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(4n+2)q^n &= 2 \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} \psi(q^4) \\
 &= 2(q^2; q^2)_{\infty}^2 \psi(q^4) \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^5} \phi(q) \\
 &= 2 \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^3} \psi(q^4) \phi(q) \\
 (3.9) \qquad \qquad \qquad &= 2 \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^3} \psi(q^4) (\phi(q^4) + 2q\psi(q^8)).
 \end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+2)q^n = 2 \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^3} \psi(q^2) \phi(q^2) = 2 \frac{(q^4; q^4)_{\infty}^7}{(q; q)_{\infty}^3 (q^2; q^2)_{\infty} (q^8; q^8)_{\infty}^2}$$

and

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+6)q^n = 4 \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^3} \psi(q^2) \psi(q^4) = 4 \frac{(q^2; q^2)_{\infty} (q^4; q^4)_{\infty} (q^8; q^8)_{\infty}^2}{(q; q)_{\infty}^3},$$

which is our (3.4) and (3.6). □

We have the following congruences.

Corollary 3.2. *For all $n \geq 0$,*

$$(3.10) \qquad \qquad \qquad \overline{\mathcal{E}\mathcal{O}}(2n+1) = 0,$$

$$(3.11) \qquad \qquad \qquad \overline{\mathcal{E}\mathcal{O}}(4n+2) \equiv 0 \pmod{2},$$

$$(3.12) \qquad \qquad \qquad \overline{\mathcal{E}\mathcal{O}}(8n+4) \equiv 0 \pmod{2},$$

$$(3.13) \qquad \qquad \qquad \overline{\mathcal{E}\mathcal{O}}(8n+6) \equiv 0 \pmod{4}.$$

Remark 3.3. The congruences (3.11)–(3.13) were obtained earlier by Andrews et al. [4]. Andrews et al. [3] introduced a partition function $p_{\nu}(n)$ which counts the number of partitions of n in which the parts are distinct and all odd parts are less than twice the smallest part.

$$(3.14) \qquad \qquad \qquad \sum_{n=0}^{\infty} p_{\nu}(n)q^n = \nu(-q),$$

where $\nu(q)$ is a mock theta function. Andrews [2, Corollary 5.2] noted that

$$(3.15) \qquad \qquad \qquad p_{\nu}(2n) = \overline{\mathcal{E}\mathcal{O}}(2n).$$

He proved the congruences using the properties of mock theta function, whereas we use the q -series identities.

4. Congruences Modulo 5, 10 and 20 for $\overline{\mathcal{E}\mathcal{O}}(n)$

In this section we prove some congruences modulo 5, 10 and 20 for $\overline{\mathcal{E}\mathcal{O}}(n)$. In the next theorem, we give a simple proof of the Andrews' result [2, Eqn. 1.6], which can be tracked back to [3, Thrm. 6.7]. He used the properties of mock theta functions to prove the congruence, whereas we manipulate the q-series identities to get the result.

Theorem 4.1. *For all $n \geq 0$,*

$$(4.1) \quad \overline{\mathcal{E}\mathcal{O}}(10n + 8) \equiv 0 \pmod{5}.$$

Proof. Applying (2.9) in (1.4), we obtain

$$(4.2) \quad \begin{aligned} \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(2n)q^n &= \frac{(q^2; q^2)_{\infty}^3}{(q; q)_{\infty}^2} = \frac{(q^2; q^2)_{\infty}^3 (q; q)_{\infty}^3}{(q; q)_{\infty}^5} \\ &\equiv \frac{(q^2; q^2)_{\infty}^3 (q; q)_{\infty}^3}{(q^5; q^5)_{\infty}} \pmod{5}. \end{aligned}$$

From (2.7), we have

$$(4.3) \quad (q; q)_{\infty}^3 \equiv J_0 + J_1 \pmod{5},$$

where J_i contains terms in which the power of q is congruent to i modulo 5, then

$$(4.4) \quad (q^2; q^2)_{\infty}^3 \equiv J_0^* + J_2^* \pmod{5},$$

where J_i^* contains terms in which the power of q is congruent to i modulo 5. Substituting (4.3) and (4.4) in (4.2), we have

$$(4.5) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(2n)q^n \equiv \frac{1}{(q^5; q^5)_{\infty}} (J_0 + J_1)(J_0^* + J_2^*) \pmod{5}.$$

There are no terms on the right in which the power of q is 4 modulo 5, so

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(2(5n + 4))q^{5n+4} \equiv 0 \pmod{5},$$

from which we deduce (4.1). \square

In the next theorem, we derive two congruences modulo 10 from the generating functions (3.2) and (3.5).

Theorem 4.2. *For all $n \geq 0$,*

$$(4.6) \quad \overline{\mathcal{E}\mathcal{O}}(20n + 18) \equiv 0 \pmod{10},$$

$$(4.7) \quad \overline{\mathcal{E}\mathcal{O}}(40n + 28) \equiv 0 \pmod{10}.$$

Proof. Using (2.9) in (3.2), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(4n+2)q^n &= 2 \frac{1}{(q; q)_{\infty}^2} \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q^4; q^4)_{\infty}} \\
&= 2 \frac{(q; q)_{\infty}^3 (q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q; q)_{\infty}^5 (q^4; q^4)_{\infty}} \\
(4.8) \qquad \qquad \qquad &\equiv 2 \frac{(q; q)_{\infty}^3 (q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q^5; q^5)_{\infty} (q^4; q^4)_{\infty}} \pmod{10}.
\end{aligned}$$

Replacing q by q^2 in (2.8), we have

$$(4.9) \quad \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q^4; q^4)_{\infty}} = \sum_{n=-\infty}^{\infty} (3n+1)q^{6n^2+4n} \equiv R_0^* + R_1^* + R_2^* \pmod{5},$$

where R_i^* contains terms in which the power of q is congruent to i modulo 5. Substituting (4.3) and (4.9) in (4.8), we obtain

$$(4.10) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(4n+2)q^n \equiv 2 \frac{1}{(q^5; q^5)_{\infty}} (J_0 + J_1) (R_0^* + R_1^* + R_2^*) \pmod{10}.$$

There are no terms on the right in which the power of q is 4 modulo 5, so

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(4(5n+4)+2)q^{5n+4} \equiv 0 \pmod{10},$$

from which we deduce (4.6). Using (2.9) in (3.5), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+4)q^n &= 2 \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^5 (q^4; q^4)_{\infty}^2} \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q^4; q^4)_{\infty}} \\
&= 2(q^4; q^4)_{\infty}^3 \frac{(q^2; q^2)_{\infty}^5}{(q; q)_{\infty}^5 (q^4; q^4)_{\infty}^5} \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q^4; q^4)_{\infty}} \\
(4.11) \qquad \qquad \qquad &\equiv 2(q^4; q^4)_{\infty}^3 \frac{(q^{10}; q^{10})_{\infty}}{(q^5; q^5)_{\infty} (q^{20}; q^{20})_{\infty}} \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q^4; q^4)_{\infty}} \pmod{10}.
\end{aligned}$$

From (2.7), we have

$$(4.12) \quad (q^4; q^4)_{\infty}^3 \equiv J_0^{**} + J_4^{**} \pmod{5},$$

where J_i^{**} contains terms in which the power of q is congruent to i modulo 5. Substituting (4.9) and (4.12) in (4.11), we obtain

$$(4.13) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+4)q^n \equiv 2 \frac{(q^{10}; q^{10})_{\infty}}{(q^5; q^5)_{\infty} (q^{20}; q^{20})_{\infty}} (J_0^{**} + J_4^{**}) (R_0^* + R_1^* + R_2^*) \pmod{10}.$$

There are no terms on the right in which the power of q is 3 modulo 5, so

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8(5n+3)+4)q^{5n+3} \equiv 0 \pmod{10},$$

from which we deduce (4.7). \square

In the next theorem, we derive a congruences modulo 20 from the generating function (3.6).

Theorem 4.3. *For all $n \geq 0$,*

$$(4.14) \quad \overline{\mathcal{E}\mathcal{O}}(40n+38) \equiv 0 \pmod{20}.$$

Proof. Using (2.9) in (3.6), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+6)q^n &= 4 \frac{1}{(q; q)_{\infty}^3} \frac{(q^4; q^4)_{\infty}^2 (q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q^2; q^2)_{\infty} (q^4; q^4)_{\infty}} \\ &= 4 \frac{1}{(q; q)_{\infty}^5} \frac{(q; q)_{\infty}^2 (q^2; q^2)_{\infty}^4 (q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q^2; q^2)_{\infty} (q^4; q^4)_{\infty}} \\ (4.15) \quad &\equiv 4 \frac{1}{(q^5; q^5)_{\infty}} \frac{(q; q)_{\infty}^2 (q^2; q^2)_{\infty}^4 (q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q^2; q^2)_{\infty} (q^4; q^4)_{\infty}} \pmod{20}. \end{aligned}$$

From (2.8), we have

$$(4.16) \quad \frac{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}} = \sum_{n=-\infty}^{\infty} (3n+1)q^{3n^2+2n} \equiv R_0 + R_2 + R_3 \pmod{5},$$

where R_i contains terms in which the power of q is congruent to i modulo 5. Substituting (4.9) and (4.16) in (4.15), we obtain

$$(4.17) \quad \sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8n+6)q^n \equiv 4 \frac{1}{(q^5; q^5)_{\infty}} (R_0 + R_2 + R_3) (R_0^* + R_1^* + R_2^*) \pmod{20}.$$

There are no terms on the right in which the power of q is 4 modulo 5, so

$$\sum_{n=0}^{\infty} \overline{\mathcal{E}\mathcal{O}}(8(5n+4)+6)q^{5n+4} \equiv 0 \pmod{20},$$

from which we deduce (4.14). \square

5. Congruences for $\mathcal{EO}_e(n)$

In this section we prove some congruences modulo 2 for $\mathcal{EO}_e(n)$.

Theorem 5.1.

$$(5.1) \quad \sum_{n=0}^{\infty} \mathcal{EO}_e(4n)q^n = \frac{(q^4; q^4)_{\infty}^5}{(q; q)_{\infty}^3 (q^8; q^8)_{\infty}^2},$$

$$(5.2) \quad \sum_{n=0}^{\infty} \mathcal{EO}_e(4n+2)q^n = 2 \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q; q)_{\infty}^3 (q^4; q^4)_{\infty}^2}.$$

Proof. From (1.5), we have

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(n)q^n = \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^2},$$

since there are no terms on the right in which the power of q is odd, we have

$$\mathcal{EO}_e(2n+1) = 0,$$

by using (2.6), we obtain

$$(5.3) \quad \begin{aligned} \sum_{n=0}^{\infty} \mathcal{EO}_e(2n)q^n &= \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^2} = (q^2; q^2)_{\infty}^2 \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^5} \phi(q) \\ &= \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^3} (\phi(q^4) + 2q\psi(q^8)). \end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(4n)q^n = \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^3} \phi(q^2) = \frac{(q^4; q^4)_{\infty}^5}{(q; q)_{\infty}^3 (q^8; q^8)_{\infty}^2}$$

and

$$\sum_{n=0}^{\infty} \mathcal{EO}_e(4n+2)q^n = 2 \frac{(q^2; q^2)_{\infty}^2}{(q; q)_{\infty}^3} \psi(q^4) = 2 \frac{(q^2; q^2)_{\infty}^2 (q^8; q^8)_{\infty}^2}{(q; q)_{\infty}^3 (q^4; q^4)_{\infty}^2},$$

which is our (5.1) and (5.2). □

We have the following congruences.

Corollary 5.2. For all $n \geq 0$,

$$(5.4) \quad \mathcal{EO}_e(2n+1) = 0,$$

$$(5.5) \quad \mathcal{EO}_e(4n+2) \equiv 0 \pmod{2}.$$

6. Conclusion

Andrews [2, Problem 4], proposed to further investigate the properties of $\overline{\mathcal{E}\mathcal{O}}(n)$. We conclude the paper with the following conjecture. Using maple, we found the following congruences hold up to $n = 2000$.

Conjecture 6.1. *For all $n \geq 0$,*

$$(6.1) \quad \overline{\mathcal{E}\mathcal{O}}(50n + 18) \equiv 0 \pmod{20},$$

$$(6.2) \quad \overline{\mathcal{E}\mathcal{O}}(50n + 28) \equiv 0 \pmod{20},$$

$$(6.3) \quad \overline{\mathcal{E}\mathcal{O}}(50n + 38) \equiv 0 \pmod{20},$$

$$(6.4) \quad \overline{\mathcal{E}\mathcal{O}}(50n + 48) \equiv 0 \pmod{20}.$$

Acknowledgements. The authors would like to thank Professor Michael Hirschhorn for his valuable comments and helpful suggestions.

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