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# SOME NEW APPLICATIONS OF $S$-METRIC SPACES BY WEAKLY COMPATIBLE PAIRS WITH A LIMIT PROPERTY

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Abstract. In this note we use a generalization of coincidence point(a property which was defined by [1] in symmetric spaces) to prove common fixed point theorem on $S$-metric spaces for weakly compatible maps. Also the results are used to achieve the solution of an integral equation and the bounded solution of a functional equation in dynamic programming.

## 1. Introduction

Fixed point theorems play a principal role in solving integral equations [2, 3] arising in several areas of mathematics and other related subjects. In 1992, Dhage [4] offered the concept of a $D$-metric space. Later on, in 2006, Mustafa and Sims [8] showed that most of the results concerning Dhage$'s$ $D$-metric space are invalid. Therefore, they introduced a new notion of a generalized metric space, called $G$-metric space. Recently, Sedghi et al. [10] introduced the concept of S-metric space and some of their properties. In this note, we use a geralization of coincidence point on $S$-metric spaces to find a procedure to prove some type of fixed point theorems and applying its consequences to get a solution for an integral equation and a functional equation in dynamic programming.

## 2. Basic Concepts

First we recall some notions, lemmas and examples which will be useful later (see [10]):

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**Definition 2.1.** Let $X$ be a nonempty set. An $S$-metric on $X$ is a function $S : X^3 \to [0, \infty)$ which satisfies the following conditions for all $x, y, z, a \in X$
(i) $S(x, y, z) \geq 0$,
(ii) $S(x, y, z) = 0$ if and only if $x = y = z$,
(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
The set $X$ with an $S$-metric is called an *$S$-metric space*.

The standard examples of $S$-metric spaces are:
(a) Let $X$ be any normed space, then $S(x, y, z) = \| y + z - 2x \| + \| y - z \|$ is an $S$-metric on $X$.
(b) Let $(X, d)$ be a metric space, then $S_d(x, y, z) = d(x, z) + d(y, z)$ is an $S$-metric on $X$. This $S$-metric is called the *usual* $S$-metric on $X$.
(c) Another $S$-metric on $(X, d)$ is $S'_d(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ which is symmetric with respect to the argument.
(d) Let $X = [0, +\infty)$, then

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise}, \end{cases}$$

is an $S$-metric. This $S$-metric can not be defined by a usual $S$-metric. We call $S$ the *maximum* $S$-metric on $X$.

In this note, we will often use the following important facts.

**Lemma 2.1** ([10])**.** *In any $S$-metric space $(X, S)$, we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

**Definition 2.2.** A sequence $\{x_n\}$ in $X$ converges to $x$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. We denote this by $\lim_{n\to\infty} x_n = x$ .

**Definition 2.3.** A sequence $\{x_n\}$ in an $S$-metric space $(X, S)$ is called a *Cauchy sequence* if $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

**Lemma 2.2** ([10])**.** *Let $(X, S)$ be an $S$-metric space then,*
a. *The limit of a sequence in an $S$-metric space is unique.*
b. *Every convergent sequence in an $S$-metric space is a Cauchy sequence.*
c. *If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.*

There exists a natural topology on an $S$-metric spaces. First, let us define the notion of (open) ball.

**Definition 2.4.** Let $(X, S)$ be an $S$-metric space. For $r > 0$ and $x \in X$ we define a ball with center $x$ and radius $r$ as follows:

$$B_s(x, r) = \{y \in X : S(y, y, x) < r\}.$$

This is a quite different concept of a ball in a usual metric space. We have:

**Example 2.1.** Let $S_d(x, y, z) = d(x, z) + d(y, z)$ be the usual $S$-metric on $(X, d)$ and let $x_0 \in X$. Then:

$$\begin{aligned} B_s(x_0, 2) = \{y \in X : S(y, y, x_0) < 2\} &= \{y \in X : 2d(y, x_0) < 2\} \\ &= \{y \in X : d(y, x_0) < 1\} = B_d(x_0, 1). \end{aligned}$$

By using this notion of a ball, we can introduce the standard topology on $S$-metric space.

**Definition 2.5.** The $S$-metric space $(X, S)$ is said to be *complete* if every Cauchy sequence converges.

We have the following result:

**Lemma 2.3** ([7])**.** *Any $S$-metric space is Hausdorff.*

**Remark 2.1.** We have:
$x_n \to x$ in $(X, d)$ if and only if $d(x_n, x) \to 0$, if and only if $S_d(x_n, x_n, x) = 2d(x_n, x) \to 0$, that is, $x_n \to x$ in $(X, S_d)$.

**Definition 2.6** ([10])**.** Let $(X, S)$ be an $S$-metric space. A self-map $T : X \to X$ is called a *contraction map* if there exists a constant $0 \leq k < 1$ such that

$$S(Tx, Tx, Ty) \leq kS(x, x, y), \quad \text{for all} \quad x, y \in X.$$

**Theorem 2.1** ([10])**.** *Let $(X, S)$ be a complete $S$-metric space and $T : X \to X$ be a contraction map. Then, $F$ has a unique fixed point.*

**Definition 2.7** ([1])**.** Let $L$ and $T$ be two self-maps on a $S$-metric space $(X, S)$. Then, the pair $(L, T)$ is said to be *weakly compatible* if they commute at their coincidence points, that is, if $Lu = Tu$ for some $u \in X$, then $TLu = LTu$.

**Definition 2.8** ([5])**.** Let $L$ and $T$ be two self-maps on an $S$-metric space $(X, S)$. We say the pair $(L, T)$ generalize the *coincidence point* if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to+\infty} Lx_n = \lim_{n\to+\infty} Tx_n = t$, for some $t \in X$. We call it *Limit Property.* We were mentioned that, this property had called by [1] (E.A.).

The following examples are from [5]:

**Example 2.2.** Let $X = [2, +\infty)$. Define $L, T : X \to X$ by $L(x) = 2x + 1$ and $T(x) = x + 1$ for all $x \in X$. Suppose that the property $E.A.$ holds. Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to+\infty} Lx_n = \lim_{n\to+\infty} Tx_n = t$ for some $t \in X$. It follows that $\lim_{n\to+\infty} x_n = \frac{t-1}{2}$ and $\lim_{n\to+\infty} x_n = t - 1$ and so, by uniqueness of limit, $t = 1$ but $t \notin X$. Therefore, $L$ and $T$ do not satisfy the Limit Property.

**Example 2.3.** Let $X = [0, +\infty)$. Define $L, T : X \to X$ by $L(x) = \frac{3}{4}x$ and $T(x) = \frac{x}{4}$ for all $x \in X$. Consider the sequence $x_n = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ in $X$. Clearly, $\lim_{n\to+\infty} Lx_n = \lim_{n\to+\infty} Tx_n = 0 \in X$ , so $L$ and $T$ satisfy the Limit Property.

Let the nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ satisfies following properties (see [6]):
$(M1)$ $\lim_{n\to+\infty} \phi^n(t) = 0$, for all $t \in (0, +\infty)$,
$(M2)$ $\phi(t) < t$ for all $t \in (0, +\infty)$,
$(M3)$ $\phi(0) = 0$.
The set of all functions such as $\phi$ is denoted by $\Phi$.

## 3. Main Results

The following theorem is our main result :

**Theorem 3.1.** *Let $(X, S)$ be an $S$-metric space and $A, B, H, T : X \to X$ be four self-mappings such that:*
*(a) $S(Ax, Ax, By) \le \phi(max\{S(Hx, Hx, Ty), S(Hx, Hx, By), S(Ty, Ty, By)\})$, for all $x, y \in X$ and $\phi \in \Phi$,*
*(b) $B(X) \subseteq H(X)$ and $A(X) \subseteq T(X)$,*
*(c) $(A, H)$ or $(B, T)$ satisfies the Limit Property,*
*(d) $A(X), B(X), H(X)$ or $T(X)$ is a closed subset of $X$.*
*Then $(A, H)$ and $(B, T)$ have a coincidence point. Further, if $(A, H)$ and $(B, T)$ are weakly compatible, then $A, B, H$ and $T$ have a unique common fixed point in $X$.*

*Proof.* Suppose $(B, T)$ satisfies the Limit Property. Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to+\infty} Bx_n = \lim_{n\to+\infty} Tx_n = t$, for some $t \in X$. Since $B(X) \subseteq H(X)$, there exists a sequence $\{y_n\}$ in $X$ such that $Bx_n = Hy_n$. Hence

$\lim_{n\to+\infty} Hy_n = t$. We will show that $\lim_{n\to+\infty} Ay_n = t$. We have:

$$\begin{aligned}&S(Ay_n, Ay_n, Bx_n)\\ &\leq \phi(\max\{S(Hy_n, Hy_n, Tx_n), S(Hy_n, Hy_n, Bx_n), S(Tx_n, Tx_n, Bx_n)\}),\end{aligned}$$

so,

$$\begin{aligned}&\max\{S(Hy_n, Hy_n, Tx_n), S(Hy_n, Hy_n, Bx_n), S(Tx_n, Tx_n, Bx_n)\}\\ &= \begin{cases} S(Hy_n, Hy_n, Tx_n) & or \\ S(Hy_n, Hy_n, Bx_n) & or \\ S(Tx_n, Tx_n, Bx_n). \end{cases}\end{aligned}$$

Hence,

$$\begin{aligned}&S(Ay_n, Ay_n, Bx_n)\\ &\leq \phi(\max\{S(Hy_n, Hy_n, Tx_n), S(Hy_n, Hy_n, Bx_n), S(Tx_n, Tx_n, Bx_n)\})\\ &= \begin{cases} \phi(S(Hy_n, Hy_n, Tx_n)) & or \\ \phi(S(Hy_n, Hy_n, Bx_n)) & or \\ \phi(S(Tx_n, Tx_n, Bx_n)). \end{cases}\end{aligned}$$

Assume that $S(Ay_n, Ay_n, Bx_n) \leq \lim_{n\to+\infty} S(Hy_n, Hy_n, Tx_n)$. Since $\phi(t) \leq t$ for all $t \in [0, \infty)$, then, by taking limit we have:

$$\lim_{n\to+\infty} S(Ay_n, Ay_n, Bx_n) \leq \lim_{n\to+\infty} S(Hy_n, Hy_n, Tx_n),$$

by Lemma 2.2, $\lim_{n\to+\infty} S(Hy_n, Hy_n, Tx_n) = S(t, t, t) = 0$, that is,

$$\lim S(Ay_n, Ay_n, Bx_n) = 0.$$

The above equality holds similarly for other cases.
By Definition 2.1(iii) and Lemma 2.1, we have:

$$\begin{aligned}S(Ay_n, Ay_n, t) &\leq 2S(Ay_n, Ay_n, Bx_n) + S(t, t, Bx_n)\\ &= 2S(Ay_n, Ay_n, Bx_n) + S(Bx_n, Bx_n, t).\end{aligned}$$

Now by taking limit and using third part of Lemma 2.2 we have, $\lim S(Ay_n, Ay_n, t) = 0$, hence by Definition 2.2, $\lim_{n\to+\infty} Ay_n = t$. That is,

$$\lim_{n\to+\infty} Ay_n = \lim_{n\to+\infty} Bx_n = \lim_{n\to+\infty} Hy_n = \lim_{n\to+\infty} Tx_n = t.$$

Suppose that $H(X)$ is a closed subset of $X$, then, $t = Hu$ for some $u \in X$. We show that $Au = Hu = t$. From (a), we have:

$$S(Au, Au, Bx_n) \leq \phi(\max\{S(Hu, Hu, Tx_n), S(Hu, Hu, Bx_n), S(Tx_n, Tx_n, Bx_n)\}).$$

Without loss of generality, assume that

$$\phi(\max\{S(Hu,Hu,Tx_n),S(Hu,Hu,Bx_n),S(Tx_n,Tx_n,Bx_n)\}) \leq \phi(S(Hu,Hu,Tx_n)),$$

then,

$$\begin{aligned} S(Au,Au,Bx_n) &\leq \phi(\max\{S(Hu,Hu,Tx_n),S(Hu,Hu,Bx_n),S(Tx_n,Tx_n,Bx_n)\}) \\ &\leq \phi(S(Hu,Hu,Tx_n)) \\ &\leq S(Hu,Hu,Tx_n), \end{aligned}$$

by taking limit, we have:

$$\begin{aligned} \lim_{n\to+\infty} S(Au,Au,Bx_n) \leq \lim_{n\to+\infty} & S(Hu,Hu,Tx_n) \\ &\overset{Lemma 2.1}{=\!=\!=} \lim_{n\to+\infty} S(Tx_n,Tx_n,Hu) \\ &=\!=\!= \lim_{n\to+\infty} S(Tx_n,Tx_n,t) \\ &\overset{Lemma 2.4}{=\!=\!=} \lim_{n\to+\infty} S(t,t,t) = 0, \end{aligned}$$

Hence, $\lim_{n\to+\infty} S(Au,Au,Bx_n) = 0$. Now, observe that

$$\begin{aligned} S(Au,Au,t) &\leq S(Au,Au,Bx_n) + S(Au,Au,Bx_n) + S(t,t,Bx_n) \\ &= 2S(Au,Au,Bx_n) + S(Bx_n,Bx_n,t), \end{aligned}$$

by taking limit and the fact that $\lim_{n\to+\infty} S(Bx_n,Bx_n,t) = S(t,t,t) = 0$, we have, $S(Au,Au,t) = 0$, therefore $Au = t$.

Hence, u is a coincidence point of the pair $(A,H)$. Since $A(X) \subseteq T(X)$, there exists $v \in X$ such that $Au = Tv$. We claim that $Tv = Bv$. Suppose that $Tv \neq Bv$, by hypothesis (a) and by (M2), we have:

$$\begin{aligned} S(Au,Au,Bv) &\leq \phi(\max\{S(Hu,Hu,Tv),S(Hu,Hu,Bv),S(Tv,Tv,Bv)\}) \\ &= \phi(\max\{0,S(Au,Au,Bv),S(Au,Au,Bv)\}) \\ &= \phi(S(Au,Au,Bv)) \\ &< S(Au,Au,Bv). \end{aligned}$$

This is a contradiction. Hence $Au = Bv$ and $Tv = Bv$. So $(B,T)$ has a coincidence point. Therefore, we have $Bv = Tv = Hu = Au$.
Now, if $B$ and $T$ are weakly compatible, then we have $BTv = TBv = TTv = BBv$ and the weak compatibility of $A$ and $H$ implies that $AHu = HAu$. Hence, $AAu = AHu = HAu = HHu$. We show that $Au$ is a common fixed point of $A,B,H$ and

$T$. Suppose that $AAu \neq Au$. By hypothesis (a) and by (M2), we have:

$$\begin{aligned} S(AAu, AAu, Au) &= S(AAu, AAu, Bv) \\ &\leq \phi(\max\{S(HAu, HAu, Tv), S(HAu, HAu, Bv), S(Tv, Tv, Bv)\}) \\ &= \phi(\max\{S(AAu, AAu, Bv), S(AAu, AAu, Bv), S(Bv, Bv, Bv)\} \\ &= \phi(\max\{S(AAu, AAu, Bv), S(AAu, AAu, Bv), 0\} \\ &= \phi(S(AAu, AAu, Bv)) \\ &< S(AAu, AAu, Bv). \end{aligned}$$

This is a contradiction. Hence, $Au = AAu = Bv$. Therefore, $Au = AAu = HAu$ is a common fixed point of $A$ and $H$. By a similar argument, $Bv$ is a common fixed point of $B$ and $T$. Since $Au = Bv$, we deduce that $Au$ is a common fixed point of $A, B, H$ and $T$. Only uniqueness of common fixed point has remained. Suppose that $w$ and $z$ are two different common fixed points of $A, B, H$ and $T$, then, by hypothesis (a) and by (M2), we have:

$$\begin{aligned} S(w, w, z) &= S(Aw, Aw, Bz) \\ &\leq \phi(\max\{S(Hw, Hw, Tz), S(Hw, Hw, Bz), S(Tz, Tz, Bz)\}) \\ &= \phi(\max\{S(w, w, z), S(w, w, z), S(z, z, z)\}) \\ &= \phi(S(w, w, z)) \\ &< S(w, z, z), \end{aligned}$$

which is a contradiction. Hence, $w = z$. Therefore, $A, B, H$ and $T$ have a unique common fixed point. □

By taking $H = T$ in Theorem 3.1, the results for three self-mappings $A, B$ and $T$ are satisfied. We have the following corollary:

**Corollary 3.1.** *Let $(X, S)$ be an $S$-metric space and $A, B, H : X \to X$ be three self mappings such that:*
*(a) $S(Ax, Ax, By) \leq \phi(\max\{S(Hx, Hx, Hy), S(Hx, Hx, By), S(Hy, Hy, By)\})$, where $\phi \in \Phi$, for all $x, y \in X$.*
*(b) $A(X) \subseteq H(X)$ and $B(X) \subseteq H(X)$,*
*(c) $(A, H)$ or $(B, H)$ satisfies the Limit Property,*
*(d) $A(X), B(X)$ or $H(X)$ is a closed subset of $X$.*
*Then the pairs $(A, H)$ and $(B, H)$ have a coincidence point. Further, if $(A, H)$ and*

*$(B,H)$ are weakly compatible, then $A,B$ and $H$ have a unique common fixed point in $X$.*

**Example 3.1.** Equip $X=[1,+\infty]$ with the maximum $S$-metric. Define $A,B,H : X\to X$ by $Ax=x$, $Bx=2x-1$ and $Hx=x^2$ for all $x\in X$ and $\phi:[0,+\infty)\to[0,+\infty)$ by $\phi(t)=t$ for all $t\geq 0$. The pair $(A,H)$ satisfies the Limit Property. Also, the hypotheses (b) and (d) of Corollary 3.1 hold trivially. We have:

$$S(Ax,Ax,By)=\begin{cases} 0 & \text{if } x=2y-1,\\ 2y-1 & \text{if } x<2y-1,\\ x & \text{if } 2y-1<x.\end{cases}$$

$$S(Hx,Hx,Hy)=\begin{cases} y^2 & \text{if } x<y,\\ x^2 & \text{if } y<x,\\ 0 & \text{if } x=y.\end{cases}$$

$$S(Hx,Hx,By)=\begin{cases} x^2 & \text{if } 2y-1<x^2,\\ 2y-1 & \text{if } x^2<2y-1,\\ 0 & \text{if } x^2=2y-1.\end{cases}$$

$$S(Hy,Hy,By)=\begin{cases} 0 & \text{if } y=1,\\ y^2 & \text{if } y\neq 1,\end{cases}$$

So for $x<2y-1$, $x<y$, $x^2<2y-1$, $y\neq 1$, we have:

$$2y-1\leq \max\{y^2,2y-1\}=y^2.$$

For $x<2y-1$, $x<y$, $2y-1<x^2$ and $y\neq 1$, we have:

$$2y-1\leq \max\{y^2,x^2\}=y^2.$$

For $y<x$, $2y-1<x$, $y\neq 1$, we have:

$$x\leq \max\{x^2,y^2\}=x^2.$$

So, the inequality (a) in Corollary 3.1 is correct(other cases are trivial). Hence, the pairs $(B,H)$ and $(A,H)$ have a coincidence point. In addition, since $(B,H)$ and $(A,H)$ are weakly compatible, so $A,B$ and $H$ have the unique common fixed point 1.

The major result of this paper is finding a solution for the following integral equation by applying Corollary 3.1.

Let $X=[0,1]$ and $C(X)$ be the space of all the real valued continuous functions defined on X. Also, suppose that the $S$-metric on this space is as follows:

$$S(x,y,z)=\sup_{t\in X}|x(t)-z(t)|+\sup_{t\in X}|y(t)-z(t)|,\quad \text{for all}\quad x,y,z\in C(X).$$

Clearly (C(X),S) is a complete $S$-metric space.
Let $p : X\times\mathbb{R} \to \mathbb{R}$ and $q : X\times X\times\mathbb{R} \to \mathbb{R}$ be two continuous functions and consider the following integral equation:

$$p(t,x(t)) = \int_X q(t,r,x(r))dr, \quad x \in C(X). \tag{1}$$

We have the following theorem:

**Theorem 3.2.** *Suppose $T : X\times\mathbb{R} \to [0,+\infty)$ is a function such that:*
*(a) $T(t,v(t)) \le \int_X q(t,r,u(r))dr \le p(t,v(t)) \quad$ for all $\quad r,t \in X$,*
*(b) $p(t,v(t)) - T(t,v(t)) \le k|p(t,v(t)) - v(t)|$, where $k \in (0,1)$.*
*Then the integral equation (1) has a solution in C(X).*

*Proof.* Define $(Ax)(t) = \int_X q(t,r,x(r))dr$ and $(Bx)(t) = p(t,x(t))$. Now we have:

$$\begin{aligned} S(Ax,Ax,By) &= 2\sup_{t\in X}|(Ax)(t)-(By)(t)| \\ &= 2\sup_{t\in X}\left|p(t,y(t)) - \int_X q(t,r,x(t))dt\right| \\ &\le 2\sup_{t\in X}|p(t,y(t)) - T(t,y(t))| \\ &\le 2k\sup_{t\in X}|p(t,y(t)) - y(t)| = kS(y,y,By). \end{aligned}$$

We put $H = id_{C(X)}$ and $\phi(l) = kl$ for all $l \ge 0$ and $k \in (0,1)$, so we have:

$$\begin{gathered} S(Ax,Ax,By) \le kS(y,y,By) = \phi(S(y,y,By)) \le \\ \phi(\max\{S(x,x,y), S(x,x,By), S(y,y,By)\}), \end{gathered}$$

hence, hypothesis (a) of Corollary 3.1 is satisfied.
To prove the Limit Property, let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to+\infty} Ax_n = t$, assume $y_n = Ax_n$. We show that for every $n \in \mathbb{N}$, $\quad By_n = y_n$. Hence we have $\lim_{n\to+\infty} y_n = t = \lim_{n\to+\infty} By_n$. We have:

$$\begin{aligned} S(Ax_n,Ax_n,By_n) &\le kS(y_n,y_n,By_n) \\ &\Rightarrow S(y_n,y_n,By_n) \le kS(y_n,y_n,By_n) \\ &\Rightarrow kS(y_n,y_n,By_n) = 0. \end{aligned}$$

Then, $y_n = By_n$ for every $n \in \mathbb{N}$.
Also, since $H(X) = X$, both hypotheses (b) and (d) are satisfied. Obviously, $(A, id_{C(X)})$ and $(B, id_{C(X)})$ are weakly compatible, hence there is a unique solution of integral equation (1) in $C(X)$. □

The problem of dynamic programming related to a multistage process reduces to the subject of solving functional equations. In this part, we want to solve the following functional equation (2) by Corollary 3.1. Suppose that $U$ and $V$ are Banach spaces, $W \subseteq U$ is a state space, which is the set of the initial state, actions and transition model of the process and $D \subseteq V$ is a decision space, which is the set of possible actions that are allowed for the process, we set:

$$Q : W \to \mathbb{R}$$

$$Q(x) = \sup_{y \in D}\{f(x,y) + K(x,y,Q(\tau(x,y)))\}, \quad x \in W, \tag{2}$$

where $\tau : W \times D \to W$, $f : W \times D \to \mathbb{R}$, $K : W \times D \times \mathbb{R} \to \mathbb{R}$. Let $B(W)$ denote the space of all bounded real-valued functions on $W$. We equip $B(W)$ with the following $S$-metric, which is obviously a complete $S$-metric space,

$$S(h,k,p) = \sup_{x \in W} |h(x) - p(x)| + \sup_{x \in W} |k(x) - p(x)| \quad \text{for all} \quad h,k,p \in B(W).$$

Now, we state the main result of this part.

**Theorem 3.3.** Let $f : W \times D \to \mathbb{R}$ and $K : W \times D \times \mathbb{R} \to \mathbb{R}$ be two bounded functions and also $\tau : W \times D \to W$ be a function. Let $A : B(W) \to B(W)$ be defined by

$$(A(h))(x) = \sup_{y \in D}\{f(x,y) + K(x,y,(h)(\tau(x,y)))\},$$

for all $h \in B(W)$ and $x \in W$. Suppose that the following condition holds:

$$|K(x,y,h(\tau(x,y))) - K(x,y,k(\tau(x,y)))| \leq \frac{1}{2}\phi(|h(x) - k(x)|), \tag{3}$$

where $x \in W$, $y \in D$ and $\phi \in \Phi$. Then the functional equation (2) has a unique bounded solution.

*Proof.* We like to remind that $(B(W), S)$ is a complete $S$-metric space. Let $\epsilon$ be an arbitrary positive number, $x \in W$ and $h_1, h_2 \in B(W)$, then there exist $y_1, y_2 \in D$ such that

$$(A(h_1))(x) < f(x,y_1) + K(x,y_1,h_1(\tau(x,y_1))) + \frac{\epsilon}{2}, \tag{4}$$

$$(A(h_2))(x) < f(x,y_2) + K(x,y_2,h_2(\tau(x,y_2))) + \frac{\epsilon}{2}, \tag{5}$$

$$(A(h_1))(x) \geq f(x,y_2) + K(x,y_2,h_1(\tau(x,y_2))), \tag{6}$$

$$(A(h_2))(x) \geq f(x,y_1) + K(x,y_1,h_2(\tau(x,y_1))). \tag{7}$$

Then by (4), (7) and (3) we have: (inequalities (6),(7) are true for all $y_1, y_2 \in D$),

$$\begin{aligned}(A(h_1))(x) - (A(h_2))(x) &< K(x, y_1, h_1(\tau(x, y_1))) - K(x, y_1, h_2(\tau(x, y_1))) + \frac{\epsilon}{2}\\ &\leq |K(x, y_1, h_1(\tau(x, y_1))) - K(x, y_1, h_2(\tau(x, y_1)))| + \frac{\epsilon}{2}\\ &\leq \frac{1}{2}(\phi(|h_1(x) - h_2(x)|) + \epsilon).\end{aligned}$$

Therefore we get:

$$(A(h_1))(x) - (A(h_2))(x) \leq \frac{1}{2}(\phi(|h_1(x) - h_2(x)|) + \epsilon). \tag{8}$$

Similarly, by (5) , (6) and (3), we obtain:

$$(A(h_2))(x) - (A(h_1))(x) \leq \frac{1}{2}(\phi(|h_1(x) - h_2(x)|) + \epsilon). \tag{9}$$

Therefore, by(8) and (9), we have:

$$2|(A(h_1))(x) - (A(h_2))(x)| \leq \phi(|h_1(x) - h_2(x)|) + \epsilon. \tag{10}$$

which implies

$$S\big((A(h_1))(x), (A(h_1))(x), (A(h_2))(x)\big) < \phi\big(S(h_1(x), h_1(x), h_2(x))\big) + \epsilon. \tag{11}$$

Since $\epsilon > 0$ is arbitrary, we can deduce that

$$S\big((A(h_1))(x)), (A(h_1))(x)), (A(h_2))(x))\big) \leq \phi\big(S(h_1(x), h_1(x), h_2(x))\big).$$

Thus, all the hypothesis of Corollary 3.1 are satisfied with $A = B$ and $H = id_{B(W)}$, the identity map on $B(W)$. Therefore, functional equation (2) has a unique bounded solution. □

**Example 3.2.** Let consider the following functional equation

$$(A(h))(x) = \sup_{y \in D} \Big\{ \arctan(x + 3|y|) + \frac{1}{2} \ln \big(1 + x + \frac{1}{1 + |y|} + |h(x)|\big)\Big\} \tag{12}$$

for $x \in [0, 1]$, where $W = [0, 1]$, $D = \mathbb{R}$. Then,

$f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is defined by $f(x, y) = \arctan(x + 3|y|)$,

$\tau : [0, 1] \times \mathbb{R} \to [0, 1]$ is defined by $\tau(x, y) = x$, and

$K : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by $K(x, y, t) = \frac{1}{2} \ln(1 + x + \frac{1}{1+|y|} + |t|)$.

It's clear that $|f(x, y)| \leq \frac{\pi}{2}$ and $|K(x, y, 0)| = \big|\frac{1}{2} \ln(1 + x + \frac{1}{1+|y|})\big| < \ln 3$ for all $x \in [0, 1]$ and all $y \in \mathbb{R}$.

Hence the first assumption of Theorem 3.3 is satisfied. Furthermore, consider the continuous function $\phi(h) = \ln(1 + h)$ for all $h \in [0, \infty]$. Therefore, for all $x \in [0, 1]$

and all $y, k \in \mathbb{R}$ (we can assume that $|h| > |k|$ without loss of generality), it follows that:

$$\begin{aligned}
&\big|K(x,y,h(x)) - K(x,y,k(x))\big| \\
=&\left|\frac{1}{2}\ln(1+x+\frac{1}{1+|y|}+|h(x)|) - \frac{1}{2}\ln(1+x+\frac{1}{1+|y|}+|k(x)|)\right| \\
=&\frac{1}{2}\left|\ln\frac{1+x+\frac{1}{1+|y|}+|h(x)|}{1+x+\frac{1}{1+|y|}+|k(x)|}\right| \\
=&\frac{1}{2}\left|\ln\frac{1+x+\frac{1}{1+|y|}+|k(x)|+(|h(x)|-|k(x)|)}{1+x+\frac{1}{1+|y|}+|k(x)|}\right| \\
=&\frac{1}{2}\left|\ln\left(1+\frac{(|h(x)|-|k(x)|)}{1+x+\frac{1}{1+|y|}+|k(x)|}\right)\right| \\
\leq&\frac{1}{2}\left|\ln(1+(|h(x)|-|k(x)|)\right| \\
=&\frac{1}{2}\ln(1+(|h(x)|-|k(x)|) = \frac{1}{2}\ln(1+\big||h(x)|-|k(x)|\big| \\
\leq&\frac{1}{2}\ln(1+|h(x)-k(x)|) = \frac{1}{2}\phi(|h(x)-k(x)|.
\end{aligned}$$

Then inequality (3) in theorem (3.3) also holds where $x \in [0,1]$, $y \in \mathbb{R}$ and $\phi \in \Phi$, which implies functional equation (12) has a unique bounded solution $h \in B[0,1]$.


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