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SEMI-INVARINAT SUBMANIFOLDS OF CODIMENSION 3 SATISFYING $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$ IN A COMPLEX SPACE FORM

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Abstract. Let M be a semi-invariant submanifold of codimension 3 with almost contact metric structure (ϕ, ξ, η, g) in a complex space form $M_{n+1}(c)$, $c \neq 0$. We denote by $R_\xi = R(\cdot, \xi)\xi$ and $A^{(i)}$ be Jacobi operator with respect to the structure vector field ξ and be the second fundamental form in the direction of the unit normal $C^{(i)}$, respectively. Suppose that the third fundamental form t satisfies $dt(X, Y) = 2\theta g(\phi X, Y)$ for certain scalar θ ($\neq 2c$)and any vector fields X and Y and at the same time R_ξ is $\phi\nabla_\xi\xi$-parallel, then M is a Hopf hypersurface in $M_n(c)$ provided that it satisfies $R_\xi A^{(1)} = A^{(1)} R_\xi$, $R_\xi A^{(2)} = A^{(2)} R_\xi$ and $\bar{r} - 2(n-1)c \leq 0$, where $\bar{r}$ denotes the scalar curvature of M.

1. Introduction

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold with parallel almost complex structure J and a Riemannian metric tensor G. A submanifold M is called a *CR submanifold* of $\tilde{M}$ if it is endowed with a pair of mutually orthogonal and complementary differentiable distribution $(T, T^\perp)$ such that for any $p \in M$ we have $JT_p = T_p$, $JT_p^\perp \subset T_p^\perp M$, where $T_p^\perp M$ denotes the normal space of M at p ([1],[33]). In particular, M is said to be a *semi-invariant submanifold* if $\dim T^\perp = 1$, and the unit normal in $JT^\perp$ is called a *distinguished normal* to M ([5],[31]). In this case, M induces an almost contact metric structure (ϕ, ξ, η, g). A typical example of a semi-invariant submanifolds is a real hypersurface. And new examples of nontrivial semi-invariant submanifold in a complex projective space are contructed in [21] and [28]. Accordingly, we may expect to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold. When the ambient manifold $\tilde{M}$ is a complex space form $M_n(c)$ with constant holomorphic sectional curvature $4c$, a real hypersurface was investigated by many authors, ([2], [6], [12], [22], [23], [25], [29] and [30] etc.). The structure vector ξ is said to be *principal* if $A\xi = \alpha\xi$, where A is denote by the shape operators of real hypersurface and $\alpha = \eta(A\xi)$. A real

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hypersurface is said to a *Hopf hypersurface* if the structure vector ξ is principal. One of them, Takagi ([29], [30]) classified all homogeneous real hypersurfaces of a complex projective space as six model spaces which are said to be of type A_1, A_2, B, C, D and E, and Cecil-Ryan([6]) and Kimura ([22]) prove that they are realized as the tubes of constant radius over Kaehlerian submanifolds when the structure vector field ξ is principal.

On the other hand, real hypersurfaces of a complex hyperbolic space have been also studied by Berndt [2], Berndt and Tamura [3], Montiel and Romero [21] and so on. Berndt [2] classified all Hopf hypersurfaces of a complex hyperbolic space and showed that they are realized as the tubes of constant radius over certain submanifolds. Also such kinds of tubes are said to be real hypersurfaces of type A_0, A_1, A_2 and type B.

Relate to the structure vector field ξ the Jacobi operator R_ξ defined by $R_\xi = R(\cdot, \xi)\xi$ for the curvature tensor R on a real hypersurface in a complex space form is said to be a *sturcture Jacobi operator* on the hypersurface. The sturcture Jacobi operator has a fundamental role in contact geometry. Some works have recentry studied several conditions on the structure Jacobi operator R_ξ and given some results on the characterization of real hypersurfaces in a complex space form ([8], [9], [15] $\sim$ [18], etc.). Recently Ortega et al. [27] have proved that there are no real hypersurface in a complex space form with parallel structure Jacobi operator R_ξ, that is, $\nabla_X R_\xi = 0$ for any vector field X. In this situation it naturally leads us to be consider another condition weaker than parallelness. In the proceding work we investigate real hypersurfaces in a complex space form with the weaker condition ξ-parallelness, that is, $\nabla_\xi R_\xi = 0$ ([8], [9]) or $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$ ([16], [17]).

For $\phi\nabla_\xi\xi$-parallelness, we introduce the following theorem without proof.

Theorem KK ([16]). *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. If it satisfies $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$ and at the same time $R_\xi A = AR_\xi$, then M is a Hopf hypersurface in $M_n(c)$, where A denote the shape operator of M.*

On the other hand, semi-invariant submanifolds of codimension 3 in a complex space form $M_{n+1}(c)$ have been studied in [13], [14], [19] $\sim$ [21] and so on by using properties of induced almost contact metric structure and those of the third fundamental form of submanifolds.

In the preceding work, Takagi, Song and the present author assert the following :

Theorem KST([21]). *Let M be a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ with constant holomorphic sectional curvature $4c$. If the structure vector ξ is an eigenvector for the shape operator in the direction of the distinguished normal and the third fundamental form t satisfies $dt = 2\theta\omega$ for a certain scalar $\theta(< 2c)$, where $\omega(X, Y) = g(\phi X, Y)$ for any vector fields X and Y on M, then M is a Hopf hypersurface in a complex projective space $P_n\mathbb{C}$.*

In continuting work [19], Lee and the present author proved that if a semi-invariant submanifold M satisfying hypotheses of above theorem in a complex hyperbolic space $H_{n+1}\mathbb{C}$, then M is a Hopf hypersurface in $H_n\mathbb{C}$.

In this paper, we consider a semi-invariant submanifold M of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ which satisfies $R_\xi A = AR_\xi$ and $R_\xi A^{(2)} = A^{(2)}R_\xi$, where A is the second fundamental tensor in the direction of the distinguished normal and R_ξ is the structure Jacobi operator defined in a semi-invariant submanifold.

In the present paper, we also prove that if the structure Jacobi operator R_ξ is $\phi\nabla_\xi\xi$- parallel and the third fundamental form t satisfies $dt(X,Y) = 2\theta g(\phi X, Y)$ for a scalar θ ($\neq 2c$) and any vector fields X and Y on M, then M is a Hopf hypersurface in $M_n(c)$ provided that the scalar curvature $\bar{r}$ of M holds $\bar{r} - 2(n-1)c \leq 0$.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the semi-invariant are supposed to be orientable.

2. Semi-invariant submanifolds

At first we review fundmental facts on a semi-invariant submanifold of a complex space form.

Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold with parallel almost complex structure J and a Riemannian metric tensor G. Let M be a real $(2n-1)$-dimensional Riemannian manifold immersed isometrically in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$. In the sequel we identify $i(M)$ with M itself. We denote by g the Riemannian metric tensor on M from that of $\tilde{M}$.

We denote by $\tilde{\nabla}$ the operator of covariant differentiation with respect to the metric tensor G on $\tilde{M}$ and by ∇ the one on M. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{3} g(A^{(i)}X, Y)\mathcal{C}^{(i)}, \tag{2.1}$$

$$\tilde{\nabla}_X \mathcal{C}^{(i)} = -A^{(i)}X + \sum_{j=1}^{3} l_j^{(i)}\mathcal{C}^{(j)} \tag{2.2}$$

for any vector fields tangent to X and Y on M and any vector normal vector $C^{(i)}$ to M, where $A^{(i)}$ is called a *second fundamental forms* with respect to the normal vector $\mathcal{C}^{(i)}$.

As is well-known, a submanifold M of a Kaehlerian manifold $\tilde{M}$ is said to be a CR *submanifold* ([1], [31]) if it is endowed with a pair of mutually orthogonal and complementary differentiable distribution $(T, T^\perp)$ such that for any point $p \in M$ we have $JT_p = T_pM$, $JT_p^\perp \subset T_p^\perp M$, where $T_p^\perp$ denote the normal space of M at p. In particular, M is said to be a *semi-invariant submanifold* ([5], [29]) provided that $dim T^\perp = 1$ or to be a *CR submanifold* with *CR dimension* $n-1$

([24]). In this case the unit normal vector field in $JT^{\perp}$ is called a *distinguished normal* to the semi-invariant submanifold and denote by $C^{\perp}$ ([5], [29]).

From now on we consider that M is a real $(2n\text{-}1)$-dimensional semi-invariant submanifold of codimension 3 in a Kaehlerain manifold $\tilde{M}$ of real dimension $2(n{+}1)$. Then we can choose a local orthonomal frame field $\{e_1, \cdots, e_{n-1}, Je_1, \cdots, Je_{n-1}, e_0 = \xi, J\xi = C, D = JE, E\}$ on the tangent space $T_p\tilde{M}$ of $\tilde{M}$ for any point $P \in M$ such that $e_1, \cdots, e_{n-1}, Je_1, \cdots, Je_{n-1},\ \xi \in T_p$ and C, D and $E \in T_p^{\perp}$. So, (2.1) can be written as

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)C + g(KX, Y)D + g(LX, Y)E$$

for any vector fields X and Y on M, where we put $A^{(1)} = A$, $A^{(2)} = K$ and $A^{(3)} = L$. If we put ${l_2}^{(1)} = l$, ${l_3}^{(1)} = m$ and ${l_{(3)}}^{2} = l$, then equations of Weingarten are also given by

$$\begin{aligned} \tilde{\nabla}_X C &= -AX + l(X)D + m(X)E, \\ \tilde{\nabla}_X D &= -KX - l(X)C + t(X)E, \\ \tilde{\nabla}_X E &= -LX - m(X)C - t(X)D \end{aligned} \tag{2.3}$$

because C, D and E are mutually orthogonal.

Now, let ϕ be the restriction of J on M, then we have

$$JX = \phi X + \eta(X)C, \quad \eta(X) = g(\xi, X), \quad JC = -\xi \tag{2.4}$$

for any vector field X on M. From this, we see, using Hermitian property of J, that the aggregate (ϕ, ξ, η, g) is an *almost contact metric structure* on M, that is, we have (cf. [32])

$$\begin{gathered} \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\xi, X) = \eta(X), \\ \phi\xi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \end{gathered}$$

for any vector fields X and Y on M.

In the sequel, we denote the normal components of $\nabla_X C$ by $\nabla^{\perp} C$. The distinguished normal C is said to be *parallel* in the normal bundle if we have $\nabla^{\perp} C = 0$.

From the Kaehler condition $\tilde{\nabla} J = 0$ and using the Gauss and Weingarten formulas, we obtain from (2.4)

$$\nabla_X \xi = \phi AX, \tag{2.5}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.6}$$

$$KX = \phi LX - m(X)\xi, \quad K\phi X = LX - \eta(X)L\xi \tag{2.7}$$

$$LX = -\phi KX + l(X)\xi, \quad L\phi X = -KX + \eta(X)K\xi \tag{2.8}$$

for any vector fields X and Y on M. From the last two relationships (2.7) and (2.8), we have

$$g(K\xi, X) = -m(X), \tag{2.9}$$

$$g(L\xi, X) = l(X). \tag{2.10}$$

which implies $g(K\xi, K\xi) = -m(K\xi)$, $g(L\xi, L\xi) = l(L\xi)$. In the following we denote $\|K\xi\| = \|m\|$ and $\|L\xi\| = \|l\|$ for simplicty, where $\|F\|^2 = g(F, F)$ for any tensor filed F on M.

Using the frame field $\{e_0 = \xi, e_1, \cdots, e_{n-1}, \phi e_1, \cdots, \phi e_{n-1}\}$ on M, it follows from (2.7) $\sim$ (2.10) that

$$T_rK = \eta(K\xi) = -m(\xi), \quad T_rL = \eta(L\xi) = l(\xi). \tag{2.11}$$

Now we retake D and E, there is no loss of generality such that we may assume $T_rL = 0$ (cf. [18]). So we have

$$l(\xi) = 0. \tag{2.12}$$

Notation. To write our formulas in a convention form, in the sequel we denote by $\alpha = \eta(A\xi), \beta = \eta(A^2\xi), \gamma = \eta(A^3\xi), h = T_rA, k = T_rK$, $h_{(2)} = T_r({}^tAA)$ and for a function f we denote by ∇f the gradient vector field of f.

From (2.11) we have

$$m(\xi) = -k. \tag{2.13}$$

Using (2.7) and (2.8) we have

$$m(X)\eta(Y) - m(Y)\eta(X) = \eta(Y)l(\phi X) - \eta(X)l(\phi Y).$$

If we put $Y = \xi$ in this, and take account of (2.13), then we find

$$l(\phi X) = m(X) + k\eta(X), \tag{2.14}$$

which tells us, using (2.12), that

$$m(\phi X) = -l(X). \tag{2.15}$$

Taking the inner product with LY to (2.7) and using (2.10),we get

$$g(KLX,Y)+g(LKX,Y)=-\{l(X)m(Y)+l(Y)m(X)\}. \tag{2.16}$$

If we take the inner product with KY to (2.7) and make use of (2.9), we obtain

$$g(K^2X,Y)=g(\phi LX,KY)+m(X)m(Y).$$

Similary, taking the inner product with LY to (2.8) and using (2.10), we also find

$$g(L^2X,Y)=g(\phi LX,KY)+l(X)l(Y).$$

Combining above equations, it follows that

$$g(L^2X,Y)-g(K^2X,Y)=l(X)l(Y)-m(X)m(Y). \tag{2.17}$$

In the rest of this paper we shall suppose that $\tilde{M}$ is a Kaehlerian manifold of contant holomorphic sectional curvature 4c, which is called a *complex space form* and denote by $M_{n+1}(c)$, that is,

$$\begin{aligned}\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z}=c\{&G(\tilde{Y},\tilde{Z})\tilde{X}-G(\tilde{X},\tilde{Z})\tilde{Y}\\&+G(J\tilde{Y},\tilde{Z})J\tilde{X}-G(J\tilde{X},\tilde{Z})J\tilde{Y}-2G(J\tilde{X},\tilde{Y})J\tilde{Z}\}\end{aligned}$$

for any vector $\tilde{X},\tilde{Y},\tilde{Z}$ on $M_{n+1}(c)$, where $\tilde{R}$ is the curvature tensor of $M_{n+1}(c)$. Then equations of Gauss and Coddzzi are given by

$$\begin{aligned}R(X,Y)Z=c\{&g(Y,Z)X-g(X,Z)Y+g(\phi Y,Z)\phi X-g(\phi X,Z)\phi Y\\&-2g(\phi X,Y)\phi Z\}+g(AY,Z)AX-g(AX,Z)AY\\&+g(KY,Z)KX-g(KX,Z)KY+g(LY,Z)LX-g(LX,Z)LY,\end{aligned} \tag{2.18}$$

$$\begin{aligned}(\nabla_XA)Y-(\nabla_YA)X-l(X)KY+l(Y)KX-m(X)LY&\\+m(Y)LX=c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\},&\end{aligned} \tag{2.19}$$

$$(\nabla_XK)Y-(\nabla_YK)X=-l(X)AY+l(Y)AX+t(X)LY-t(Y)LX, \tag{2.20}$$

$$(\nabla_XL)(Y)-(\nabla_YL)(X)=-m(X)AY+m(Y)AX-t(X)KY+t(Y)KX, \tag{2.21}$$

where R is the Riemannian curvature tensor of M, and those of the Ricci tensor by

$$\begin{aligned}(\nabla_Xl)(Y)-(\nabla_Yl)(X)+g((KA-AK)X,Y)&\\+m(X)t(Y)-m(Y)t(X)=0,&\end{aligned} \tag{2.22}$$

$$(\nabla_X m)(Y)-(\nabla_Y m)(X)+g((LA-AL)X,Y)+t(X)l(Y)-t(Y)l(X)=0, \tag{2.23}$$

$$\begin{aligned}(\nabla_X t)Y-(\nabla_Y t)X+g((LK-KL)X,Y)\\ =l(Y)m(X)-l(X)m(Y)+2cg(\phi X,Y).\end{aligned} \tag{2.24}$$

Now, we put $\nabla_\xi \xi = U$ in the sequel. Then U is orthogonal to ξ because of (2.5). We put

$$A\xi = \alpha\xi + \mu W, \tag{2.25}$$

where W is a unit vector field orthogonal to ξ. Then we have

$$U = \mu\phi W \tag{2.26}$$

by virtue of (2.5). So, W is also orthogonal to U. Further, we have

$$\mu^2 = \beta - \alpha^2. \tag{2.27}$$

From (2.25) and (2.26) we have

$$\phi U = -A\xi + \alpha\xi. \tag{2.28}$$

If we take account of (2.5), (2.25) and the last equation, then we find

$$g(\nabla_X \xi, U) = \mu g(AW, X). \tag{2.29}$$

Since W is orthogonal to ξ, we see, using (2.5) and (2.26), that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{2.30}$$

Differentiating (2.28) covariantly along M and using (2.5) and (2.6), we find

$$(\nabla_X A)\xi = -\phi\nabla_X U + g(AU+\nabla\alpha, X)\xi - A\phi AX + \alpha\phi AX. \tag{2.31}$$

Taking the inner product with ξ to this and using (2.9), (2.10), (2.12), (2.19) and (2.28), we find

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha - 2kL\xi. \tag{2.32}$$

Applying (2.31) by ϕ and making use of (2.29), we obtain

$$\phi(\nabla_X A)\xi = \nabla_X U + \mu g(AW,X)\xi - \phi A\phi AX - \alpha AX + \alpha g(A\xi, X)\xi, \tag{2.33}$$

which enables us to obtain

$$\nabla_U U = \phi(\nabla_U A)\xi + \phi A\phi AU + \alpha AU. \tag{2.34}$$

Finally, we introduce the structure Jacobi operator R_ξ with respect to the structure vector field ξ which is defined by $R_\xi X = R(X,\xi)\xi$ for any vector field X, Then we have from (2.18)

$$\begin{aligned} R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + \eta(K\xi)KX - \eta(KX)K\xi \\ + \eta(L\xi)LX - \eta(LX)L\xi. \end{aligned}$$

Since l and m are dual 1-forms of $L\xi$ and $-K\xi$ respectively because of (2.9) and (2.10), the last equation can be written as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX + m(X)K\xi - l(X)L\xi, \tag{2.35}$$

where we have used (2.9)$\sim$(2.13).

3. The third fundamental forms of semi-invariant submannfolds

In this section we will suppose that M is a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ and that the third fundamental form t satisfies

$$dt = 2\theta\omega, \quad \omega(X,Y) = g(\phi X, Y) \tag{3.1}$$

for any vector fields X and Y on M and a certain scalar θ, where d denotes the exterior differential operator. Then (2.24) reformed as

$$g((LK - KL)X, Y) + l(X)m(Y) - l(Y)m(X) = -2(\theta - c)g(\phi X, Y),$$

or, using (2.16)

$$g(LKX, Y) + l(X)m(Y) = -(\theta - c)g(\phi X, Y), \tag{3.2}$$

which together with (2.9)$\sim$(2.12) gives

$$l(KX) = kl(X), \quad m(LX) = 0.$$

for any vector X on M, that is

$$KL\xi = kL\xi, \quad LK\xi = 0. \tag{3.3}$$

Further, putting $X = L\xi$ in (3.2), and using (2.10) and the first equation of the last relationship, we find

$$kl(LY) + \|L\xi\|^2 m(Y) = (\theta - c)l(\phi Y),$$

which together with (2.14) gives

$$kl(LX) = ((\theta - c) - \|L\xi\|^2)m(X) + k(\theta - c)\eta(X). \tag{3.4}$$

Differentiating (3.1) covariantly along M and using (2.6) and the first Banchi identity, we find

$$(X\theta)\omega(Y, Z) + (Y\theta)\omega(Z, X) + (Z\theta)\omega(X, Y) = 0,$$

which implies $(n-2)X\theta = 0$. Thus θ is constant if $n > 2$.

For the case where $\theta = c$ in (3.1) we have $dt = 2c\omega$. In this case, the normal connection of M is said to be *L-flat*([24]).

By properties of the almost contact metric structure we have from (3.2) the following :

$$T_r({}^tKK) - \|K\xi\|^2 + \|L\xi\|^2 = 2(n-1)(\theta - c),$$

where we have used (2.7), (2.10) and (2.11), which together with (2.9) implies that

$$\|K - k\eta \otimes \xi\|^2 + \|L\xi\|^2 = 2(n-1)(\theta - c). \tag{3.5}$$

Thus, $\theta - c$ is nonnegative.

In the same way, we have (2.8), (2.12), (2.15) and (3.2)

$$-\|L\xi\|^2 + \|K\xi - k\xi\|^2 - T_r({}^tLL) = 2(n-1)(\theta - c).$$

Lemma 3.1. *Let M be a semi-invariant submanifold with L-flat normal connection in $M_{n+1}(c)$, $c \neq 0$. If $A\xi = \alpha\xi$, then we have $\nabla^\perp C = 0$ and $A^{(2)} = A^{(3)} = 0$.*

Proof. Since $\theta - c = 0$, we have $L = 0$ and $KX = k\eta(X)\xi$ because of (3.4) and (3.5). By virtue of (2.11), it follows that $m(X) = -k\eta(X)$. We also have $l = 0$ because of (2.10). Thus, it suffices to show that $k = 0$. Using these facts, (2.22) reformed as

$$k\{\eta(X)A\xi - g(A\xi, X)\xi\} = k(\eta(X)t - t(X)\xi),$$

which together with $A\xi = \alpha\xi$ gives $k(t - t(\xi)\xi) = 0$. If we suppose that $k \neq 0$ on M, then we have $t = t(\xi)\xi$ on this open subset. Differentiating this covariantly and using (2.5) and (3.1) with $\theta = c$, we find

$$2cg(\phi X, Y) = t(\xi)g((A\phi - \phi A)X, Y)$$

by virtue of $A\xi = \alpha\xi$, which implies

$$2c(n-1) = t(\xi)(h - \alpha).$$

On the other side, from (2.21) we have

$$k\{\eta(X)(AY+t(Y)\xi)-\eta(Y)(AX+t(X)\xi)\}=0,$$

which implies $k(h-\alpha)=0$, a contradiction. Hence $k=0$ on M. □

Transforming (3.2) by ϕ and using (2.7) and (2.15), we find

$$K^2X+\eta(X)K^2\xi+l(X)L\xi=(\theta-c)(X-\eta(X)\xi),$$

which shows $m(KX)\eta(Y)-m(KY)\eta(X)=0$. Therefore, we have $m(KX)=-\|K\xi\|^2\eta(X)$, that is

$$K^2\xi=\|K\xi\|^2\xi. \tag{3.6}$$

Thus, it follows that

$$K^2X+l(X)L\xi-\|K\xi\|^2\eta(X)\xi=(\theta-c)(X-\eta(X)\xi). \tag{3.7}$$

In the same way we have from (3.2)

$$l(LX)=-km(X)+(\|K\xi\|^2)\eta(X),$$

where we have used (2.8), (2.14) and (3.3).

Since we have (2.15) and the second equation of (3.3), we see from (3.2)

$$(\theta-c-\|K\xi\|^2)L\xi=0.$$

On the other hand, taking an inner product $L\xi$ to (3.2) and using (3.3), we obtain

$$kl(LX)=(\theta-c-\|L\xi\|^2)m(X)+k(\theta-c)\eta(X)$$

because of (2.14) and (3.3), which together with the last two equations implies that

$$(\theta-c-\|L\xi\|^2-k^2)(\|K\xi\|^2-k^2)=0.$$

We are now going to prove that $L\xi=0$ on M.

Let Ω_0 be a set of points such that $\|L\xi\|\neq 0$ on M and suppose that Ω_0 be nonvoid. Then we have

$$\|K\xi\|^2=\theta-c, \quad \|L\xi\|^2+k^2=\theta-c \tag{3.8}$$

on Ω_0. In fact, if not, then we have $m(X)=-k\eta(X)$, which together with (2.14) gives $l(\phi X)=0$ and hence $L\xi=0$ because of (2.12), a contradiction. Thus, the second relationship of (3.8) is valid by virtue of (2.11). From now on, we discuss our arguments on the open set Ω_0 of M. Then (3.7) turns out to be

$$K^2X = (\theta - c)X - l(X)L\xi. \tag{3.9}$$

Differentiating this covariantly and using (2.20), (2.21) and other equations already obtained, we find (see, (2.23) and (2.24) of [21])

$$(\nabla_X K)Y = t(X)LY + l(Y)AX + g(AX, Y)L\xi, \tag{3.10}$$

$$\nabla_X L\xi = -t(X)K\xi - AKX - kAX. \tag{3.11}$$

If we differentiate (2.8) covariantly and using (2.5), (2.6), (2.9) and (2.14), we find

$$(\nabla_X L)Y = -t(X)KY + m(Y)AX - g(AX, Y)K\xi. \tag{3.12}$$

Since $T_rL = 0$ because of (2.11) and (2.12), taking the trace of this, and using (2.9), we get

$$kt(X) + 2m(AX) = 0. \tag{3.13}$$

Differentiating (2.9) covariantly and taking account of (2.5), (2.6) and (3.10), we find

$$(\nabla_X m)Y = -t(X)l(Y) - g(AX, LY). \tag{3.14}$$

On the other hand, differentiating (2.11) covariantly and using (2.14) and (3.14), we find $\nabla k = 2AL\xi$, that is, $Yk = 2l(AY)$, which implies

$$\begin{aligned} X(Yk) = 2l((\nabla_X A)Y) + 2\{t(X)m(AY) - g(KAX, AY) \\ - kg(A^2X, Y)\} + 2l(A\nabla_X Y). \end{aligned}$$

If we take the skew-symmetric part with respect to X and Y and making use of (3.13), then we obtain

$$l((\nabla_X A)Y - (\nabla_Y A)X) = 0,$$

which together with (2.12), (2,14) and (2.19) yields

$$\begin{aligned} l(Y)l(KX) - l(X)l(KY) + m(Y)l(LX) - m(X)l(LY) \\ = c\{\eta(X)m(Y) - \eta(Y)m(X)\}. \end{aligned} \tag{3.15}$$

By the way, putting $Y = \xi$ in (3.2) and using (2.10) and (2.11), we have $l(KX) = kl(X)$. Thus, (3.15) can be written as

$$m(Y)l(LX) - m(X)l(LY) + c(m(X)\eta(Y) - m(Y)\eta(X)) = 0. \tag{3.16}$$

which together with (3.8) and the next equation of (3.7) gives

$$(\theta-2c)(m(X)\eta(Y)-m(Y)\eta(X))=0.$$

From this and (2.11) we see that $(\theta-2c)(m(X)+k\eta(X))=0$ and hence $(\theta-2c)l(X)=0$ by virtue of (2.12) and (2.14), a contradiction if $\theta-2c\neq 0$.

Developed above, we have

Lemma 3.2. *Let M be a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c\neq 0$ satisfying $dt=2\theta\omega$ for a scalar $\theta(\neq 2c)$. Then we have $l=0$ on M.*

In the rest of this paper, we assume that M satisfies (3.1) with $\theta-2c\neq 0$. Then we have $l=0$ and hence

$$m=-k\xi \tag{3.17}$$

because of (2.14). Hence (2.9) and (2.10) reformed respectively as

$$K\xi=k\xi, \quad L\xi=0. \tag{3.18}$$

It is, using (3.17), clear that (2.7), (2.8) and (3.2) are reduced respectively to

$$KX=\phi LX+k\eta(X)\xi, \tag{3.19}$$

$$LX=-\phi KX, \tag{3.20}$$

$$g(LKX,Y)+(\theta-c)g(\phi X,Y)=0. \tag{3.21}$$

From the last two relationships, we obtain

$$LK+KL=0, \tag{3.22}$$

$$L^2X=(\theta-c)(X-\eta(X)\xi). \tag{3.23}$$

Further, if we take account of (3.17) and the fact that $l=0$, then the structure equations (2.19)$\sim$(2.23) reformed respectively as

$$\begin{aligned}(\nabla_X A)Y-(\nabla_Y A)X=k\{&\eta(Y)LX-\eta(X)LY\}\\ &+c\{\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi\},\end{aligned} \tag{3.24}$$

$$(\nabla_X K)Y-(\nabla_Y K)X=t(X)LY-t(Y)LX, \tag{3.25}$$

$$(\nabla_X L)Y-(\nabla_Y L)X=k\{\eta(X)AY-\eta(Y)AX\}-t(X)KY+t(Y)KX, \tag{3.26}$$

$$g((KA - AK)X, Y) = k\{\eta(X)t(Y) - t(X)\eta(Y)\}, \tag{3.27}$$

$$g((LA - AL)X, Y) = (Xk)\eta(Y) - \eta(X)(Yk) + kg((\phi A + A\phi)X, Y). \tag{3.28}$$

Putting $X = \xi$ in (3.27) and using (3.18), we find

$$g(KA\xi, X) = kg(A\xi, X) + k(t(X) - t(\xi)\eta(Y)). \tag{3.29}$$

If we replace X by ϕX and make use of (2.26) and (3.20), then we get

$$g(KU, X) = k(t(\phi X) - u(X)), \tag{3.30}$$

where $u(X) = g(U, X)$ for any vector field X.

Replacing X by ξ in (3.28) and using (2.5), (3.18) and (3.20), we find

$$KU = (\xi k)\xi - \nabla k + kU, \tag{3.31}$$

which together with (3.30) gives

$$Xk = (\xi k)\eta(X) + k(2u(X) - t(\phi X)). \tag{3.32}$$

If we replace Y by ϕY in (3.28) and make use of (3.19) and the last equation, then we find

$$\begin{aligned} g(\phi ALX - KAX, Y) = &-k\{(t(Y) - t(\xi)\eta(Y))\eta(X) \\ &+ 2\eta(X)(g(A\xi, Y) - \alpha\eta(Y)) + 2g(A\xi, X)\eta(Y) - g(AX, Y) + g(\phi A\phi X, Y)\}, \end{aligned}$$

form which, take the skew-symmetric part with respect to X and Y,

$$\phi ALX = -LA\phi X. \tag{3.33}$$

Sine θ is constant if $n > 2$, differentiating (3.23) covariantly, we find

$$L(\nabla_X L)Y + (\nabla_X L)LY = (c - \theta)\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\}.$$

Using the quite same method as that used to (3.10) from (3.9), we can drive from the last equation the following :

$$\begin{aligned} 2L(\nabla_X L)Y = (\theta - c)\{&2t(X)\phi Y - \eta(Y)(A\phi + \phi A)X \\ &+ g((A\phi - \phi A)X, Y)\xi - \eta(X)(\phi A - A\phi)Y\} \\ &- k\{\eta(Y)(LA + AL)X - g((AL + LA)X, Y)\xi \\ &- \eta(X)(AL - LA)Y\}, \end{aligned} \tag{3.34}$$

where we have used (3.21) and (3.26), which together with (3.18), (3.28) and (3.32) yields

$$\begin{aligned}(\theta - c)(\phi A - A\phi)X + (k^2 + \theta - c)(\eta(X)U + u(X)\xi)\\ + k\{LA + AL - k(t(\phi X)\xi + \eta(X)t \circ \phi)\} = 0.\end{aligned} \tag{3.35}$$

In the previous paper [14] and [21], the following proposition was proved for the case where $c > 0$.

Proposition 3.3. *If M satisfies $dt = 2\theta\omega$ for a scalar θ and $\mu = 0$ in $M_{n+1}(c)$, $c \neq 0$, then we have $k = 0$ on M.*

Proof outline. This proved for $c > 0$ (see, Proposition 3.5 of [18]). But, regardless of the sign of c this one is established. However, only $\xi k = 0$ and $\xi\alpha = 0$ should be newly certified. We are now going to prove that $\xi k = 0$.

Differentiating (3.17) covariantly and using (2.5), we find

$$\nabla_X m = -(Xk)\xi + k\phi AX,$$

from which, taking the skew-symmetric part and using (3.28),

$$LAX - ALX - k(\phi A + A\phi)X = (Xk)\xi - \eta(X)\nabla k.$$

If we put $X = \xi$ in this and make use of (3.18), then we find

$$\nabla k = (\xi k)\xi \tag{3.36}$$

because $A\xi = \alpha\xi$ was assumed. From the last two equations, it follows that

$$LA - AL = k(\phi A + A\phi). \tag{3.37}$$

Differentiating (3.36) covariantly, and taking the skew-symmetric part obtained, we obtain

$$(\xi k)(\phi A + A\phi) = 0, \tag{3.38}$$

where we have used (2.5).

Since we have $A\xi = \alpha\xi$ because of (2.25), we can write (2.33) as

$$(\nabla_X A)\xi = -A\phi AX + \alpha\phi AX + (X\alpha)\xi,$$

which together with (3.18) and (3.24) gives

$$2A\phi AX + \alpha(A\phi + \phi A)X + 2c\phi X = \eta(X)\nabla\alpha - (X\alpha)\xi. \tag{3.39}$$

Putting $X = \xi$ in this, we also find

$$\nabla\alpha = (\xi\alpha)\xi. \tag{3.40}$$

Using the quite same method as that used to (3.38) from (3.36), we can derive from the last equation the following :

$$(\xi\alpha)(\phi A + A\phi) = 0. \tag{3.41}$$

Now, if we suppose that $\xi k \neq 0$. Then we have

$$\phi A + A\phi = 0, \quad LA = AL$$

on this open subset because of (3.37) and (3.38). We discuss our arguments on such a place. By virtue of (3.40) and the last relationship, we can write (3.39) as

$$A^2\phi + c\phi = 0.$$

If we apply this by ϕ. then we obtain

$$A^2X + cX = (\alpha^2 + c)\eta(X)\xi, \tag{3.42}$$

where we have used $A\xi = \alpha\xi$.

Since $A\xi = \alpha\xi$, that is $U = 0$ was assumed, (3.35) can be written as

$$(\theta - c)A\phi X + kALX = 0,$$

which together with (2.7) yields

$$(\theta - c)AX + kAKX = \alpha(\theta - c + k^2)\eta(X)\xi.$$

Combining this to (3.42), we find $kKX + (\theta - c)X = (\theta - c + k^2)\eta(X)\xi$, which shows $(n-1)(\theta - c) = 0$. Thus we have $\theta - c = 0$ if $n > 2$. This contradicts Lemma 3.1. Thus $\xi k = 0$ is proved on M.

By the same as above we can prove $\xi\alpha = 0$ by virtue of (3.40) and (3.41). This completes the proof.

We set $\Omega = \{p \in M : k(p) \neq 0\}$ and suppose Ω is not empty. In the rest of this paper, we discuss our arguments on the open subset Ω of M. So, by Proposition 3.3 we see that $\mu \neq 0$ on Ω.

4. Semi-invariant submanifolds satisfying $R_\xi A^{(2)} = A^{(2)} R_\xi$ and $R_\xi A = AR_\xi$

We will continue our arguments under the same hypotheses $dt = 2\theta\omega$ for a scalar $\theta(\neq 2c)$ as those stated in section 3. Further suppose, throughout this paper, that $R_\xi A^{(2)} = A^{(2)} R_\xi$.

By virtue of (3.17) and (3.18) we can write (2.35) as

$$R_\xi X = c(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi + kKX - k^2\eta(X)\xi, \tag{4.1}$$

which implies

$$R_\xi KX = c(KX - k\eta(X)\xi) + \alpha AKX - \eta(AKX)A\xi + kK^2X + k^3\eta(X)\xi,$$

where we have used the first equation of (3.18), which together with (2.25), (3.27) and (3.29) gives

$$(R_\xi K - KR_\xi)X = k\mu\{t(X)W - w(X)t - t(\xi)(\eta(X)W - w(X)\xi)\}, \tag{4.2}$$

where $g(W, X) = w(X)$ for any vector field X.

According to (4.2) and Proposition 3.3, we then have

Lemma 4.1. *$R_\xi A^{(2)} = A^{(2)} R_\xi$ holds on Ω if and only if $t \in f(\xi, W)$, where $f(\xi, W)$ is denoted by a linear subspace spanned by ξ and W.*

Because of Lemma 4.1, we have

$$t(X) = t(\xi)\eta(X) + t(W)w(X) \tag{4.3}$$

for any vector field X.

From (2.26) and (4.3) we obtain $t(\phi X) = -\frac{1}{\mu} t(W)u(X)$, which together with (3.30) yields

$$KU = \tau U, \tag{4.4}$$

where τ is defined by $\mu\tau = -k(\mu + t(W))$, or using (3.19),

$$LU = \mu\tau W. \tag{4.5}$$

By virtue of (3.21) and the last two relationships, it follows that

$$\tau^2 = \theta - c. \tag{4.6}$$

τ is a nonnegative constant on Ω if $n > 2$.

In a direct consequence of (2.8), (3.20) and (4.4), we verify that

$$\mu LW = \tau U. \tag{4.7}$$

Using (2.25) and (3.18), we can write (3.29) as

$$\mu KW = k\mu W + k(t - t(\xi)\xi),$$

which together with (4.2) and (4.3) gives

$$KW = -\tau W \tag{4.8}$$

because of Proposition 3.3.

Now, by using (3.30) and (4.4) it is verified that

$$t(\phi X) = (1 + \frac{\tau}{k})u(X) \tag{4.9}$$

on Ω, or using the property of the almost contact metric structure,

$$t(X) = t(\xi)\eta(X) - \mu(1 + \frac{\tau}{k})w(X) \tag{4.10}$$

for any vector field X.

If we take account of (4.4), then (3.31) can be written as

$$Xk = (\xi k)\eta(X) + (k - \tau)u(X) \tag{4.11}$$

for any vector field X.

On the other hand, if we use (2.28) and (3.24), then (2.31) implies that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha + 2\eta(L\xi) - 2\eta(K\xi)L\xi,$$

which together with (3.18) implies that

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha. \tag{4.12}$$

Putting $X = \xi$ in (2.31) and making use of (2.25) and (2.27), we get

$$\phi(\nabla_\xi A)\xi = \nabla_\xi U + \beta\xi - \alpha A\xi + \phi AU,$$

which together with (4.12) yields

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{4.13}$$

In the following, we see, using (2.25) and (2.28), that $\phi U = -\mu W$. Differentiating this covariantly and using (2.6), we find

$$g(AU, X)\xi - \phi\nabla_X U = (X\mu)W + \mu\nabla_X W. \tag{4.14}$$

Putting $X = \xi$ in this and using (4.13), we get

$$\mu\nabla_\xi W = 3AU - \alpha U + \nabla\alpha - (\xi\alpha)\xi - (\xi\mu)W, \tag{4.15}$$

which tells us that

$$W\alpha = \xi\mu. \tag{4.16}$$

Now, if we take account of (4.9), then (3.35) turns out to be

$$\tau^2(A\phi - \phi A)X + \tau(\tau - k)(u(X)\xi + \eta(X)U) + k(AL + LA)X = 0. \tag{4.17}$$

On the other hand, we have from (4.1)

$$R_\xi AX = cAX - (k^2 + c)g(A\xi, X)\xi + \alpha A^2 X - g(A^2\xi, X)A\xi + kKAX.$$

Thus, the second assumption $R_\xi A = AR_\xi$ gives

$$\begin{aligned} g(A^2\xi, X)g(A\xi, Y) - g(A^2\xi, Y)g(A\xi, X) + (k^2 + c)\{g(A\xi, X)\eta(Y) - g(A\xi, Y)\eta(X)\} \\ = k^2(t(Y)\eta(X) - t(X)\eta(Y)), \end{aligned}$$

where we have used (3.27). Putting $X = \xi$ in this, we find

$$-\alpha A^2\xi + (\beta - k^2 - c)A\xi = k^2 t - \{k^2 t(\xi) + \alpha(k^2 + c)\}\xi. \tag{4.18}$$

Combining the last two equations, we obtain

$$g(A^2\xi, X)(A\xi - \alpha\xi) - (g(A\xi, X) - \alpha\eta(X))A^2\xi = \beta(\eta(X)A\xi - g(A\xi, X)\xi).$$

If we put $X = A\xi$ in this, and take account of (2.27), then we have

$$\mu^2 A^2\xi = (\gamma - \beta\alpha)A\xi + (\beta^2 - \alpha\gamma)\xi,$$

which together with Proposition 3.3 implies that

$$A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi,$$

where we have defined the function ρ by $\mu^2\rho = \gamma - \beta\alpha$. Hence, (4.18) is reformed as

$$(\beta - \rho\alpha - k^2 - c)(A\xi - \alpha\xi) = k^2(t - t(\xi)\xi),$$

which connected to (4.4) gives

$$k\tau = c + \rho\alpha - \beta. \tag{4.19}$$

Accordingly, it follows that

$$A^2\xi = \rho A\xi + (c - k\tau)\xi. \tag{4.20}$$

In the next step, we see from (2.25) and (4.20)

$$AW = \mu\xi + (\rho - \alpha)W \tag{4.21}$$

since we have $\mu \neq 0$ on Ω, where we have used (2.27). Differentiating this covariantly along Ω, we find

$$(\nabla_X A)W + A\nabla_X W = (X\mu)\xi + \mu\nabla_X\xi + X(\rho - \alpha)W + (\rho - \alpha)\nabla_X W. \tag{4.22}$$

If we take the inner product W to this and using (2.30) and (4.21), we find

$$g((\nabla_X A)W, W) = -2g(AU, X) + X\rho - X\alpha \tag{4.23}$$

since W is orthogonal to ξ. Taking the inner product with ξ to (4.22) and using (2.30), we also find

$$\mu g((\nabla_X A)W, \xi) = (\rho - 2\alpha)(g(AU, X) + \mu(X\mu)), \tag{4.24}$$

or, using (3.24)

$$\mu(\nabla_\xi A)W = (\rho - 2\alpha)AU + \mu\nabla\mu - k\mu LW - cU. \tag{4.25}$$

From this and (3.24) we verify that

$$\mu(\nabla_W A)\xi = (\rho - 2\alpha)AU - 2cU + \mu\nabla\mu. \tag{4.26}$$

Putting $X = \xi$ in (4.23) and using (4.24), we get

$$W\mu = \xi\rho - \xi\alpha. \tag{4.27}$$

Replacing X by ξ in (4.22) and using (4.7) and (4.25), we find

$$\begin{aligned}(\rho - 2\alpha)AU - k\tau U - cU + \mu\nabla\mu + \mu(A\nabla_\xi W - (\rho - \alpha)\nabla_\xi W)\\ = \mu(\xi\mu)\xi + \mu^2 U + \mu(\xi\rho - \xi\alpha)W,\end{aligned} \tag{4.28}$$

which together with (4.15) and (4.16) implies that

$$\begin{aligned}3A^2U - 2\rho AU + (\alpha\rho - \beta - k\tau - c)U + A\nabla\alpha + \frac{1}{2}\nabla\beta - \rho\nabla\alpha\\ = 2\mu(W\alpha)\xi + (2\alpha - \rho)(\xi\alpha)\xi + \mu(\xi\rho)W.\end{aligned} \tag{4.29}$$

Lemma 4.2. *If $A\nabla_\xi\xi = \lambda\nabla_\xi\xi$, then $\xi\lambda = 0$ and $W\lambda = 0$ on Ω.*

Proof. Differentiating $AU = \lambda U$ covariantly along Ω, we find

$$(\nabla_X A)U + A\nabla_X U = (X\lambda)U + \lambda\nabla_X U.$$

If we take the inner with a vector field Y, and take the skew-symmetric part with respect to X and Y, then we obtain

$$\begin{aligned}\mu(k\tau + c)(\eta(X)w(Y) - \eta(Y)w(X)) + g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\\ = (X\lambda)u(Y) - (Y\lambda)u(X) + \lambda(g(\nabla_X U, Y) - g(\nabla_Y U, X)),\end{aligned}$$

where we have used (2.25), (2.28), (3.24) and (4.5). Replacing X by U in this and taking account of $AU = \lambda U$, we get

$$A\nabla_U U - \lambda\nabla_U U = (U\lambda)U - \mu^2\nabla\lambda. \tag{4.30}$$

If we take the inner product with ξ and remember (4.21), then we have

$$\mu g(\nabla_U U, \xi) + \mu^2(W\lambda) + (\rho - \alpha - \lambda)g(\nabla_U U, W) = 0. \tag{4.31}$$

By the way, from (4.4) we get

$$(\nabla_X K)U + K\nabla_X U = \tau\nabla_X U, \tag{4.32}$$

which implies that $g((\nabla_X K)U, U) = 0$. Because of (3.21), (4.8) and the last equation gives $(\nabla_U K)U = 0$, which connected to (4.8) and (4.32) yields $g(W, \nabla_U U) = 0$. Thus, (4.31) reformed as $\mu g(\nabla_U U, \xi) + \mu^2(W\lambda) = 0$. However, the first term of this vainshes identically by virtue of (2.29) and (4.21), which shows $\mu(W\lambda) = 0$ and hence

$$W\lambda = 0. \tag{4.33}$$

In the same way, we verify, using (2.29) and (4.21), that

$$\xi\lambda = 0. \tag{4.34}$$

This completes the proof. □

If we put $X = \mu W$ in (4.17) and use (2.20), (3.18), (4.21) and Lemma 4.1, then we get

$$(k+\tau)AU + (k-\tau)(\rho-\alpha)U = 0 \tag{4.35}$$

Because of (4.11) and Proposition 3.3, it is clear that $k+\tau \neq 0$ on Ω. Therefore, (4.35) implies that

$$AU = \lambda U, \tag{4.36}$$

where we have put $(k+\tau)\lambda = -(k-\tau)(\rho-\alpha)$, which implies

$$\lambda(k+\tau) + (k-\tau)(\rho-\alpha) = 0. \tag{4.37}$$

Finally, differentiating (2.25) covariantly and using (2.5), we find

$$(\nabla_X A)\xi + A\phi AX = (X\alpha)\xi + \alpha\phi AX + (X\mu)W + \mu\nabla_X W.$$

If we put $X = \mu W$ in this and take account of (2.26), (4.21) and (4.25), then we get

$$\begin{aligned}&\mu^2\nabla_W W - \mu\nabla\mu\\ &\quad = (2\rho - 3\alpha)AU + (\alpha^2 - \rho\alpha - 2c)U - \mu(W\alpha)\xi - \mu(W\mu)W.\end{aligned} \tag{4.38}$$

5. Semi-invariant submanifolds satisfying $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$

We will continue our arguments under the same hypotheses as those stated in section 4. That is, we consider a semi-invariant submanifold of codimension 3 in $M_{n+1}(c)$, $c \neq 0$ which satisfies $dt = 2\theta\omega$, and at the same time $R_\xi A = AR_\xi$ and $R_\xi A^{(2)} = A^{(2)}R_\xi$.

Differentiating (4.1) covariantly, we find

$$\begin{aligned}g((\nabla_X R_\xi)Y, Z) = &-(k^2+c)\{\eta(Z)g(\nabla_X\xi, Y) + \eta(Y)g(\nabla_X\xi, Z)\}\\ &+ (X\alpha)g(AY,Z) + \alpha g((\nabla_X A)Y, Z) - g(A\xi, Z)\{g((\nabla_X A)\xi, Y)\\ &- g(A\phi AY, X)\} - g(A\xi, Y)\{g((\nabla_X A)\xi, Z) - g(A\phi AZ, X)\}\\ &+ (Xk)g(KY,Z) + kg((\nabla_X K)Y, Z) - 2k(Xk)\eta(Y)\eta(Z).\end{aligned}$$

If we put $X = W$ in this and make use of (4.11), we find

$$\begin{aligned}(\nabla_W R_\xi)Y =&(W\alpha)AY - (k^2+c)\{g(\phi AW, Y)\xi + \eta(Y)\phi AW\}\\ &\alpha(\nabla_W A)Y - \{g((\nabla_W A)\xi, Y) + g(A\phi AW, Y)\}A\xi\\ &+ k(\nabla_W K)Y - \{(\nabla_W A)\xi + A\phi AW\}\eta(AY).\end{aligned}$$

From now on we assume that $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ holds on M, Then we have

$$\begin{aligned}&\alpha(\nabla_W A)X + k(\nabla_W K)X\\ &\quad= -(W\alpha)AX + (k^2+c)\{g(\phi AW, X)\xi + \eta(X)\phi AW\}\\ &\quad+ g((\nabla_W A)\xi, X) + g(A\phi AW, X)A\xi + \{(\nabla_W A)\xi + A\phi AW\}\eta(AX)\end{aligned} \tag{5.1}$$

for any vector field X.

We notice here the following fact :

Remark 1. $\tau \neq 0$ on Ω.

If not, then we have $\theta - c = 0$. Thus, (3.23) implies that $L = 0$ and hence $KX = k\eta(X)\xi$ because of (3.19) and $\tau^2 = \theta - c$. Thus, (3.26) is reformed as

$$k\{\eta(X)AY - \eta(Y)AX + (\eta(X)t(Y) - \eta(Y)t(X))\xi\} = 0,$$

Putting $Y = \xi$ in this and $\sigma = \alpha + t(\xi)$, we have $t(X) + g(A\xi, X) - \sigma\eta(X) = 0$. Combining the last two equations, we obtain $AX = \eta(X)A\xi + g(A\xi, X)\xi - \alpha\eta(X)\xi$. From this we have

$$AU = 0, \quad AW = \mu\xi. \tag{5.2}$$

On the other hand, putting $X = W$ in (5.1) and using (5.2), we find

$$\alpha(\nabla_W A)W + k(\nabla_W K)W = g((\nabla_W A)\xi, W) + \mu(\nabla_W A)\xi,$$

which together with (4.26) and (5.2) yields

$$\alpha(\nabla_W A)W + k(\nabla_W K)W = -2cU + \mu\nabla\mu. \tag{5.3}$$

From the second equation of (5.2), we find

$$(\nabla_X A)W + A\nabla_X W = \mu\phi AX$$

because of $\mu^2 = c$, which is obtained by (4.19) and (5.2). Putting $X = W$ in the last relationship, and using $\mu^2 = c$, (4.38) and (5.2), we have $(\nabla_W A)W = 0$.

Since we have from (4.8) $KW = 0$, in the same way as above it is seen that $(\nabla_W K)W = 0$. Thus, (5.3) will produce a contradiction because $\mu^2 = c$. Therefore $\Omega = \emptyset$ is proved provided that $\tau = 0$.

Now, replacing X by ξ in (5.1), we find

$$k(\nabla_W K)\xi = \alpha A\phi AW + (k^2 + c)\phi AW,$$

where we have used (2.32) and Lemma 3.2.

On the other hand, differentiating the first equation of (3.18) covariantly with respect to W and using (2.8) and (4.11), we get

$$(\nabla_W K)\xi + K\phi AW = k\phi AW.$$

Substituting this into the last equation, we obtain

$$\alpha A\phi AW + c\phi AW + kK\phi AW = 0. \tag{5.4}$$

From this and (4.21), we have $(\rho-\alpha)\{\alpha AU + (k\tau + c)U\} = 0$, where we have used (4.4) and (4.21). So we have

$$\alpha AU + (k\tau + c)U = 0. \tag{5.5}$$

In fact, if not, then we have $\rho - \alpha = 0$. Therefore (4.19) and (4.21) are reduced respectively to $\mu^2 = c - k\tau$ and $AW = \mu\xi$ on this subset. We restrict our discussions on this subset. We also have from (4.35) the following : $(k+\tau)AU = 0$, which together with (4.11) and Proposition 3.3, gives $AU = 0$.

On the other hand, differentiating (4.8) covariantly, we find

$$(\nabla_X K)W + K\nabla_X W + \tau\nabla_X W = 0.$$

If we take the inner product with a vector field Y and taking the skew-symmetric part and using (3.25) and (4.8), then we find

$$\frac{\tau}{\mu}\{t(Y)u(X) - t(X)u(Y)\} + g(K\nabla_X W, Y) - g(K\nabla_Y W, X)$$
$$= \tau\{(\nabla_Y W)X - (\nabla_X W)Y\}.$$

Replacing X by W in this and making use of (4.8), we have

$$\frac{\tau}{\mu}t(W)U + K\nabla_W W + \tau\nabla_W W = 0, \tag{5.6}$$

which together with (4.38) and the fact that $AU = 0$ gives

$$\mu\tau t(W)U + \mu(K\nabla\mu + \tau\nabla\mu) - 4c\tau U - \mu(k+\tau)(W\alpha)\xi = 0. \tag{5.7}$$

By the way, we have from (4.19) $\mu^2 = c - k\tau$. Differentiating this and using (4.11), we find $\mu\nabla\mu = -\tau\{(\xi k)\xi + (k-\tau)U\}$, which together with (3.18) and (4.4) yields $\mu K\nabla\mu = -\tau\{k(\xi k)\xi + (k-\tau)U\}$. Substituting the last two relationships into (5.7), we obtain

$$\tau\{\mu t(W) - 2\tau(k-\tau) - 4c\}U - (k+\tau)\{\tau(\xi k) + \mu(W\alpha)\}\xi = 0,$$

which together with (4.10) gives

$$\mu^2(1 + \frac{\tau}{k})\tau + 2\tau^2(k-\tau) - 4c\tau = 0,$$

or using (4.19) and Remark 5.1 implies that $3\tau k^2 - \tau^2 k + 3ck - c\tau = 0$. Thus, k is a constant and hence $k = \tau$ because of (4.11). So we get $\tau^2 + c = 0$ with the aid of Remark 5.1. From this and (4.19) with $\rho = \alpha$ will produce a contradiction. Accordingly $\rho - \alpha \neq 0$ on Ω is proved.

Remark 2. $\alpha \neq 0$ on Ω.

In fact, if not, then we have $\alpha = 0$ and hence

$$\tau^2 - c + \beta = 0 \tag{5.8}$$

on this subset because of (4.19). We also have $k\tau + c = 0$ on the set by virtue of (5.5) and Proposition 3.3. Thus, k is a nonzero constant and consequently $\tau^2 + c = 0$ because of (4.11). It is contradictory to (5.8). Therefore $\alpha = 0$ is impossible on Ω.

From the previously obtained formula, it seen that

$$(\nabla_W K)W + K\nabla_W W + \tau\nabla_W W = 0,$$

which together with (5.6) yields $\mu(\nabla_W K)W = \tau t(W)U$. Thus, it follows that

$$k(\nabla_W K)W = -\tau(k+\tau)U, \tag{5.9}$$

where we have used (4.10)

If we use (4.21) and (4.26), then (5.1) can be written as

$$\begin{aligned}
&\alpha(\nabla_W A)X + k(\nabla_W K)X + (W\alpha)AX \\
&\quad = \frac{1}{\mu}\{(\rho - 2\alpha)g(AU, X) - 2cu(X) + \mu(X\mu)\}A\xi \\
&\quad + \frac{1}{\mu}\{(\rho - 2\alpha)AU - 2cU + \mu\nabla\mu\}\eta(AX) + \varepsilon(k^2 + c)(u(X)\xi + \eta(X)U) \\
&\quad + \varepsilon\alpha\{g(AU, X)\xi + \eta(X)AU\} + \varepsilon\{g(AU, X)W + w(X)AU\},
\end{aligned} \tag{5.10}$$

where we have put $\varepsilon = \rho - \alpha$.

Since we have from (4.23)

$$g((\nabla_X A)W, W) = 2g(AU, X) + X\varepsilon,$$

if we replace X by μW in (3.24) and make use of (2.26) and (4.7), then we find

$$(\nabla_W A)W = -2AU + \nabla\varepsilon. \tag{5.11}$$

Putting $X = W$ in (5.10), we have

$$\begin{aligned}
&\alpha(\nabla_W A)W + k(\nabla_W K)W + (W\alpha)AW \\
&\quad = \varepsilon AU + (W\mu)A\xi + (\rho - 2\alpha)AU - 2cU + \mu\nabla\mu,
\end{aligned}$$

or, using (2.27), (4.27), (5.9) and (5.11),

$$\alpha\nabla\rho - \frac{1}{2}\nabla\beta = \rho AU + \varepsilon AU + \{\tau(k+\tau) - 2c\}U + (\xi\varepsilon)A\xi - (W\alpha)AW.$$

If we take account of (5.5), then we can write this as

$$\frac{1}{2}\nabla\beta - \alpha\nabla\rho = c(2+\frac{\varepsilon}{\alpha})U - \tau(k+\tau+\frac{\varepsilon}{\alpha}k)U \\ - \rho AU + (W\alpha)AW - (\xi\varepsilon)A\xi. \tag{5.12}$$

Applying this by W and using (4.21) and (4.27), we get

$$\frac{1}{2}W\beta - \alpha(W\rho) = \varepsilon(W\alpha) - \mu(W\mu),$$

which together with (2.27) yields

$$W\beta = \alpha(W\rho) + \rho(W\alpha). \tag{5.13}$$

Differentiating (2.27) with respect to W and making use of the last equation, we obtain

$$\alpha W\varepsilon = 2\mu(\xi\varepsilon) - \varepsilon(W\alpha). \tag{5.14}$$

Lemma 5.1. *$k - \tau \neq 0$ on Ω.*

Proof. If not, then we have $k - \tau = 0$ on this open subset. So we have $(k+\tau)AU = 0$ on the set because of (4.35) and hence $AU = 0$ with the aide of Remark 5.1. We discuss our arguments on such a place. Thus, (5.5) gives $\tau^2 + c = 0$ and hence $\theta = 0$ by virtue of (4.6). Therefore we get $dt = 0$ because of (3.1). We also have from (4.10) $t(X) = t(\xi)\eta(X) - 2\mu w(X)$ for any vector field X. Differentiating this covariantly and using (2.5), and taking the skew-symmetric part obtained,

$$X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) - t(\xi)g((\phi A + A\phi)X, Y) \\ = 2((X\mu)w(Y) - (Y\mu)w(X) + 2\mu dw(X, Y),$$

which together with (2.24), (4.15) and the fact that $AU = 0$ gives

$$2\mu dw(X, Y) \\ = (t(\xi) + 2\alpha)(u(X) - \eta(X)u(Y)) - 2\mu((X\alpha)\eta(X) - (Y\alpha)\eta(X)) \\ + t(\xi)g((A\phi + \phi A)X, Y) - 2\mu((X\mu)w(Y) - (Y\mu)w(X)).$$

Putting $Y = W$ in this, and using (3.19), (4.21) and (4.38) yields

$$\varepsilon\{t(\xi) + 2\alpha\} + 4c = 0. \tag{5.15}$$

In the next step, if we differentiate (4.21) covariantly and taking the skew-symmetric part obtained,

$$\frac{\tau}{\mu}(u(X)t(Y) - t(X)u(Y)) + g(K\nabla_X W, Y) \\ - g(K\nabla_Y W, X) = \tau dw(Y, X),$$

where we have used (3.32) and (4.7). If we put $X = \xi$ or W in this and take account of previously obtained formulas (for detail, see [13], [14])

$$(\rho - \alpha)U\alpha = -2c\mu^2, \quad \mu(U\mu) = 2(\rho\alpha - \alpha^2 + c)\mu^2. \tag{5.16}$$

On the other hand, from $AU = 0$, we have $(\nabla_X A)U + A\nabla_X U = 0$, which shows $g(\nabla_X AU, U) = 0$ and hence $(\nabla_U A)U = 0$ because of (3.31) and (4.5). Thus, it follows that $A\nabla_U U = 0$.

We also have above equation that that $(\nabla_\xi A)U + A\nabla_\xi U = 0$, which connected to (4.13) implies that

$$(\nabla_\xi A)U + \alpha A^2\xi - \beta A\xi + A\phi\nabla\alpha = 0,$$

or, using (4.20)

$$\phi(\nabla_\xi A)U + (\rho\alpha - \beta)U + \phi A\phi\nabla\alpha = 0.$$

Because of (2.34) and the fact that $AU = 0$, we have $\nabla_U U = \phi(\nabla_U A)\xi$.

By the way, it is, using (2.19), seen that $\phi(\nabla_U A)\xi = \phi(\nabla_\xi A)U$.

Combining the last three relationships, we obtain

$$\nabla_U U = (\beta - \rho\alpha)U - \phi A\phi\nabla\alpha.$$

If we take the inner product with U to this and make use of (4.21), and the facts that $AU = 0$ and $\tau^2 + c = 0$, then we obtain

$$\mu(U\mu) = 2c\mu^2 - (\rho - \alpha)U\alpha,$$

which together with (5.16) yields $\alpha(\rho - \alpha) = 0$, a contradiction because of $\varepsilon \neq 0$ and Remark 5.2, Thus, $k - \tau \neq 0$ on Ω is proved. □

Remark 3. $\xi k = 0$ on Ω.

In fact, if we take the inner product with W to (4.11), then we have $Wk = 0$. From (4.36) and (5.5) we have $\alpha\lambda + k\tau + c = 0$. Differentiating this with respect to W and making use of (4.33) and the fact that $Wk = 0$, we find $\lambda W\alpha = 0$. If $W\alpha \neq 0$ on Ω, then we get $k\tau + c = 0$, which connected to (4.11) gives $k - \tau = 0$, a contradiction because of Remark 5.1. Thus, it follows that $W\alpha = 0$ on Ω.

Differentiation (4.37) with respect to W gives $(k - \tau)W\rho = 0$ because of (4.33) and the fact that $W\alpha = 0$, which together with Remark 5.1 implies that $W\rho = 0$. Thus, (4.19) gives $W\beta = 0$, which connected to (2.27) yields $W\mu = 0$. Thus, (4.27) becomes $\xi\rho - \xi\alpha = 0$. By differentiating (4.37) with respect to ξ, we then have

$$(\lambda + \rho - \alpha)\xi k = 0,$$

where we have used (4.34). If $\xi k \neq 0$ on Ω, then $\lambda = \alpha - \rho$, which together with (4.37) implies that $\tau\lambda = 0$. Because of Remark 5.1, we see that $\lambda = 0$. Accordingly we have $k\tau + c = 0$ because (4.36) and (5.5), which connected to

(4.11) gives $k = \tau$, a contradiction by virtue of Lemma 5.1. Therefore $\xi k = 0$ on Ω is proved.

Because of Remark 5.2, we can write (4.11) as

$$\nabla k = (k - \tau)U. \tag{5.17}$$

Differentiating this covariantly and taking the skew-symmetric, we find $du = 0$ because of Lemma 5.1. So we have

$$g(\nabla_X U, \xi) - g(\nabla_\xi U, X) = 0$$

for any vector X, which connected to (2.29) and (4.13) yields

$$3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha + \mu AW = 0.$$

From this, and using (2.25), (2.28), (4.21) and (4.36), we have $\nabla\alpha = (\xi\alpha)\xi) + (\rho - 3\lambda)U$.

Differentiating the second equation of (4.36) with respect to ξ and using Remark 5.2 and (4.34), we find $\lambda\xi\alpha = 0$. But, the function λ does not vanish by virtue of (4.36), (5.17), Lemma 5.2 and Remark 5.1. Thus $\xi\alpha = 0$ on Ω, which implies that

$$\nabla\alpha = (\rho - 3\lambda)U. \tag{5.18}$$

Now, differentiating (4.4) covariantly along Ω, and taking the skew-symmetric part and using (3.27), we find

$$g(K\nabla_X U, Y) - g(K\nabla_Y U, X) + \mu\tau\{t(X)w(Y) - t(Y)w(X)\} = 0,$$

where we have used the fact that $du = 0$. Putting $X = \xi$ in this and using (2.29), (3.18) and (4.13), we get

$$K(3\lambda\phi U + \alpha A\xi - \beta\xi + \phi\nabla\alpha) + k\mu AW + \mu\tau t(\xi)W = 0,$$

which connected to (2.25), (2.28), (3.20), (4.5) and (4.21) gives

$$\tau t(\xi) + (\rho - \alpha)(k + \tau) = 0, \tag{5.19}$$

or, using (4.37),

$$\tau(k - \tau)t(\xi) = \lambda(k + \tau)^2. \tag{5.20}$$

Because of (2.26), we can write (4.10) as

$$t(Y) = t(\xi)\eta(Y) - \mu(1 + \frac{\tau}{k})w(Y)$$

for any vector field Y. Differentiating this covariantly along Ω and using (2.5), (2.6), (4.14), (4.36) and (5.17), we find

$$
\begin{aligned}
X(t(Y)) =& X(t(\xi))\eta(Y) + t(\xi)g(\phi AX, Y) + \frac{\tau}{k^2}(k-\tau)\mu u(X)w(Y) \\
&- (1+\frac{\tau}{k})\{\lambda u(X)\eta(Y) - g(\phi\nabla_X U, Y)\} + t(\nabla_X Y),
\end{aligned}
$$

from which, taking the skew-symmetric part, and making use of (2.28), (2.31), (3.1), (4.36) and (5.17) implies that

$$
\begin{aligned}
&2\theta g(\phi X, Y) - t(\xi)\{g(\phi AX, Y) - g(\phi AY, X)\} + \frac{\tau}{k^2}\mu(w(X)u(Y) - w(Y)u(X)) \\
&= X(t(\xi))\eta(Y) - Y(t(\xi))\eta(X) + (1+\frac{\tau}{k^2})\{2cg(\phi X, Y) + (\rho - 3\lambda)(u(X)\eta(Y) \\
&- u(Y)\eta(X)) - 2g(A\phi AX, Y) + \alpha(g(\phi AX, Y) - g(\phi AY, X))\}.
\end{aligned} \tag{5.21}
$$

Putting $Y = \xi$ in this and using (2.5) and (4.36), we find

$$
X(t(\xi)) = \xi(t(\xi))\eta(X) + \{t(\xi) + (1+\frac{\tau}{k})(\lambda + \alpha - \rho)u(X),
$$

which together with (5.19) gives

$$
\nabla t(\xi) = \xi(t(\xi))\xi + (1+\frac{\tau}{k})(\lambda + t(\xi))U. \tag{5.22}
$$

On the other hand, differentiating (5.19) and using (4.37) and (5.17), we get $\tau\nabla t(\xi) = (k+\tau)(\nabla\alpha - \nabla\rho + \lambda U)$.

By the way, differentiating (4.37) with respect to ξ and using (4.34), Lemma 5.1 and Remark 5.2, we have $\xi\rho - \xi\alpha = 0$. Thus, it follows that $\xi(t(\xi)) = 0$. Combining above two equations, we obtain $k(\nabla\alpha - \nabla\rho) = 2\tau(\lambda + \alpha - \rho)U$, where we have used (4.37) and the fact that $k - \tau \neq 0$ on Ω.

On the other hand, if we differentiate (5.20), and using (5.17) and itself, we find

$$
\lambda(k+\tau)^2 U + \tau(k-\tau)\nabla t(\xi) = (k+\tau)^2\nabla\lambda + 2\lambda(k^2 - \tau^2)U,
$$

or, using (5.22) and the last equation,

$$
(k+\tau)\nabla\lambda = 6\tau\lambda U. \tag{5.23}
$$

Now, from (2.27) and (4.19) we have

$$
\mu^2 = c - k\tau + \alpha(\rho - \alpha).
$$

By the way we see, using (4.36) and (5.5), that

$$
\alpha\lambda + k\tau + c = 0. \tag{5.24}
$$

Combining the last two equations, we obtain

$$
(k-\tau)\mu^2 = 2\theta k, \tag{5.25}
$$

where we have used (4.6).

If we put $X = U$ and $Y = W$ in (5.21) and make use of (4.21), (4.36) and (5.25), then we find

$$\begin{aligned}&2\theta(1+\frac{\tau}{k}) - t(\xi)(\rho-\alpha+\lambda)\\&\quad = (1+\frac{\tau}{k})\{2c - 2\lambda(\rho-\alpha) + \alpha(\rho-\alpha+\lambda)\},\end{aligned}$$

where we have used Proposition 3.3.

Multiflying $k(k-\tau)$ to this and making use of (4.37), we get

$$\theta(k^2-\tau^2) + \lambda k\tau t(\xi) = c(k^2-\tau^2) + \lambda^2(k+\tau)^2 - \tau\lambda\alpha(k+\tau),$$

Since $k+\tau \neq 0$ because of (5.17), if we use (5.19) and (5.24), then we find

$$\theta(k-\tau) - \lambda k(\rho-\alpha) = (\tau^2+c)k + \lambda^2(k+\tau),$$

which together with (4.6) and (4.37) yields $\theta(k-\tau) = (k+\tau)\lambda^2$.

Differentiating this and taking account of (5.17) and itself, we obtain

$$\lambda^2(k+\tau)U = (k-\tau)\lambda^2 U + 2(k+\tau)\lambda\nabla\lambda,$$

which tells us, using (5.23), that $\lambda = 0$. From this fact and (5.24) we verify that k is a constant. Therefore, it follows that $k-\tau = 0$ because of (5.17), a contradiction. Hence, we conclude that $k = 0$ on M. From this fact, and (3.17) and (3.18) we have $m = 0$ and $K\xi = 0$. We also have from (3.21)

$$K^2X = (\theta-c)(X-\eta(X)\xi) \tag{5.26}$$

because of (3.19) with $k = 0$. Further, (3.25) turns out to be

$$(\theta-c)\{(\phi A - A\phi)X + \eta(X)U + u(X)\xi\} = 0.$$

By the way, if we use (3.21) and (5.21), then we see that $A\xi = \alpha\xi$, that is $U = 0$ on M (for detail, sec[21]). Thus, the last relationship can be written as

$$(\theta-c)(\phi A - A\phi) = 0.$$

Now, we assume that $\theta - c \neq 0$ on M. Then we have $A\phi = \phi A$, which together with (3.24) with $k = 0$ yields $A\xi = \alpha\xi$ and

$$A^2X = \alpha AX + c(X-\eta(X)\xi). \tag{5.27}$$

From this we have

$$h_{(2)} = \alpha h + 2(n-1)c. \tag{5.28}$$

If we take account of (3.24) with $k = 0$ and (5.27), then we can verify that (see, Theorem 4.3 of [24])

$$(\nabla_X A)Y = -c\{\eta(Y)\phi X + g(\phi X, Y)\xi\}. \tag{5.29}$$

Because of the fact that $A\phi = \phi A$ and $k = 0$, (3.34) turns out to be

$$L(\nabla_X L)Y = (\theta - c)\{t(X)\phi Y - \eta(Y)A\phi X - \eta(X)\phi AY\},$$

or, using (3.19), (3.23) and (3.33),

$$\begin{aligned} &g((\nabla_X L)Y, Z) \\ &\quad = -t(X)g(KY, Z) + \eta(X)g(AKY, Z) + g(AX, KY)\eta(Z). \end{aligned} \tag{5.30}$$

Differentiating (5.26) covariantly along M and using (2.5), we find

$$(\nabla_X K)KY + K(\nabla_X K)Y = (c - \theta)\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\}.$$

As in the quite same method as that used from (3.23) to derive (3.34), we can deduce from the last equation the following :

$$\begin{aligned} 2(\nabla_X K)KY = (\theta - c)\{&-2t(X)\phi Y + \eta(X)(\phi A - A\phi)Y \\ &+ g((\phi A - A\phi)X, Y)\xi + \eta(Y)(\phi A + A\phi)X\}, \end{aligned}$$

where we have used (3.21) and (3.25), which together with the fact that $\phi A = A\phi$ gives

$$K(\nabla_X K)Y = (\theta - c)\{-t(X)\phi Y + \eta(X)\phi AY + g(\phi AX, Y)\xi\}.$$

If we transform this by K and make use of (3.20), (3.26) and (5.26), then we can write the last equation as

$$(\nabla_X K)Y = t(X)LY - \eta(X)ALX - \eta(Y)LAX - g(ALX, Y)\xi.$$

Differentiating this covariantly along M and making use of (2.5), (3.19), (5.27), (5.29), (5.30) and itself, we find

$$\begin{aligned} (\nabla_Z\nabla_X K)Y = Z(t(X)) &- c\{\eta(X)g(Z, KY)\xi + \eta(X)\eta(Y)KZ\} + T(Z, X, Y) \\ &- \alpha\{\eta(X)\eta(Y)AKZ + g(AZ, KY)\eta(X)\xi\} \\ &- g(\nabla_Z X, \xi)ALY - g(X, \phi AZ)ALY - \eta(Y)AL\nabla_Z X \\ &- g(Y, \phi AZ)ALX - g(A\nabla_Z X, LY)\xi - g(AX, LY)\phi AZ, \end{aligned}$$

where $T(Z, X, Y)$ is a certain tensor field with $T(Z, X, Y) = T(X, Z, Y)$, from which, taking the skew-symmetric part with respect to Z and X, and the fact that $A\phi = \phi A$ and using the Ricci identity for K,

$$\begin{aligned}
&(R(Z,X)\cdot K)(Y) \\
&= 2\theta g(\phi Z,X)LY - c\{\eta(X)g(Z,KY)\xi - \eta(Z)g(X,KY)\xi \\
&\quad + \eta(Y)(\eta(X)KZ - \eta(Z)KX)\} \\
&\quad - \alpha\{\eta(Y)(\eta(X)AKZ - \eta(Z)AKX) \\
&\qquad + g(AZ,KY)\eta(X)\xi - g(AX,KY)\eta(Z)\xi\} \\
&\quad + 2g(Z,\phi AX)ALY - g(Y,\phi AZ)ALX + g(Y,\phi AX)ALZ \\
&\quad - g(AX,LY)\phi AZ + g(AZ,LY)\phi AX.
\end{aligned} \tag{5.31}$$

Putting $Z = \phi e_i$ and $X = e_i$ and summing for i, and using (3.1), (3.19), (3.20) and (5.27), we find

$$\sum_{i=0}^{2n-1}(R(\phi e_i,e_i)\cdot K)(Y) = 4\{c-(n-1)\theta\}LY + 2(h+\alpha)LAY. \tag{5.32}$$

On the other hand, from (2.18) we see, using (3.20), (5.26) and (5.27), that

$$\sum_{i=0}^{2n-1}(R(\phi e_i,e_i)\cdot K)(Y) = 4\{2\theta-(2n+3)c\}LY - 4\alpha LAY,$$

where we have used the fact that $A\phi = \phi A$, which connected to (5.32) implies that

$$(h+3\alpha)LAX = 2\{(n+1)\theta - 2(n+2)c\}LX,$$

which together with (3.23) yields

$$(h+3\alpha)(g(AX,Y)-\alpha\eta(X)\eta(Y)) = 2\{(n+1)\theta-2(n+2)c\}(X-\eta(X)\eta(Y)).$$

If we put $X = Y = e_i$ and summing up to $i = 0, 1, \cdots, 2n-2$, we have

$$(h+3\alpha)(h-\alpha) = 4(n-1)\{(n+1)\theta - 2(n+2)c\}. \tag{5.33}$$

If we put $Y = e_i$, $Z = Ae_i$ in (5.31) and summing up to $i = 0, 1, \cdots, 2n-2$, then we have

$$\sum_{i=0}^{2n-1}(R(Ae_i,X)\cdot K)(e_i) = (2\theta - 3\alpha^2 - 4c)AKX - 3c\alpha KX.$$

By the way, from (2.18) we have

$$\sum_{i=0}^{2n-1}(R(Ae_i,X)\cdot K)(e_i) = (8c-2\theta+h_{(2)})AKX - \{(\theta-2c)(h-\alpha)-c\alpha\}KX.$$

Thus, above two equations gives

$$(4\theta - 12c - h_{(2)} - 3\alpha^2)AKX = \{4c\alpha - (\theta - 2c)(h - \alpha)\}KX,$$

which connected to (5.26) yields

$$(4\theta - 12c - h_{(2)} - 3\alpha^2)(h - \alpha) = 2(n - 1)\{4c\alpha - (\theta - 2c)(h - \alpha)\}.$$

Combining this to (5.33) and take account (5.28), we obtain

$$(\theta - 3c)(h - \alpha) = 2(n - 1)(\theta - 2c)\alpha. \tag{5.34}$$

If we compare with (5.33) and (5.34), then we get

$$\{(n - 1)(\theta - 2c) - 2c\}\{(\theta - 3c)^2 - (\theta - 2c)\alpha^2\} = 0. \tag{5.35}$$

For the case where $c > 0$, if $\theta - 2c < 0$, then the last equation led to

$$(\theta - 3c)^2 - (\theta - 2c)\alpha^2 = 0, \tag{5.36}$$

a contradiction because $\theta - 2c < 0$. Hence we arrive at $\theta - c = 0$ and consequently $A^{(2)} = A^{(3)} = 0$ because of (3.23) and (5.27)

Let $N_0(p) = \{\nu \in T_p^{\perp}(M) : A\nu = 0\}$ and $H_0(p)$ be the maximal J-invariant subspace of $H_0(p)$. Since $A^{(2)} = A^{(3)} = 0$, the orthogonal complement of $H_0(p)$ is invariant under parallel translation with respect to the normal connection because of $\nabla^{\perp}\mathcal{C} = 0$ was assumed. Thus, by the reduction theorem in [10] and by Proposition 3.3, we conclude that

Theorem 5.2. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codimension 3 in a complex projective space $P_{n+1}\mathbb{C}$ such that the third fundmental form t satisfies $dt = 2\theta\omega$ for a scalar $\theta - 2c < 0$. Suppose that M satisfies $R_\xi A^{(2)} = A^{(2)} R_\xi$ and at the same time $R_\xi A = AR_\xi$. Then M is a real hypersurface in a complex projective space $P_n\mathbb{C}$ provided that $\nabla_{\phi\nabla_\xi\xi} R_\xi = 0$ holds on M.*

If we consider $c < 0$, then we have $(n - 1)(\theta - 2c) - 2c > 0$ because $\theta - c$ is nonnegative. Hence we obtain (5.36) with the aid of (5.35). Thus, it is, using (5.31) and (5.36), seen that

$$h(h - \alpha) = 2(n - 1)(2n - 1)(\theta - 2c) - 2(n - 1)c. \tag{5.37}$$

On the other hand, from (2.18) the Ricci tensor S of M is given by

$$SX = c\{(2n + 1)X - 3\eta(X)\xi\} + hAX - A^2X - K^2X - L^2X$$

because of $k = l = 0$, which together with (3.23) and (5.26) gives

$$SX = \{c(2n + 1) - 2(\theta - c)\}X + (2\theta - 5c)\eta(X)\xi + hAX - A^2X.$$

Therefore, the scalar curvature $\bar{r}$ of M is given by

$$\bar{r} = 2(n-1)(2n+1)c - 4(n-1)(\theta - c) + h(h-\alpha),$$

which connected to (5.37) yields

$$\bar{r} = 2(n-1)c + 2(n-1)(2n-3)(\theta - c).$$

If we assume that $\bar{r} - 2(n-1)c \leq 0$, then we obtain $\theta - c = 0$, a contradiction. Consequently we have $A^{(2)} = A^{(3)} = 0$ because of (3.23) and (5.26).

In the same way as that used the proof of Theorem 5.2, we have

Theorem 5.3. *Let M be a real (2n-1)-dimensional ($n > 2$) semi-invariant submanifold of codimension 3 in a complex hyperbolic space $H_{n+1}\mathbb{C}$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar θ and the scalar curvature $\bar{r}$ of M is satisfies $\bar{r} - 2c(n-1) \leq 0$. Suppose that M satisfies $R_\xi A^{(2)} = A^{(2)} R_\xi$ and at the same time $R_\xi A = AR_\xi$. Then M is a real hypersurface in a complex space $H_n\mathbb{C}$ provided that $\nabla_{\phi\nabla_\xi \xi} R_\xi = 0$ holds on M.*

6. The proof of Hopf hypersurfaces

In this section, we devote to prove that the real hypersurface satisfying hypotheses stated in Theorem 5.2 is a Hopf hypersurface in $M_n(c)$.

From the hypothesis $R_\xi A = AR_\xi$ we have

$$A^2\xi = \rho A\xi + c\xi \tag{6.1}$$

because of (4.20), which shows

$$\beta = \rho\alpha + c. \tag{6.2}$$

In the rest of this paper, let Ω' be the open subset of M defined by $\Omega' = \{p \in M : \mu(p) \neq 0\}$. We discuss our arguments on the set Ω' and suppose that Ω' be nonvoid. Then, by virtue of (2.25) and (6.1) we get

$$AW = \mu\xi + (\rho - \alpha)W. \tag{6.3}$$

Differentiating (6.1) covariantly along M and using (2.5), we find

$$\begin{aligned} &g((\nabla_X A)A\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) - \rho g(A\phi AX, Y) \\ &\quad = (X\rho)g(A\xi, Y) + \rho((\nabla_X A)\xi, Y) + cg(\phi AX, Y), \end{aligned}$$

which together with (2.32) with $k = 0$ and (3.24) with $k = 0$ gives

$$(\nabla_\xi A)A\xi = \rho AU - cU + \frac{1}{2}\nabla\beta.$$

Putting $X = \xi$ in above equation and using (2.32), (3.24) and the last relationship, we obtain

$$3A^2U - 2\rho AU - 2cU = (\xi\rho)A\xi - A\nabla\alpha + \rho\nabla\alpha - \frac{1}{2}\nabla\beta. \tag{6.4}$$

Another hypothesis $\nabla_{\phi\nabla_\xi\xi}R_\xi = 0$ tells us that

$$\alpha AU + cU = 0. \tag{6.5}$$

with the aid of (5.3), and hence $\alpha \neq 0$ on Ω' because of Remark 5.2.

Using (6.2) and (6.5) we can write (5.12) as

$$\frac{1}{2}(\alpha\nabla\rho - \rho\nabla\alpha) = -c(1 + \frac{2\rho}{\alpha}U) + (\xi\varepsilon)A\xi - (W\alpha)AW, \tag{6.6}$$

which shows that

$$\alpha^2(U\rho) - \rho\alpha(U\alpha) = -2c(2\rho + \alpha)\mu^2. \tag{6.7}$$

Furthermore, using (6.5) we can write (5.10) as

$$\begin{aligned}\alpha(\nabla_W A)X =& -(W\alpha)AX - \frac{c\varepsilon}{\alpha}\{u(X)W + w(X)U\}\\ &+ g(\nabla\mu - \frac{c\rho}{\alpha\mu}U, X)A\xi + g(A\xi, X)(\nabla\mu - \frac{c\rho}{\alpha\mu}U).\end{aligned}$$

Replacing X by U in this and using (6.5), we get

$$\alpha(\nabla_W A)U = (U\mu)A\xi + \frac{c}{\alpha}\{(W\alpha)U - \varepsilon\mu^2W - \mu\rho A\xi\}. \tag{6.8}$$

If we take the inner product with U to (4.22) and make use of (3.24) with $k = 0$, (2.29) and (6.3), then we get

$$(\alpha\varepsilon + c)g(\nabla_X W, U) = \alpha g((\nabla_X A)W, U) + c\alpha\mu\eta(X) - \alpha^2 g(AW, X),$$

which together with (2.27), (6.2) and (6.8) yields

$$\begin{aligned}\mu^2 g(\nabla_X W, U) = g\{(U\mu)A\xi + &\frac{c}{\alpha}(W\alpha)U - \varepsilon\mu^2W - \mu\rho A\xi\\ &+ c\alpha\mu\xi - \alpha\mu^2AW, X\}.\end{aligned} \tag{6.9}$$

Putting $X = U$ in this, we have

$$g(\nabla_U W, U) = \frac{c}{\alpha}W\alpha. \tag{6.10}$$

On the other hand, applying (2.31) by ϕ and using (2.29), we find

$$\phi(\nabla_X A)\xi = \nabla_X U - \mu w(AX)\xi - \phi A\phi AX - \alpha AX + \alpha\eta(AX)\xi.$$

If we put $X = U$ in this and use (2.26), (4.21) and (6.5), then we find

$$\nabla_U U = \phi(\nabla_U A)\xi + c(\frac{\varepsilon}{\alpha} - 1)U. \tag{6.11}$$

Further, taking the inner product with U to (2.31), we also get

$$g(\nabla_X W, U) = \frac{1}{\mu} g((\nabla_U A)X, \xi) - (\frac{c}{\alpha} + \alpha) g(AW, X) - 2cw(X),$$

where we have used (2.28), (3.24) with $k = 0$ and (6.5), which connected to (6.11) yields

$$\begin{aligned}&\mu(\nabla_U A)\xi - \mu^2\{(\alpha + \frac{c}{\alpha})AW + 2cW\}\\ &\quad = (U\mu)A\xi + \frac{c}{\alpha}\{(W\alpha)U - \varepsilon\mu^2 W - \mu\rho A\xi\} + c\alpha\xi - \alpha\mu^2 AW.\end{aligned}$$

Applying this by ϕ and using (4.21) and (6.11), we obtain

$$\nabla_U U = -\frac{c}{\alpha}(W\alpha)W + \delta U \tag{6.12}$$

for some function δ on Ω'.

In the next step, differentiating (6.5) covariantly, and using itself, we find

$$-\frac{c}{\alpha}(X\alpha)U + \alpha(\nabla_X A)U + \alpha A\nabla_X U + c\nabla_X U = 0,$$

from which, taking the skew-symmetric part and using (2.28) and (3.24),

$$\begin{aligned}&\frac{c}{\alpha}\{(Y\alpha)u(X) - (X\alpha)u(Y)\} + c\alpha\mu\{\eta(X)w(Y) - \eta(Y)w(X)\}\\ &\quad + \alpha\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + cdu(X, Y) = 0,\end{aligned} \tag{6.13}$$

where du is the exterior derivative of 1-form u. If we put $X = \xi$ in this and making use of (2.25) and (2.29), then we find

$$\begin{aligned}\alpha\mu g(W, \nabla_Y U) = &-\frac{c}{\alpha}(\xi\alpha)u(Y) + c\alpha\mu w(Y) + \mu(\alpha^2 + c)g(AW, Y)\\ &+ g(\alpha A\nabla_\xi U + c\nabla_\xi U, Y).\end{aligned}$$

If we put $Y = U$ in this and use (6.5), then we get $\alpha^2 g(\nabla_U U, W) = -c\mu(\xi\alpha)$, which together with (6.10) gives

$$\alpha(W\alpha) = \mu(\xi\alpha). \tag{6.14}$$

Combining this to (6.12), we have

$$\nabla_U U = -\frac{c}{\alpha^2}\mu(\xi\alpha)W + \delta U,$$

which together with (6.5) yields

$$\alpha A\nabla_U U + c\nabla_U U = -\frac{c}{\alpha}\mu(\xi\alpha)(AW + \frac{c}{\alpha}W).$$

If we take account of (2.25), (4.21) and (6.2), then we can write this as

$$\alpha A\nabla_U U + c\nabla_U U == -\frac{c}{\alpha^2}\mu^2(\xi\alpha)A\xi.$$

Putting $X = U$ in (6.13) and using the last relationship, we find

$$\alpha\nabla\alpha = \frac{\alpha(U\alpha)}{\mu^2}U + (\xi\alpha)A\xi. \tag{6.15}$$

By the way, if we use (6.2) and (6.5), then we can write (6.4) as

$$\frac{1}{2}(\alpha\nabla\rho - \rho\nabla\alpha) = -c(2 - \frac{2\rho}{\alpha} - \frac{3\rho}{\alpha^2})U + (\xi\rho)A\xi - A\nabla\alpha.$$

Combining this to (6.6), we have

$$A\nabla\alpha + 3c(\frac{c}{\alpha^2} - 1)U = (\xi\alpha)A\xi + (W\alpha)AW.$$

If we take the inner product with U to this and make use of (6.5), then we find $\alpha(U\alpha) = 3(c - \alpha^2)\mu^2$, which together with (6.15) gives

$$\alpha\nabla\alpha = 3(c - \alpha^2)U + (\xi\alpha)A\xi. \tag{6.16}$$

Differentiating this covariantly and using (2.5) and taking the skew-symmetric part obtained, we have

$$\begin{aligned} &X(\xi\alpha)\eta(AY) - Y(\xi\alpha)\eta(AX) + 3(c - \alpha^2)du(X,Y) \\ &\quad = (\xi\alpha)\{6(\eta(AX)u(Y) - \eta(AY)u(X\xi) - 2g(A\phi AX, Y) + 2cg(\phi X, Y)\} \end{aligned}$$

for any vector fields X and Y.

From this, it is verified that $\xi\alpha = 0$ (for detail, see [16]). Thus, the last equation can be written as

$$(\alpha^2 - c)du(X,Y) = 0. \tag{6.17}$$

By the way, we see, (2.29) and (4.13), that

$$-du(X,\xi) = g(\mu AW + 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha, X).$$

If we use (2.25), (2.26), (6.3) and (6.5), then the last two equations deduce that

$$(\alpha^2 - c)\{\mu(\frac{3c}{\alpha} + \rho)W + \phi\nabla\alpha\} = 0.$$

Combining this to (6.16) with $\xi\alpha = 0$, we obtain $(\alpha^2 - c)(\rho + 3\alpha) = 0$.

Now, suppose that $\alpha^2 - c \neq 0$. Then we have $\rho + 3\alpha = 0$ and hence $\nabla\rho + 3\nabla\alpha = 0$ on this open subset. Hence (6.6) reformed as

$$c(1 + \frac{2\rho}{\alpha})U = (\xi\varepsilon)A\xi - (W\alpha)AW$$

on the set. Accordingly we see that $\alpha + 2\rho = 0$, a contradiction because of Remark 5.2. Therefore, we have $\alpha^2 = c$ on whole space. Thus, α is constant on Ω'. Accordingly it is seen, using (6.6), that $\xi\rho = 0$. Using these facts, (6.4) can be written as

$$3A^2U - 2\rho AU - 2cU + \frac{1}{2}\nabla\beta = 0,$$

which connected to (6.5) and $\alpha^2 = c$ gives $\nabla\beta = -2(2\rho\alpha + c)U$.

Using the same method as that used from (6.16) to (6.17) we can deduce from the last equation $(2\rho\alpha + c)(\rho + 3\alpha) = 0$, which shows that $2\rho\alpha + c = 0$ and hence $2\mu^2 + c = 0$ because of (6.2) and the fact that $\alpha^2 = c$, a contradiction. Therefore we conclude that $\Omega' = \emptyset$, that is $\mu = 0$ on M. Thus, M is a Hopf hypersurface on $M_n(c)$.

Owing to Theorem 5.2 and Theorem 5.3, we have

Theorem 6.1. *Let M be a real $(2n-1)$-dimensional $(n > 2)$ semi-invariant submanifold of codemension 3 in a complex space form $M_{n+1}(c)$, $c \neq 0$ such that the third fundamental form t satisfies $dt = 2\theta\omega$ for a scalar θ. Suppose that M satisfies $R_\xi A = AR_\xi$ and at the same time $R_\xi A^{(2)} = A^{(2)} R_\xi$. If $\theta - 2c < 0$ for $c > 0$ or if the scalar curvature $\bar{r}$ of M satisfies $\bar{r} \leq 2(n-1)c$ for $c < 0$, then M is a Hopf hypersurface in a complex space form in $M_n(c)$ provided that R_ξ is $\phi\nabla_\xi\xi$-parallel.*

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