J. Appl. Math. & Informatics Vol. **39***(2021), No. 1 - 2, pp. 173 - 196*
https://doi.org/10.14317/jami.2021.173

ANALYSIS OF $M/M/c$ RETRIAL QUEUE WITH THRESHOLDS, PH DISTRIBUTION OF RETRIAL TIMES AND UNRELIABLE SERVERS

SRINIVAS R. CHAKRAVARTHY*, SERIFE OZKAR, AND SHRUTI

Abstract. This paper treats a retrial queue with phase type retrial times and a threshold type-policy, where each server is subject to breakdowns and repairs. Upon a server failure, the customer whose service gets interrupted will be handed over to another available server, if any; otherwise, the customer may opt to join the retrial orbit or depart from the system according to a Bernoulli trial. We analyze such a multi-server retrial queue using the recently introduced threshold-based retrial times for orbiting customers. Applying the matrix-analytic method, we carry out the steady-state analysis and report a few illustrative numerical examples.

AMS Mathematics Subject Classification : 60K20, 60J28.
Key words and phrases : Retrial queue, phase type distribution, thresholds, breakdowns and repairs, matrix analytical method.

1. Introduction

Queueing systems with repeated calls, referred to as retrial queueing systems in the literature, are characterized by the feature that an arriving customer not able to get into service immediately will join the retrial orbit. These orbiting customers attempt, after random interval times, to enter into the service facility. Such classical retrial systems have been widely used to model many applications in telecommunication networks, computer and communication systems, cellular mobile networks and local area networks (see, e.g.,[2, 11, 20]).

There are several variations and generalizations of the classical retrial queues. A significant part of the existing literature on retrial queues assume the service environment to be one hundred percent reliable. However, in reality, the servers may be subject to random breakdowns for many reasons and hence require recovery (repair) time. During such times the servers are unavailable. Such situations

Received August 30, 2020. Revised September 12, 2020. Accepted November 26, 2020.
*Corresponding author.

are common, especially in manufacturing and communications systems. Consequently, the performance of the system gets significantly affected, leading to the study of retrial queueing systems with servers subject to breakdowns and repairs (see e.g., [6, 7, 17, 22, 28]).

Most of the works on retrial queues with unreliable environment for the service facility deal with a single server. Indeed, it is well known that investigating multi-server queues is, in general, more complicated than investigating single server queues. Nevertheless, multi-server unreliable retrial queues are much more flexible and applicable in practice than single server unreliable retrial queues. The literature devoted to multi-server retrial queues with servers subject to breakdowns is not very rich. We can refer only to the following papers. Kim et al. [16] were the first to investigate a multi-server retrial queue with breakdowns. It was assumed in [16] that the breakdowns occur according to a MAP, the server recovery time has a phase type distribution and that the retrial times are exponentially distributed. Note that MAP is a versatile Markovian point process introduced by Neuts [18] and studied extensively in the literature.

Using the direct truncation method, an $M/M/c$ retrial queue under breakdowns and repairs of the servers was studied by Subramanian [26]. In order to deal with the huge state space in the study of a multi-server retrial queue with finite source and unreliable servers, Ghrabi and Dutheillet [12] proposed an algorithm for directly computing the infinitesimal generator without a need to generate the Markov chain. Later, Raiah and Oukid [21] provided approximations for the reliability indices, in particular, server availability and failure frequency, of an unreliable $M/M/2$ retrial queue.

An $M/M/c$ retrial queue with Bernoulli feedback, geometric loss, and servers subject to breakdowns and repairs was examined by Ke et al. [15] under the assumption of exponential retrial times. Chang et al. [5] analyzed a feedback retrial queue with multiple unreliable servers, balking and reneging, by assuming exponential distribution for all random variables.

Recently, Dudin and Dudina [9] studied a retrial multi-server queueing system with MAP arrivals and so-called PHF (phase-type with failures) service time distribution as a model of a channel with unreliable transmission of information. Specifically, the authors in [9] use a finite-state Markov chain with two absorbing states for the service time distribution such that one absorbing state corresponds to a service completion and the other absorbing state corresponds to the failure of a service as opposed to the failure of the server. That is, there is no concept of the server being sent for a repair; instead, the customer who is interrupted through such a failure will either (a) start a new service; or (b) sent back to the orbit to retry again; or (c) resume the service from the phase from where the failure occurred. One of these three possibilities occur with certain probabilities. Thus, there is no need for the model studied in [9] to have a repair facility. Assuming retrial times to be exponential, the authors use asymptotically quasi-Toeplitz Markov chain approach to analyze their model in steady-state.

In much of the literature on retrial queues, on the other hand, the retrial times are assumed to be exponentially distributed with the exception of a few papers (see e.g., [1, 3, 8, 14, 27]). Also, Shin [23] investigated an $M/M/c$ retrial queue with retrial times of phase type ($PH-$) restricted to order two (due to inherent complexity created by size of the underlying state space). Later, relaxing the assumption of PH_2 retrial time, Shin and Moon [24] presented an approximation for the distribution of the number of customers in the orbit as well as for those under service. Recall that $PH-$type distributions were introduced by Neuts [19].

The few studies in which non-exponential retrial times such as phase type retrial times are assumed, develop a variety of approximation methods or put a bound on the retrial orbit. These are mainly due to the complexity involved in keeping track of the elapsed retrial time for each of possibly extremely large number of customers in the orbit. As pointed out in [10], threshold-type policy is also optimal for retrial queues as for the ordinary queues. Assuming a finite capacity retrial orbit with MAP arrivals and $PH-$services, Efrosinin and Breuer [10] show that the optimal policy that minimizes the number of customers in the system is of a threshold-type. It is worth pointing out that in the context of a general retrial distribution but with the restriction that only the customer, if any, at the head of the retrial orbit is allowed to retry once the server becomes idle, Gomez-Corral [13] studied a retrial queueing model in steady-state.

With the aim of including $PH-$retrial times while not significantly increasing the complexity of the retrial queueing model, recently Chakravarthy [4] proposed a different approach introducing the concept of threshold based $PH-$retrial times. This threshold based approach enabled to study the model without the worry of exploding state space. It was shown in [4] how that threshold model can be used to approximate the classical retrial queueing model with phase type retrials.-type policy for the customers waiting in the retrial orbit. It should be pointed out that the threshold-type policy adopted in [10] is only at the points of arrivals (either new or from retrial orbit), whereas the one adopted in [4] is based on those waiting in the retrial orbit. To the best of our knowledge there is no literature that employs a threshold-type policy for the waiting customers in the retrial orbit until the recent publication of [4].

The queueing model considered in the present paper generalizes the threshold-type retrial queueing model investigated in [4] to the case of unreliable servers. Note that when the failure rate goes to zero or the repair rate goes to infinity we get the model in [4]. Upon a server failure, the interrupted customer is allowed to handoff, that is, to access another available server so as to complete its ongoing service. Moreover, taking into consideration their impatience behavior, the interrupted customers either may choose to enter into the retrial orbit or abandon forever according to a Bernoulli trial.

The rest of the paper is organized as follows. In Section 2, we describe the threshold-type unreliable retrial queueing model considered here in full detail. Section 3 deals with the steady-state analysis of the proposed model. The stability condition and the steady-state probability vector are obtained using the matrix-analytic method. In Section 4, some important performance measures and reliability indices of this model are derived. Some illustrative numerical examples are presented in Section 5 and a few concluding remarks are given in Section 6.

2. Model description

In this paper we study a threshold-type retrial queueing model that consists of c ($c \geq 1$) identical and parallel servers subject to random breakdowns and repairs as demonstrated in Figure 1.

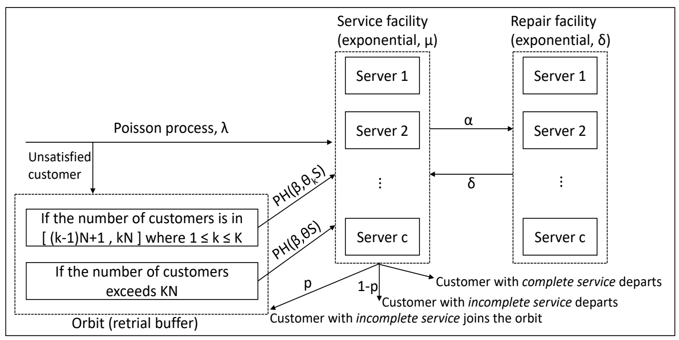


FIGURE 1. Threshold retrial unreliable queueing system.

Each server can be in down (non-operational) or up (operational) state, and it can be idle or busy (on service). The basic assumptions of the model under study are as follows.

- Customers arrive according to a Poisson process with rate λ.
- Service times of customers are independent of each other and have a common exponential distribution with parameter μ.
- An arriving customer, finding all the servers busy or down, will join a retrial orbit of infinite capacity. The customer in orbit repeats its request after random amount of time until it gets into the service facility.
- The retrial time distribution is of phase type with rate dictated by two threshold parameters, N, $1 \leq N < \infty$, and K, $1 \leq K < \infty$. That is, when the number of retrial customers in the orbit is in the interval $[(k-1)N+1, kN]$, $1 \leq k \leq K$, the retrial times are of phase type with representation $(\beta, \theta_k S)$ of order n. Once the number in the retrial orbit exceeds KN, the retrial times follow $PH-$ distribution with representation $(\beta, \theta S)$ of order n. This is adopted from [4].
- While serving a customer, the servers are subject to accidental breakdowns, independently of each other. The breakdowns are assumed to occur according to a Poisson process with failure rate of α per server. Thus, if i servers are busy the failure rate will be $i\alpha$.

- As soon as a server fails, the failed server undergoes for repair immediately. The repair time of the servers is assumed to be exponentially distributed with a repair rate of δ per server. Also, each server has its own repairman. The logic for this assumption is as follows. When the system studied is such that the facility is located in different places and when the server fails the fixing is done in that local station while the customer can be (virtually) transferred to another idle server if any. Later on, we will point out how this assumption can be relaxed.
- An interrupted customer due to the server failure will be processed by an idle server, if any; otherwise, the customer joins the orbit with probability $p, 0 \le p \le 1$, or leaves the system with probability $1-p$.
- We assume that the inter-arrival time, the service time, the retrial time, and the repair time are all mutually independent.

Suppose that we define $N(t)$ to be the number of customers in the retrial orbit; $J_1(t)$ to be the number of failed servers; $J_2(t)$ to be the number of busy servers; and $J_3(t)$ to be the phase of the retrial times (when $N(t) > 0$) at time t. It is easy to verify that the process $\{N(t), J_1(t), J_2(t), J_3(t) : t \ge 0\}$ is a Markov process possessing the QBD-structure on the state space given by $\Omega = \bigcup_{i=0}^{\infty} r(i)$, where
$r(0) = \{(0, j_1, j_2) :\ 0 \le j_1 \le c,\ 0 \le j_2 \le c - j_1\}$, and
$r(i) = \{(i, j_1, j_2, j_3) :\ 0 \le j_1 \le c,\ 0 \le j_2 \le c - j_1,\ 1 \le j_3 \le n\},\ i \ge 1.$

Let $\mathbf{0} = \{(0, j_1, j_2),\ 0 \le j_1 \le c,\ 0 \le j_2 \le c - j_1\}$ denote the set of states corresponding to the system in which the retrial buffer is empty, j_1 servers are in breakdown and j_2 servers are busy. The set of states $\boldsymbol{i} = \{(i, j_1, j_2, j_3) :\ 0 \le j_1 \le c,\ 0 \le j_2 \le c - j_1,\ 1 \le j_3 \le n\}$, for $i \ge 1$, denotes the system that i customers are in the retrial orbit with j_1 failed servers and j_2 busy servers, and the underlying phase-type distribution is in phase j_3. To make notation simpler to understand when displaying the infinitesimal generator Q below, we group the set of levels 1 through KN into K groups of N levels. The generator Q of the Markov chain governing the system has a block-tridiagonal matrix structure and is given by

$$Q = \begin{pmatrix} B_0 & A_0 & & & & & & & & \\ C_0 & B_{1,1} & A_1 & & & & & & & \\ & C_{2,1} & B_{2,1} & \ddots & & & & & & \\ & & C_{3,1} & \ddots & A_1 & & & & & \\ & & & \ddots & B_{K-1,1} & A_1 & & & & \\ & & & & C_{K,1} & B_{K,1} & A_2 & & & \\ & & & & & C_2 & B & A & & \\ & & & & & & C & B & A & \\ & & & & & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1}$$

Before we display the matrices appearing in Q, we need to set up some standard notation for use in sequel. The column vector, $\boldsymbol{e}$, consists of 1's; $\boldsymbol{e}_i$ is a unit column vector with 1 in the i^{th} position and 0 elsewhere; I is an identity matrix; and $\Delta(T_1, ..., T_r)$, a diagonal matrix with diagonal entries given by T_i, $i = 1, ..., r$. These entries can be scalars, vectors, or matrices, and will be clear from the context. The dimensions of these will be clear in the context where they are used, and when more clarity is needed we will indicate the dimension. The column vector $\boldsymbol{S^0}$ is such that $S\boldsymbol{e} + \boldsymbol{S^0} = \boldsymbol{0}$. The Kronecker product and Kronecker sum (see, e.g.,[25]), respectively, are represented with the symbols $\otimes$ and $\oplus$. Finally, we define $\tilde{d} = 0.5(c+1)(c+2)$.

The matrices A, A_0, A_1, and A_2 in the upper diagonal have dimensions $(n\tilde{d} \times n\tilde{d})$, $(\tilde{d} \times Nn\tilde{d})$, $(Nn\tilde{d} \times Nn\tilde{d})$ and $(Nn\tilde{d} \times n\tilde{d})$, respectively.

$$A = \begin{pmatrix} A_{1,0} & A_{0,0} & & & \\ & A_{1,1} & A_{0,1} & & \\ & & \ddots & \ddots & \\ & & & A_{1,c-1} & A_{0,c-1} \\ & & & & A_{1,c} \end{pmatrix}, \tag{2}$$

where

$$A_{0,j} = (c-j)\ \alpha\ p\ \boldsymbol{e}_{c+1-j}(c+1-j) \otimes [\boldsymbol{e'}_{c-j}(c-j) \otimes I_n],\ 0 \le j \le c-1,\ \text{and}$$

$$A_{1,j} = \lambda\ \boldsymbol{e}_{c+1-j}(c+1-j) \otimes [\boldsymbol{e'}_{c+1-j}(c+1-j) \otimes I_n],\ 0 \le j \le c.$$

$$A_0 = \boldsymbol{e'}_1(N) \otimes \hat{A},\ \text{where } \hat{A} = \begin{pmatrix} \hat{A}_{1,0} & \hat{A}_{0,0} & & & \\ & \hat{A}_{1,1} & \hat{A}_{0,1} & & \\ & & \ddots & \ddots & \\ & & & \hat{A}_{1,c-1} & \hat{A}_{0,c-1} \\ & & & & \hat{A}_{1,c} \end{pmatrix}, \tag{3}$$

with

$$\hat{A}_{0,j} = (c-j)\ \alpha\ p\ \boldsymbol{e}_{c+1-j}(c+1-j) \otimes [\boldsymbol{e'}_{c-j}(c-j) \otimes \beta],\ 0 \le j \le c-1,$$

$$\hat{A}_{1,j} = \lambda\ \boldsymbol{e}_{c+1-j}(c+1-j) \otimes [\boldsymbol{e'}_{c+1-j}(c+1-j) \otimes \beta],\ 0 \le j \le c.$$

$$A_1 = \boldsymbol{e}_N(N) \otimes [\boldsymbol{e'}_1(N) \otimes A]. \tag{4}$$

$$A_2 = \boldsymbol{e}_N(N) \otimes A. \tag{5}$$

The matrices C, C_0, $C_{r,1}$, and C_2 in the lower diagonal have dimensions $(n\tilde{d} \times n\tilde{d})$, $(Nn\tilde{d} \times \tilde{d})$, $(Nn\tilde{d} \times Nn\tilde{d})$ and $(n\tilde{d} \times Nn\tilde{d})$, respectively.

$$C = \Delta(C_0, C_1, \cdots, C_{c-1}, 0), \tag{6}$$

where

$$C_j = \theta \begin{pmatrix} 0 & S^0\beta & & & \\ & 0 & S^0\beta & & \\ & & \ddots & \ddots & \\ & & & 0 & S^0\beta \\ & & & & 0 \end{pmatrix}_{(c+1-j)n\times(c+1-j)n}, \ 0 \le j \le c-1.$$

$$C_0 = \boldsymbol{e}_1(N) \otimes \hat{C} \text{ where } \hat{C} = \Delta(\hat{C}_0, \hat{C}_1, \cdots, \hat{C}_{c-1}, 0), \tag{7}$$

with

$$\hat{C}_j = \theta_1 \begin{pmatrix} 0 & S^0 & & & \\ & 0 & S^0 & & \\ & & \ddots & \ddots & \\ & & & 0 & S^0 \\ & & & & 0 \end{pmatrix}_{(c+1-j)n\times(c+1-j)}, \ 0 \le j \le c-1.$$

$$C_{r,1} = \boldsymbol{e}_1(N) \otimes [\boldsymbol{e'}_N(N) \otimes \tilde{C}_r], \ 2 \le r \le K, \tag{8}$$

where

$$\tilde{C}_r = \Delta(\tilde{C}_{r,0}, \tilde{C}_{r,1}, \cdots, \tilde{C}_{r,c-1}, 0),$$

with

$$\tilde{C}_{r,j} = \theta_r \begin{pmatrix} 0 & S^0\beta & & & \\ & 0 & S^0\beta & & \\ & & \ddots & \ddots & \\ & & & 0 & S^0\beta \\ & & & & 0 \end{pmatrix}_{(c+1-j)n\times(c+1-j)n}, \ 0 \le j \le c-1.$$

$$C_2 = \boldsymbol{e'}_N(N) \otimes C. \tag{9}$$

The matrices B_0, B, and $B_{r,1}$ in the main diagonal have dimensions $(\tilde{d} \times \tilde{d})$, $(n\tilde{d} \times n\tilde{d})$ and $(Nn\tilde{d} \times Nn\tilde{d})$, respectively.

$$B_0 = \begin{pmatrix} \hat{B}_{1,0} & \hat{B}_{0,0} & & & \\ \hat{B}_{2,1} & \hat{B}_{1,1} & \hat{B}_{0,1} & & \\ & \hat{B}_{2,2} & \ddots & \ddots & \\ & & \ddots & \hat{B}_{1,c-1} & \hat{B}_{0,c-1} \\ & & & \hat{B}_{2,c} & \hat{B}_{1,c} \end{pmatrix}, \tag{10}$$

where

$$\hat{B}_{2,j} = \begin{pmatrix} j\delta & & & & \\ & j\delta & & & \\ & & \ddots & & \\ & & & j\delta & 0 \end{pmatrix}_{(c+1-j)\times(c+2-j)}, \ 1 \le j \le c,$$

$$\hat{B}_{0,j} = \begin{pmatrix} 0 & & & & \\ \alpha & & & & \\ & 2\alpha & & & \\ & & \ddots & & \\ & & & (c-1-j)\alpha & \\ & & & & (c-j)\alpha(1-p) \end{pmatrix}_{(c+1-j)\times(c-j)}, \ 0 \le j \le c-1,$$

and

$$\hat{B}_{1,j} = \begin{pmatrix} -\lambda - j\delta & \lambda & & & \\ \mu & -\lambda - j\delta - (\mu+\alpha) & \ddots & & \\ & 2\mu & \ddots & & \\ & & \ddots & & \lambda \\ & & & (c-j)\mu & -\lambda - j\delta - (c-j)(\mu+\alpha) \end{pmatrix}, 0 \le j \le c.$$

$$B = \Delta(\hat{B}_0, \hat{B}_1, \cdots, \hat{B}_{c-1}, \theta(S + S^0\beta)) + (B_0 \otimes I_n), \tag{11}$$

where

$$\hat{B}_j = \begin{pmatrix} \theta S & & & & \\ & \theta S & & & \\ & & \ddots & & \\ & & & \theta S & \\ & & & & \theta(S + S^0\beta) \end{pmatrix}_{(c+1-j)n\times(c+1-j)n}, \ 0 \le j \le c-1.$$

$$B_{r,1} = \begin{pmatrix} B_r & A & & \\ \tilde{C}_r & \ddots & \ddots & \\ & \ddots & B_r & A \\ & & \tilde{C}_r & B_r \end{pmatrix}, \ 1 \le r \le K, \tag{12}$$

where

$$B_r = \Delta(\tilde{B}_0, \tilde{B}_1, \cdots, \tilde{B}_{c-1}, \theta_r(S + S^0\beta)) + (B_0 \otimes I_n),$$

with

$$\tilde{B}_j = \begin{pmatrix} \theta_r S & & & & \\ & \theta_r S & & & \\ & & \ddots & & \\ & & & \theta_r S & \\ & & & & \theta_r (S + S^0 \beta) \end{pmatrix}_{(c+1-j)n \times (c+1-j)n}, \quad 0 \le j \le c-1.$$

A pictorial description of the transition diagram of the system is shown in Figure 2.

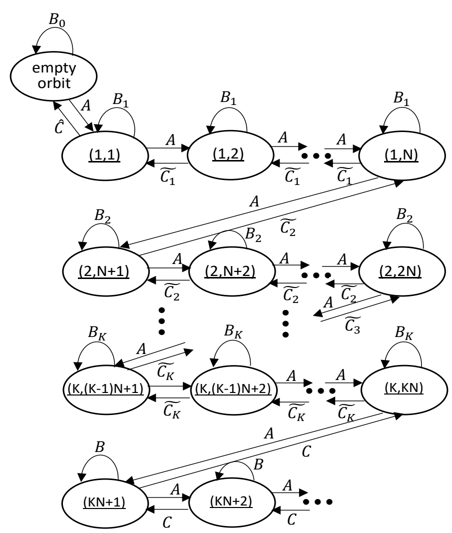


FIGURE 2. State transition diagram of the system.

3. Steady state analysis

In this section, the model described in Section 2 will be studied in steady state. We know that the stability condition of the classical retrial queueing model is same as that of the classical queueing model [2]. That is, a retrial queueing model of the type $M/M/c$ in which all the customers waiting in the retrial orbit (of infinite size) independently attempt to capture a free server is stable if and only if $\lambda < c\mu$. However, the moment one puts a restriction in the way the retrials are modeled, the stability condition needs to be modified. For example, Chakravarthy [4] established a stability condition for the threshold-type retrial queueing model with multi servers. Similarly, we establish such a stability condition for the model studied here.

3.1. Stability condition. Let $\boldsymbol{\pi}$ be the steady-state probability vector of the finite generator $F = A+B+C$. That is, $\boldsymbol{\pi} = [\pi_0^0, \pi_0^1, \cdots \pi_0^c, \pi_1^0, \pi_1^1, \cdots, \pi_1^{c-1}, \cdots, \pi_c^0]$ satisfies

$$\boldsymbol{\pi} F = \mathbf{0}, \ \boldsymbol{\pi} \boldsymbol{e} = 1. \tag{13}$$

The following theorem establishes the stability condition of the queueing system under study.

Theorem 3.1. *The threshold retrial unreliable queueing model under study with the generator given in (1) is stable if and only if*

$$\lambda < (1-p)\alpha \sum_{i=1}^{c} i\pi_{c-i}^i \boldsymbol{e} + \mu \sum_{i=1}^{c} \sum_{k=0}^{c-i} i\pi_k^i \boldsymbol{e}. \tag{14}$$

Proof. The queuing system under study with the QBD type generator given in (1) is stable (see, e.g., [19]) if and only if $\boldsymbol{\pi} A\boldsymbol{e} < \boldsymbol{\pi} C\boldsymbol{e}$. First note that the equations in (13) reduce to

$$\begin{gathered}(c-j)\alpha\pi_j^{c-j}\boldsymbol{e} + (c-j)\mu \sum_{k=0}^{j} \pi_k^{c-j}\boldsymbol{e} = \theta \sum_{k=0}^{j} \pi_k^{c-j-1}\boldsymbol{S}^0 + \lambda \sum_{k=0}^{j} \pi_k^{c-j-1}\boldsymbol{e}, \ 1 \le j \le c-1, \\ c(\alpha+\mu)\pi_0^c\boldsymbol{e} = \theta\pi_0^{c-1}\boldsymbol{S}^0 + \lambda\pi_0^{c-1}\boldsymbol{e}, \\ \sum_{k=0}^{c} \sum_{i=0}^{c-k} \pi_k^i = 1,\end{gathered} \tag{15}$$

from which it is easy to obtain that

$$\alpha \sum_{k=0}^{c-1} (c-k)\pi_k^{c-k}\boldsymbol{e} + \mu \sum_{i=0}^{c-1} \sum_{k=1}^{c-i} i\pi_k^i \boldsymbol{e} = \theta \sum_{i=0}^{c-1} \sum_{k=0}^{c-i-1} \pi_k^i \boldsymbol{S}^0 + \lambda \left[1 - \sum_{k=0}^{c} \pi_k^{c-k}\boldsymbol{e}\right]. \tag{16}$$

Further, on noting that

$$\boldsymbol{\pi} A\boldsymbol{e} = \lambda \sum_{k=0}^{c} \pi_k^{c-k}\boldsymbol{e} + \alpha p \sum_{k=0}^{c-1} (c-k)\pi_k^{c-k}\boldsymbol{e}, \ \boldsymbol{\pi} C\boldsymbol{e} = \theta \sum_{i=0}^{c-1} \sum_{k=0}^{c-i-1} \pi_k^i \boldsymbol{S}^0, \tag{17}$$

the proof follows immediately by using (16) and (17) for $\boldsymbol{\pi} A\boldsymbol{e} < \boldsymbol{\pi} C\boldsymbol{e}$. □

Let $\rho = \frac{\boldsymbol{\pi} A\boldsymbol{e}}{\boldsymbol{\pi} C\boldsymbol{e}}$ denote the traffic intensity for our threshold model here.

3.2. Steady state probability vector. Let $\boldsymbol{x} = (\boldsymbol{x}(0), \boldsymbol{x}(1), \boldsymbol{x}(2), \cdots)$ denote the steady-state probability vector of the generator Q in (1). That is, $\boldsymbol{x}$ satisfies

$$\boldsymbol{x} Q = \mathbf{0}, \ \boldsymbol{x}\boldsymbol{e} = 1. \tag{18}$$

We partition the vectors $\boldsymbol{x}(i)$, for $i \ge 0$, as $\boldsymbol{x}(i) = [\boldsymbol{x}_0(i), \boldsymbol{x}_1(i), \cdots, \boldsymbol{x}_c(i)]$ where, $\boldsymbol{x}_j(0) = [x_j^0(0), x_j^1(0), \cdots, x_j^{c-j}(0)]$ and $\boldsymbol{x}_j(i) = [\boldsymbol{x}_j^0(i), \boldsymbol{x}_j^1(i), \cdots, \boldsymbol{x}_j^{c-j}(i)], \ i \ge 1$

and $0 \leq j \leq c$. First, note that $\boldsymbol{x}_j(0)$ is of dimension $c+1-j$, while $\boldsymbol{x}_j(i),\ i \geq 1$ is of dimension $(c+1-j)n$. Secondly, the k^{th} component of the vector $\boldsymbol{x}_j(0)$ i.e., $x_j^k(0)$ gives the steady state probability vector that the retrial orbit is empty with exactly j servers failed and k servers busy serving the customers; the r^{th} component of $\boldsymbol{x}_j^k(i)$ gives the steady state probability vector that the retrial orbit has i customers with exactly j servers failed, k servers busy serving the customers, and that the underlying $PH-$distribution is in phase r, $1 \leq r \leq n$.

Under the stability condition given in (14), the steady state vector, $\boldsymbol{x}$, of the threshold retrial queueing model with unreliable servers under study with the generator given in (1) is obtained by solving the following system of linear equations:

$$\begin{gathered}
\boldsymbol{x}(0)B_0 + \boldsymbol{x}(1)\hat{C} = \mathbf{0}, \\
\boldsymbol{x}(0)\hat{A} + \boldsymbol{x}(1)B_1 + \boldsymbol{x}(2)\tilde{C_1} = \mathbf{0}, \\
\boldsymbol{x}(i-1)A + \boldsymbol{x}(i)B_1 + \boldsymbol{x}(i+1)\tilde{C_1} = \mathbf{0},\ 2 \leq i \leq N-1, \\
\boldsymbol{x}(N-1)A + \boldsymbol{x}(N)B_1 + \boldsymbol{x}(N+1)\tilde{C_2} = \mathbf{0}, \\
\boldsymbol{x}(i-1)A + \boldsymbol{x}(i)B_{j+1} + \boldsymbol{x}(i+1)\tilde{C}_{j+1} = \mathbf{0}, \\
jN+1 \leq i \leq (j+1)N-1,\ 1 \leq j \leq K-1, \\
\boldsymbol{x}(i-1)A + \boldsymbol{x}(i)B_j + \boldsymbol{x}(i+1)\tilde{C}_{j+1} = \mathbf{0},\ i = jN,\ 2 \leq j \leq K-1, \\
\boldsymbol{x}(KN-1)A + \boldsymbol{x}(KN)\big[B_K + RC\big] = \mathbf{0}, \\
\textstyle\sum_{i=0}^{KN-1} \boldsymbol{x}(i)\boldsymbol{e} + \boldsymbol{x}(KN)\big(I-R\big)^{-1}\boldsymbol{e} = 1,
\end{gathered} \tag{19}$$

where the (rate) matrix R is the minimal nonnegative solution to the matrix-quadratic equation:

$$R^2C + RB + A = 0. \tag{20}$$

The QBD-structure of the generator given in (1), under the stability condition, yields a modified matrix-geometric solution. Thus, the non-boundary states, namely, for $i \geq KN$ are given by

$$\boldsymbol{x}(i+KN) = \boldsymbol{x}(KN)R^i,\ i \geq 0. \tag{21}$$

where the matrix R satisfies the matrix-quadratic equation given in (20). One can use logarithmic reduction methods to compute R, especially when the dimension of R is of reasonable size.

The following lemma, which generalizes the one in [4] for the current model, will serve as an accuracy check in the numerical implementation of the steady-state probability vector.

Lemma 3.2. *We have*

$$\Big[\sum_{j=1}^{K}\theta_j \sum_{i=(j-1)N+1}^{jN} \boldsymbol{x}(i) + \theta\boldsymbol{x}(KN)R(I-R)^{-1}\Big](\boldsymbol{e}\otimes I) = d\boldsymbol{\beta}(-S)^{-1}, \tag{22}$$

where d is the normalizing constant and is given by

$$d = \Big[\sum_{j=1}^{K}\theta_j \sum_{i=(j-1)N+1}^{jN} \boldsymbol{x}(i)\boldsymbol{e} + \theta\boldsymbol{x}(KN)R(I-R)^{-1}\boldsymbol{e}\Big]. \tag{23}$$

Proof. First, we rewrite the equations for the steady-state probability vector, $\boldsymbol{x}$ in (19). Replacing the last equation by $\boldsymbol{x}(KN-1)A + \boldsymbol{x}(KN)B_K + \boldsymbol{x}(KN+1)C = \mathbf{0}$ and $\boldsymbol{x}(i-1)A + \boldsymbol{x}(i)B + \boldsymbol{x}(i+1)C = \mathbf{0},\ i \ge KN+1$, the results are given by

$$\begin{aligned}
&\boldsymbol{x}_0(0)\hat{B}_{1,0} + \boldsymbol{x}_1(0)\hat{B}_{2,1} + \boldsymbol{x}_0(1)\hat{C}_0 = \mathbf{0},\\
&\boldsymbol{x}_{j-1}(0)\hat{B}_{0,j-1} + \boldsymbol{x}_j(0)\hat{B}_{1,j} + \boldsymbol{x}_{j+1}(0)\hat{B}_{2,j+1} + \boldsymbol{x}_j(1)\hat{C}_j = \mathbf{0},\ 1 \le j \le c-1,\\
&\boldsymbol{x}_{c-1}(0)\hat{B}_{0,c-1} + \boldsymbol{x}_c(0)\hat{B}_{1,c} = \mathbf{0},\\
&\boldsymbol{x}_0(0)\hat{A}_{1,0} + \boldsymbol{x}_0(1)\big[\tilde{B}_0 + (\hat{B}_{1,0}\otimes I_n)\big] + \boldsymbol{x}_1(1)\big(\hat{B}_{2,1}\otimes I_n\big) + \boldsymbol{x}_0(2)\tilde{C}_{1,0} = \mathbf{0},\\
&\boldsymbol{x}_{j-1}(0)\hat{A}_{0,j-1} + \boldsymbol{x}_j(0)\hat{A}_{1,j} + \boldsymbol{x}_{j-1}(1)\big(\hat{B}_{0,j-1}\otimes I_n\big) + \boldsymbol{x}_j(1)\big[\tilde{B}_j + (\hat{B}_{1,j}\otimes I_n)\big]\\
&\quad + \boldsymbol{x}_{j+1}(1)\big(\hat{B}_{2,j+1}\otimes I_n\big) + \boldsymbol{x}_j(2)\tilde{C}_{1,j} = \mathbf{0},\ 1 \le j \le c-1,\\
&\boldsymbol{x}_{c-1}(0)\hat{A}_{0,c-1} + \boldsymbol{x}_c(0)\hat{A}_{1,c} + \boldsymbol{x}_{c-1}(1)\big(\hat{B}_{0,c-1}\otimes I_n\big) + \boldsymbol{x}_c(1)\big[\tilde{B}_c + (\hat{B}_{1,c}\otimes I_n)\big] = \mathbf{0}.
\end{aligned} \tag{24}$$

For $2 \le i \le N-1$,

$$\begin{aligned}
&\boldsymbol{x}_0(i-1)A_{1,0} + \boldsymbol{x}_0(i)\big[\tilde{B}_0 + (\hat{B}_{1,0}\otimes I_n)\big] + \boldsymbol{x}_1(i)\big(\hat{B}_{2,1}\otimes I_n\big) + \boldsymbol{x}_0(i+1)\tilde{C}_{1,0} = \mathbf{0},\\
&\boldsymbol{x}_{j-1}(i-1)A_{0,j-1} + \boldsymbol{x}_j(i-1)A_{1,j} + \boldsymbol{x}_{j-1}(i)\big(\hat{B}_{0,j-1}\otimes I_n\big) + \boldsymbol{x}_j(i)\big[\tilde{B}_j + (\hat{B}_{1,j}\otimes I_n)\big]\\
&\quad + \boldsymbol{x}_{j+1}(i)\big(\hat{B}_{2,j+1}\otimes I_n\big) + \boldsymbol{x}_j(i+1)\tilde{C}_{1,j} = \mathbf{0},\ 1 \le j \le c-1,\\
&\boldsymbol{x}_{c-1}(i-1)A_{0,c-1} + \boldsymbol{x}_c(i-1)A_{1,c} + \boldsymbol{x}_{c-1}(i)\big(\hat{B}_{0,c-1}\otimes I_n\big)\\
&+ \boldsymbol{x}_c(i)\big[\tilde{B}_c + (\hat{B}_{1,c}\otimes I_n)\big] = \mathbf{0}.
\end{aligned} \tag{25}$$

For $i = N$,

$$\begin{aligned}
&\boldsymbol{x}_0(N-1)A_{1,0} + \boldsymbol{x}_0(N)\big[\tilde{B}_0 + (\hat{B}_{1,0}\otimes I_n)\big] + \boldsymbol{x}_1(N)\big(\hat{B}_{2,1}\otimes I_n\big) + \boldsymbol{x}_0(N+1)\tilde{C}_{2,0} = \mathbf{0},\\
&\boldsymbol{x}_{j-1}(N-1)A_{0,j-1} + \boldsymbol{x}_j(N-1)A_{1,j} + \boldsymbol{x}_{j-1}(N)\big(\hat{B}_{0,j-1}\otimes I_n\big)\\
&+ \boldsymbol{x}_j(N)\big[\tilde{B}_j + (\hat{B}_{1,j}\otimes I_n)\big] + \boldsymbol{x}_{j+1}(N)\big(\hat{B}_{2,j+1}\otimes I_n\big) + \boldsymbol{x}_j(N+1)\tilde{C}_{2,j} = \mathbf{0},\ 1 \le j \le c-1,\\
&\boldsymbol{x}_{c-1}(N-1)A_{0,c-1} + \boldsymbol{x}_c(N-1)A_{1,c} + \boldsymbol{x}_{c-1}(N)\big(\hat{B}_{0,c-1}\otimes I_n\big)\\
&+ \boldsymbol{x}_c(N)\big[\tilde{B}_c + (\hat{B}_{1,c}\otimes I_n)\big] = \mathbf{0}.
\end{aligned} \tag{26}$$

For $rN+1 \le i \le (r+1)N-1,\ 1 \le r \le K-1$,

$$\begin{aligned}
&\boldsymbol{x}_0(i-1)A_{1,0} + \boldsymbol{x}_0(i)\big[\tilde{B}_0 + (\hat{B}_{1,0}\otimes I_n)\big] + \boldsymbol{x}_1(i)\big(\hat{B}_{2,1}\otimes I_n\big) + \boldsymbol{x}_0(i+1)\tilde{C}_{r+1,0} = \mathbf{0},\\
&\boldsymbol{x}_{j-1}(i-1)A_{0,j-1} + \boldsymbol{x}_j(i-1)A_{1,j} + \boldsymbol{x}_{j-1}(i)\big(\hat{B}_{0,j-1}\otimes I_n\big) + \boldsymbol{x}_j(i)\big[\tilde{B}_j + (\hat{B}_{1,j}\otimes I_n)\big]\\
&\quad + \boldsymbol{x}_{j+1}(i)\big(\hat{B}_{2,j+1}\otimes I_n\big) + \boldsymbol{x}_j(i+1)\tilde{C}_{r+1,j} = \mathbf{0},\ 1 \le j \le c-1,\\
&\boldsymbol{x}_{c-1}(i-1)A_{0,c-1} + \boldsymbol{x}_c(i-1)A_{1,c} + \boldsymbol{x}_{c-1}(i)\big(\hat{B}_{0,c-1}\otimes I_n\big)\\
&+ \boldsymbol{x}_c(i)\big[\tilde{B}_c + (\hat{B}_{1,c}\otimes I_n)\big] = \mathbf{0}.
\end{aligned} \tag{27}$$

For $i = rN,\ 2 \le r \le K-1$,

$$\begin{array}{l}
\boldsymbol{x}_0(i-1)A_{1,0} + \boldsymbol{x}_0(i)\big[\tilde{B}_0 + (\hat{B}_{1,0} \otimes I_n)\big] + \boldsymbol{x}_1(i)\big(\hat{B}_{2,1} \otimes I_n\big) + \boldsymbol{x}_0(i+1)\tilde{C}_{r+1,0} = \mathbf{0},\\
\boldsymbol{x}_{j-1}(i-1)A_{0,j-1} + \boldsymbol{x}_j(i-1)A_{1,j} + \boldsymbol{x}_{j-1}(i)\big(\hat{B}_{0,j-1} \otimes I_n\big) + \boldsymbol{x}_j(i)\big[\tilde{B}_j + (\hat{B}_{1,j} \otimes I_n)\big]\\
\ +\boldsymbol{x}_{j+1}(i)\big(\hat{B}_{2,j+1} \otimes I_n\big) + \boldsymbol{x}_j(i+1)\tilde{C}_{r+1,j} = \mathbf{0},\ 1 \le j \le c-1,\\
\boldsymbol{x}_{c-1}(i-1)A_{0,c-1} + \boldsymbol{x}_c(i-1)A_{1,c} + \boldsymbol{x}_{c-1}(i)\big(\hat{B}_{0,c-1} \otimes I_n\big)\\
+\boldsymbol{x}_c(i)\big[\tilde{B}_c + (\hat{B}_{1,c} \otimes I_n)\big] = \mathbf{0}.
\end{array} \tag{28}$$

For $i = KN$,

$$\begin{array}{l}
\boldsymbol{x}_0(KN-1)A_{1,0} + \boldsymbol{x}_0(KN)\big[\tilde{B}_0 + (\hat{B}_{1,0} \otimes I_n)\big] + \boldsymbol{x}_1(KN)\big(\hat{B}_{2,1} \otimes I_n\big)\\
+\boldsymbol{x}_0(KN+1)C_0 = \mathbf{0},\\
\boldsymbol{x}_{j-1}(KN-1)A_{0,j-1} + \boldsymbol{x}_j(KN-1)A_{1,j} + \boldsymbol{x}_{j-1}(KN)\big(\hat{B}_{0,j-1} \otimes I_n\big) + \boldsymbol{x}_j(KN)\big[\tilde{B}_j\\
\ +(\hat{B}_{1,j} \otimes I_n)\big] + \boldsymbol{x}_{j+1}(KN)\big(\hat{B}_{2,j+1} \otimes I_n\big) + \boldsymbol{x}_j(KN+1)C_j = \mathbf{0},\ 1 \le j \le c-1,\\
\boldsymbol{x}_{c-1}(KN-1)A_{0,c-1} + \boldsymbol{x}_c(KN-1)A_{1,c} + \boldsymbol{x}_{c-1}(KN)\big(\hat{B}_{0,c-1} \otimes I_n\big)\\
+\boldsymbol{x}_c(KN)\big[\tilde{B}_c + (\hat{B}_{1,c} \otimes I_n)\big] = \mathbf{0}.
\end{array} \tag{29}$$

For $i \ge KN+1$,

$$\begin{array}{l}
\boldsymbol{x}_0(i-1)A_{1,0} + \boldsymbol{x}_0(i)\big[\tilde{B}_0 + (\hat{B}_{1,0} \otimes I_n)\big] + \boldsymbol{x}_1(i)\big(\hat{B}_{2,1} \otimes I_n\big) + \boldsymbol{x}_0(i+1)C_0 = \mathbf{0},\\
\boldsymbol{x}_{j-1}(i-1)A_{0,j-1} + \boldsymbol{x}_j(i-1)A_{1,j} + \boldsymbol{x}_{j-1}(i)\big(\hat{B}_{0,j-1} \otimes I_n\big) + \boldsymbol{x}_j(i)\big[\hat{B}_j + (\hat{B}_{1,j} \otimes I_n)\big]\\
+\boldsymbol{x}_{j+1}(i)\big(\hat{B}_{2,j+1} \otimes I_n\big) + \boldsymbol{x}_j(i+1)C_j = \mathbf{0},\ 1 \le j \le c-1,\\
\boldsymbol{x}_{c-1}(i-1)A_{0,c-1} + \boldsymbol{x}_c(i-1)A_{1,c} + \boldsymbol{x}_{c-1}(i)\big(\hat{B}_{0,c-1} \otimes I_n\big)\\
++\boldsymbol{x}_c(i)\big[\hat{B}_c + (\hat{B}_{1,c} \otimes I_n)\big] = \mathbf{0}.
\end{array} \tag{30}$$

Post-multiplying the first three equations in (24) by $\boldsymbol{e}$ and adding the resulting equations, we get

$$\begin{array}{l}
-\alpha p \sum_{i=0}^{c-1}(c-i)\boldsymbol{x}_i(0)\boldsymbol{e}_{c+1-i}(c+1-i) - \lambda \sum_{i=0}^{c} \boldsymbol{x}_i(0)\boldsymbol{e}_{c+1-i}(c+1-i)\\
\ +\theta_1 \sum_{i=0}^{c-1} \boldsymbol{x}_i(1)\Big(\big(\boldsymbol{e} - \boldsymbol{e}_{c+1-i}(c+1-i)\big) \otimes \boldsymbol{S}^0\Big) = 0.
\end{array} \tag{31}$$

Post-multiplying the other equations in (24) and all equations in (25)−(30) by $(\boldsymbol{e} \otimes I)$ and adding the resulting equations, we get

$$\begin{array}{l}
\alpha p \sum_{i=0}^{c-1}(c-i)\boldsymbol{x}_i(0)\boldsymbol{e}_{c+1-i}(c+1-i) \otimes \boldsymbol{\beta} + \lambda \sum_{i=0}^{c} \boldsymbol{x}_i(0)\boldsymbol{e}_{c+1-i}(c+1-i) \otimes \boldsymbol{\beta}\\
\ -\big[\theta_1 \sum_{i=0}^{c-1} \boldsymbol{x}_i(1)\big(\big((\boldsymbol{e} \otimes S) + \boldsymbol{e}_{c+1-i}(c+1-i)\big) \otimes \boldsymbol{S}^0\boldsymbol{\beta}\big)\big](\boldsymbol{e} \otimes I)\\
\ +\big[\sum_{j=1}^{K} \theta_j \sum_{i=(j-1)N+1}^{jN} \sum_{k=0}^{c} \boldsymbol{x}_k(i)\boldsymbol{e}_{c+1-k}\\
\ +\theta \sum_{i=KN+1}^{\infty} \sum_{k=0}^{c} \boldsymbol{x}_k(i)\boldsymbol{e}_{c+1-k}\big](\boldsymbol{e} \otimes I) = \mathbf{0}.
\end{array} \tag{32}$$

Post-multiplying equation (31) by $\boldsymbol{\beta}$ and adding the equation in (32), we get

$$\begin{array}{l}
\big[\sum_{j=1}^{K} \theta_j \sum_{i=(j-1)N+1}^{jN} \sum_{k=0}^{c} \boldsymbol{x}_k(i)\boldsymbol{e}_{c+1-k}\\
\ +\theta \sum_{i=KN+1}^{\infty} \sum_{k=0}^{c} \boldsymbol{x}_k(i)\boldsymbol{e}_{c+1-k}\big](\boldsymbol{e} \otimes (S + \boldsymbol{S}^0\boldsymbol{\beta})) = \mathbf{0},
\end{array} \tag{33}$$

from which using the uniqueness of the stationary vector of the generator $(S + \boldsymbol{S}^0\boldsymbol{\beta})$ the stated result follows. □

4. System performance and reliability measures

The system measures are used to bring out the qualitative behavior of the queueing model under study, and hence we display a few for our model here.

(1) P_{idle}: This is the probability that the system is idle (i.e., all servers are idle or/and failed and the retrial orbit is empty) at an arbitrary time: $P_{idle} = \sum_{j=0}^{c} x_j^0(0)$.

(2) $P_{O-empty}$: This is the probability that the retrial orbit is empty at an arbitrary time: $P_{O-empty} = \boldsymbol{x}(0)\boldsymbol{e}$.

(3) PSM_r: This is the probability that the system is in mode r, $0 \leq r \leq K+1$. Note that the system is said to be in (i) mode 0 when the retrial orbit is empty; (ii) in mode i, $1 \leq i \leq K$, when the rate of retrials is θ_i; and (iii) in mode K+1 when the rate of retrials is given by θ.

$$PSM_r = \begin{cases} \boldsymbol{x}(0)\boldsymbol{e},\ r = 0, \\ \sum_{i=1}^{N} \boldsymbol{x}\big((r-1)N+i\big)\boldsymbol{e},\ 1 \leq r \leq K, \\ \boldsymbol{x}(KN)R(I-R)^{-1}\boldsymbol{e},\ r = K+1 \end{cases}$$

(4) PMF_b: The probability mass function of the number of busy servers is given by $\{\tilde{y_i},\ 0 \leq i \leq c\}$, where

$$\tilde{y_i} = \sum_{k=0}^{c-i} x_k^i(0) + \sum_{j=1}^{\infty}\sum_{k=0}^{c-i} \boldsymbol{x}_k^i(j)\boldsymbol{e},\ 0 \leq i \leq c-1,$$
$$\tilde{y_c} = x_0^c(0) + \sum_{j=1}^{\infty} \boldsymbol{x}_0^c(j)\boldsymbol{e}.$$

From this probability mass function, we can obtain the mean, μ_b, number of busy servers and the standard deviation, σ_b, of the number of busy servers.

(5) PMF_f: The probability mass function of the number of failed servers is given by $\{\tilde{z_i},\ 0 \leq i \leq c\}$, where

$$\tilde{z_i} = \sum_{k=0}^{c-i} x_i^k(0) + \sum_{j=1}^{\infty}\sum_{k=0}^{c-i} \boldsymbol{x}_i^k(j)\boldsymbol{e},\ 0 \leq i \leq c-1,$$
$$\tilde{z_c} = x_c^0(0) + \sum_{j=1}^{\infty} \boldsymbol{x}_c^0(j)\boldsymbol{e}.$$

Again, from this probability mass function, we can obtain the mean, μ_f, number of failed servers and the standard deviation, σ_f, of the number of failed servers.

(6) P_{block}: This is the probability of blocking when an arriving customer finds all servers busy or/and failed and is calculated as

$$P_{block} = \sum_{k=0}^{c} x_k^{c-k}(0) + \sum_{j=1}^{\infty}\sum_{k=0}^{c} \boldsymbol{x}_k^{c-k}(j)\boldsymbol{e}.$$

(7) P_{ft}: This refers to the probability of termination of an ongoing service due to a server breakdown, when no other server is available and the

interrupted service leaves the system with probability $(1-p)$. It is calculated as

$$P_{ft} = \alpha(1-p)\left[\sum_{i=1}^{c} i\ x_{c-i}^{i}(0) + \sum_{j=1}^{\infty}\sum_{i=1}^{c} i\ \boldsymbol{x}_{c-i}^{i}(j)\boldsymbol{e}\right].$$

(8) $\xi_r,\ 1 \le r \le K+1$: This refers to the rate of a successful capture of an idle server by an orbiting customer when the system is in mode r, and is calculated as

$$\xi_r = \begin{cases} \theta_r \sum_{j=1}^{N}\sum_{k=0}^{c-1}\sum_{i=0}^{c-k-1} \boldsymbol{x}_k^i((r-1)N+j)\boldsymbol{S}^0,\ 1 \le r \le K, \\ \theta \sum_{j=1}^{\infty}\sum_{k=0}^{c-1}\sum_{i=0}^{c-k-1} \boldsymbol{x}_k^i(KN+j)\boldsymbol{S}^0,\ r = K+1. \end{cases}$$

(9) μ_{RO}: Mean number of customers in retrial orbit

$$\mu_{RO} = \sum_{i=1}^{KN-1} i\ \boldsymbol{x}(i)\boldsymbol{e} + \boldsymbol{x}(KN)\big[KN\ (I-R)^{-1} + R(I-R)^{-2}\big]\boldsymbol{e}.$$

5. Illustrative numerical examples

In this section, to understand and to bring out the qualitative aspects of the threshold queueing model under study, we look at a few illustrative numerical examples. We analyze different scenarios by varying the parameters of the model including the retrial time distributions. We consider four special types of phase-type distributions for the retrial times.

ERLANG (E): This is an Erlang distribution of order 5.
EXPON (X): This is an exponential distribution.
PH-MIX (M): This is a mixture of Erlang of order 2 (with probability 0.5) and a hyperexponential with mixing probabilities $(0.45, 0.05)$ with rates $(1.9, 0.19)$.
HYPEXP (H): This is a hyperexponential distribution with mixing probabilities $(0.6, 0.2, 0.15, 0.05)$ with rates $(100, 10, 1, 0.1)$.

While we will normalize these four distributions so as to keep the means to be the same, they have a different variance structure. The ratios of the standard deviations for E, X, M and H are 0.44721, 1, 1.66415 and 4.64204, respectively.

Since the main focus of this paper is the introduction of the failures and repairs of the server, we will discuss the impact of the failure rates on the key measures. Towards this end, we compare the model presented here against the one in [4]. Hence, we look at the ratio of the measures for $\alpha > 0$ over the corresponding one for $\alpha = 0$.

Example 1: In this example, we discuss the behaviour of the measures, P_{block} and μ_{RO}. Towards this end, we fix $K = 40$, $N = 5$, $\lambda = 1$, $\delta = 1$, $\theta_k = 0.1k,\ 1 \le k \le 40$, $\theta = 4.1$, and vary c, ρ, α and p. In order to compare the measures properly (as the stability condition depends on various parameters including the type of retrial distribution), we first identified that value of μ for which the traffic intensity is close (up to three decimal places) to the given value. That is, given a value for ρ, we find the minimum value for μ which $\rho \approx \frac{\boldsymbol{\pi}_A \boldsymbol{e}}{\boldsymbol{\pi}_C \boldsymbol{e}}$. By considering the four distributions listed above for the retrials, we display the ratio of the measure P_{block} for $\alpha > 0$ over P_{block} for $\alpha = 0$ in Figure 3 and a similar ratio for the μ_{RO} in Figure 4. It should be pointed out the each cluster (that is, the set of 11 points in the form of dots) corresponds to varying α as $\alpha = 0, 0.01, 0.02, ..., 0.09, 0.1$. To avoid cluttering the x-axis with too many variables, we did not display α but it should be clear from the "dots" that α values are used in the graphs. Also, we did not plot the graphs for exponential retrials as the graphs are similar to Erlang ones, and also to not clutter with too many graphs in one Figure. From these figures, we notice the following observations.

- With regard to the ratio on P_{block}:
  (1) As is to be expected, the ratio shows a non-increasing pattern (as α is increased) when $p > 0$ under all scenarios. However, the rate of decrease depends on the values of c and the type of retrial distributions.
  (2) In the multi-server case, we see an interesting trend. For $p = 0$, this ratio appears to increase as α increases. However, for the other two values of p considered here, the ratio decreases (but with a higher rate for $p = 1$ as compared to $p = 0.5$) as α is increased. This may look counter intuitive but this phenomenon can be explained as follows. A high failure rate will result in removing customers from the system and thus results in less blocking for future customers.
  (3) The effect of p is seen when moving from $\rho = 0.5$ to $\rho = 0.95$, especially, for retrial distribution having a large variability. It is more pronounced when c is increased.
- With regard to the ratio on μ_{RO}:
  (1) For all practical purposes, we do not see a significant difference in this ratio for the single server cases under all scenarios. While we do see some changes in this ratio as a function of α for some scenarios when $c = 2$ and $p = 0$, it is only when $c > 2$, we tend to see significant effect of α as well as p. This could be due to a variety of reasons including the change in the service rate (due to the requirement that ρ is fixed across all scenarios).
  (2) We also notice that when the traffic intensity is high, this ratio appears to be the same as the model considered in [4] when $p = 1$. This indicates that letting customers leave the system whenever the servers are interrupted is somehow off-set by the failure of the server to account for a similar mean number of customers in the orbit.
  (3) The impact of high variability in the retrial distribution is clearly seen in the ratio in the case of multi-server systems. Specifically, we notice that the ratio decreases as the variability in the retrial times increases.

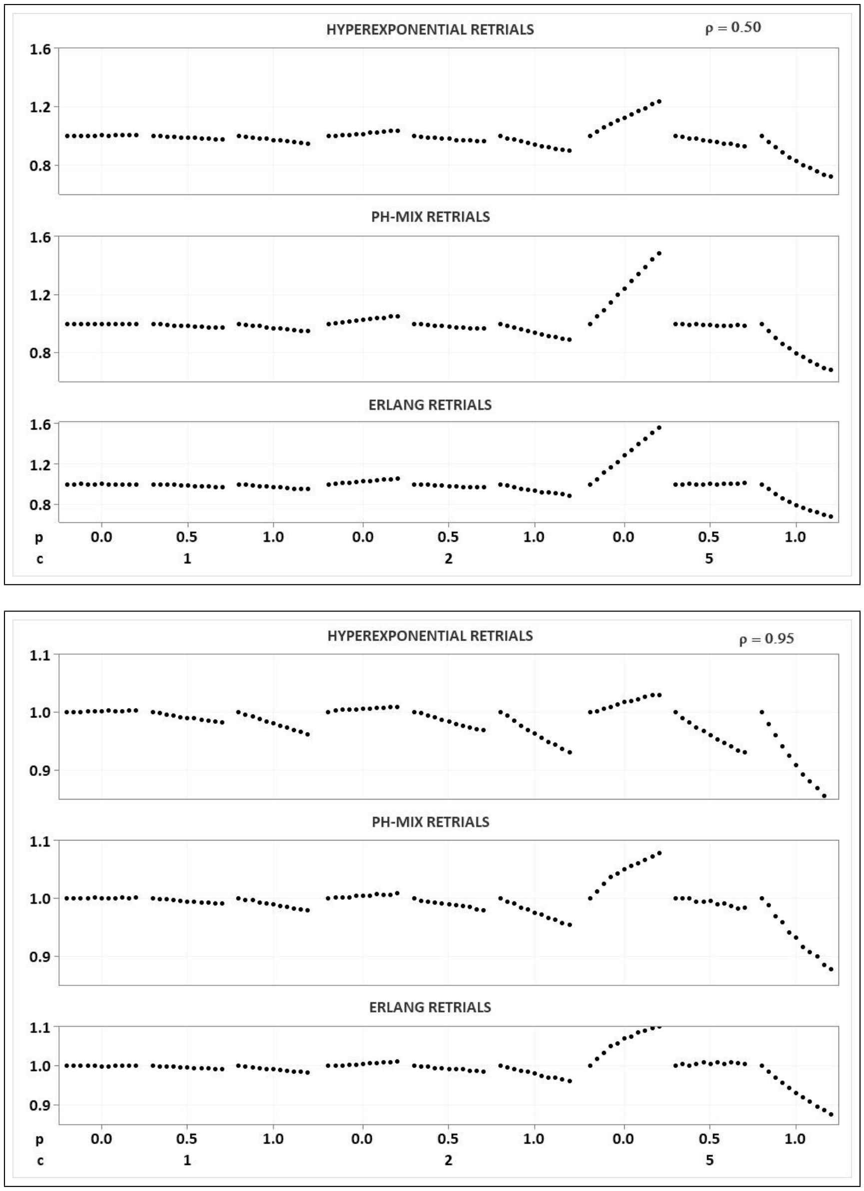
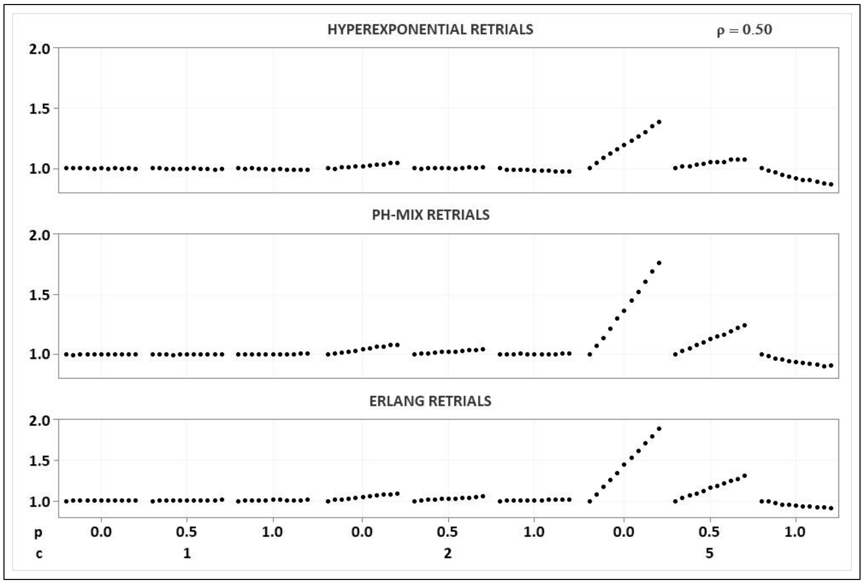


FIGURE 3. Ratios of P_{block} under various scenarios.



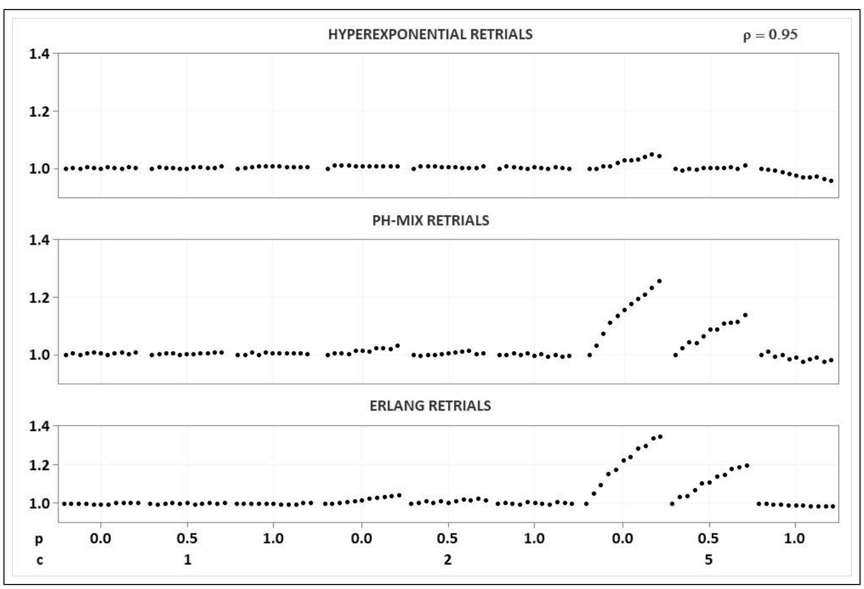


FIGURE 4. Ratios of μ_{RO} under various scenarios.

Example 2: The purpose of this example is to investigate the effect of α, ρ, p, c and the type of retrial distribution on the minimum value of θ needed so that the traffic intensity will be equal to the given value of ρ. Note that ρ does not depend on $\theta_k,\ 1 \le k \le K$. We look at the ratio of $ln(\theta)$ for $\alpha = 0.01, 0.05$ over $ln(\theta)$ for $\alpha = 0$. We fix $K = 70$, $\lambda = 1$, $\delta = 1$, $\mu = \frac{1}{c\rho}$, and vary c, ρ, α and p. That is, we are comparing the behavior of the retrial rates for our current model as compared to the corresponding model without failures.

A quick look at Table 1 reveals that for all the four retrial times and for $p = 0, 0.5, 1$, the retrial rate appears to be insensitive to the value of α, for low traffic intensity (see, e.g., $\rho = 0.5$) when $c = 1$. On the other hand, for higher values of the traffic intensity (see, e.g., $\rho = 0.95$), the retrial rate shows more sensitivity to α as well as on c. The level of sensitivity depends on the value of c and the type of retrial distribution. This is not surprising since for lower traffic intensity values, the system is idle for longer periods of time, and so there is no reason to require a higher retrial rate to capture a free server.

TABLE 1. The ratio, $Ln(\theta)_{\alpha>0}/Ln(\theta)_{\alpha=0}$, under various scenarios*

			$Ln(\theta)_{\alpha=0.01}/Ln(\theta)_{\alpha=0}$				$Ln(\theta)_{\alpha=0.05}/Ln(\theta)_{\alpha=0}$			
p	ρ	c	E	X	M	H	E	X	M	H
0	0.5	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		2	1.03	1.03	1.02	1.02	1.21	1.17	1.10	1.09
		5	-0.24	-0.20	-0.18	0.04	-0.22	-0.18	-0.16	0.05
	0.95	1	1.10	1.09	1.09	1.07	0.69	0.71	0.73	0.78
		2	0.61	0.64	0.67	0.74	0.37	0.41	0.44	0.57
		5	0.28	0.33	0.36	0.52	0.05	0.09	0.11	0.32
0.5	0.5	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		5	-0.26	-0.21	-0.19	0.03	-0.22	-0.18	-0.16	0.07
	0.95	1	0.76	0.77	0.79	0.83	0.90	0.91	0.92	0.93
		2	1.48	1.44	1.40	1.32	0.56	0.59	0.62	0.70
		5	0.48	0.52	0.56	0.67	0.25	0.29	0.32	0.49
1	0.5	1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
		2	0.93	0.97	0.98	0.98	0.81	0.85	0.91	0.92
		5	-0.27	-0.23	-0.20	0.01	-0.28	-0.24	-0.22	-0.01
	0.95	1	1.42	1.39	1.36	1.29	0.75	0.77	0.79	0.83
		2	0.62	0.65	0.68	0.75	0.83	0.84	0.85	0.88
		5	0.26	0.31	0.35	0.51	0.31	0.36	0.39	0.54

*$E = ERLANG, X = EXPON, M = PH - MIX,$ and $H = HYPEXP$

Example 3: In this example, we look at the two measures, ξ_r, $1 \le r \le K+1$ and PSM_r, $0 \le r \le K+1$. We identify the values of K, N, and r for which the customer has the largest probability of capturing a free server (ξ^*) and for which the probability that the system operates in mode r is highest among $K+1$ modes (PSM^*). That is, we look for $(K_2^*, r_2^*, N_2^*, \xi^*)$ and $(K_1^*, r_1^*, N_1^*, PSM^*)$. We fix $\lambda = 1$, $\delta = 1$, $\theta_1 = 0.5, \theta = 5.0$, and obtain μ so that ρ equals to the specific value considered. By looking at $N = 1, 2, 5$ and varying K from 1 to 40, the optimal values are displayed in Tables 2 and 3, respectively, under various scenarios.

TABLE 2. $(K_2^*, r_2^*, N_2^*, \xi^*)$ under various scenarios

α	ρ	p	ToR	$c = 1$	$c = 2$	$c = 5$
0.01	0.5	0	ERLANG	(5, 2, 5, 0.211)	(4, 1, 5, 0.212)	(3, 1, 5, 0.109)
			PH-MIX	(6, 2, 5, 0.172)	(5, 1, 5, 0.193)	(4, 1, 5, 0.112)
			HYPEXP	(8, 2, 5, 0.075)	(7, 1, 5, 0.092)	(6, 1, 5, 0.088)
		0.5	ERLANG	(5, 2, 5, 0.211)	(4, 1, 5, 0.213)	(3, 1, 5, 0.106)
			PH-MIX	(6, 2, 5, 0.172)	(5, 1, 5, 0.193)	(4, 1, 5, 0.109)
			HYPEXP	(8, 2, 5, 0.075)	(7, 1, 5, 0.093)	(6, 1, 5, 0.088)
		1	ERLANG	(5, 1, 5, 0.549)	(4, 1, 5, 0.564)	(3, 1, 5, 0.103)
			PH-MIX	(6, 2, 5, 0.172)	(5, 1, 5, 0.528)	(4, 1, 5, 0.107)
			HYPEXP	(8, 2, 5, 0.075)	(7, 1, 5, 0.093)	(6, 1, 5, 0.088)
	0.95	0	ERLANG	(27, 7, 5, 0.115)	(32, 6, 5, 0.099)	(35, 4, 5, 0.089)
			PH-MIX	(25, 8, 5, 0.106)	(28, 7, 5, 0.090)	(33, 5, 5, 0.078)
			HYPEXP	(20, 9, 5, 0.073)	(22, 8, 5, 0.063)	(27, 8, 5, 0.052)
		0.5	ERLANG	(27, 7, 5, 0.115)	(32, 6, 5, 0.101)	(40, 4, 5, 0.088)
			PH-MIX	(25, 8, 5, 0.106)	(28, 7, 5, 0.091)	(34, 5, 5, 0.078)
			HYPEXP	(20, 9, 5, 0.073)	(23, 8, 5, 0.063)	(27, 7, 5, 0.052)
		1	ERLANG	(27, 7, 5, 0.116)	(32, 6, 5, 0.101)	(40, 4, 5, 0.091)
			PH-MIX	(25, 8, 5, 0.107)	(28, 7, 5, 0.091)	(34, 5, 5, 0.080)
			HYPEXP	(20, 9, 5, 0.073)	(22, 8, 5, 0.064)	(27, 7, 5, 0.053)
0.05	0.5	0	ERLANG	(5, 2, 5, 0.211)	(4, 1, 5, 0.213)	(4, 1, 5, 0.130)
			PH-MIX	(6, 2, 5, 0.171)	(5, 1, 5, 0.194)	(4, 1, 5, 0.129)
			HYPEXP	(8, 2, 5, 0.075)	(7, 1, 5, 0.092)	(6, 1, 5, 0.093)
		0.5	ERLANG	(5, 2, 5, 0.211)	(4, 1, 5, 0.215)	(3, 1, 5, 0.115)
			PH-MIX	(6, 2, 5, 0.172)	(5, 1, 5, 0.195)	(4, 1, 5, 0.118)
			HYPEXP	(8, 2, 5, 0.075)	(7, 1, 5, 0.094)	(6, 1, 5, 0.093)
		1	ERLANG	(5, 2, 5, 0.213)	(4, 1, 5, 0.216)	(3, 1, 5, 0.102)
			PH-MIX	(6, 2, 5, 0.173)	(5, 1, 5, 0.197)	(3, 1, 5, 0.107)
			HYPEXP	(8, 2, 5, 0.075)	(7, 1, 5, 0.096)	(5, 1, 5, 0.092)
	0.95	0	ERLANG	(27, 7, 5, 0.115)	(31, 6, 5, 0.101)	(18, 3, 5, 0.138)
			PH-MIX	(25, 8, 5, 0.106)	(28, 7, 5, 0.092)	(22, 4, 5, 0.103)
			HYPEXP	(20, 9, 5, 0.072)	(22, 8, 5, 0.064)	(26, 8, 5, 0.056)
		0.5	ERLANG	(27, 7, 5, 0.117)	(31, 6, 5, 0.105)	(33, 5, 5, 0.103)
			PH-MIX	(25, 8, 5, 0.107)	(28, 7, 5, 0.095)	(30, 5, 5, 0.090)
			HYPEXP	(20, 9, 5, 0.073)	(22, 8, 5, 0.065)	(26, 8, 5, 0.058)
		1	ERLANG	(28, 7, 5, 0.119)	(31, 6, 5, 0.109)	(33, 4, 5, 0.110)
			PH-MIX	(25, 8, 5, 0.109)	(28, 7, 5, 0.098)	(32, 5, 5, 0.093)
			HYPEXP	(20, 9, 5, 0.073)	(22, 8, 5, 0.066)	(27, 8, 5, 0.059)

A look at the tables reveal the following observations:

- With respect to the probability of capturing a free server, we notice that for all combinations except $\rho = 0.5, c = 5$, ERLANG retrials have the largest probability for a customer to capture a server.
- As p is increased (for fixed ToR), except for $c = 5$ the rate of increase in the values of ξ^* appears to be less significant under all scenarios. However, when $c = 5$ there is a significant difference in the ξ^* values for all (but for HYPEXP retrials), showing that when $\rho = 0.95$, an increase in p leads to a significant increase in ξ^* whereas we see a marked drop when $\rho = 0.5$.
- As α increases (by fixing all other parameters) we observe a steep increase in ξ^* for all the four retrial times when $c = 5$.
- The measure, PSM, interestingly indicates that when $\rho = 0.5$, for HYPEXP retrials the system operates most of the times in the mode when

there is no one in the retrial orbit, under all the three values of c considered. However, for the other three retrial times, we notice such behavior notably for $c = 5$.

TABLE 3. $(K_1^*, r_1^*, N_1^*, PSM^*)$ under various scenarios

α	ρ	p	ToR	$c = 1$	$c = 2$	$c = 5$
0.01	0.5	0	ERLANG	(5, 1, 5, 0.550)	(4, 1, 5, 0.566)	(15, 0, 1, 0.778)
			PH-MIX	(6, 1, 5, 0.492)	(5, 1, 5, 0.530)	(16, 0, 1, 0.750)
			HYPEXP	(19, 0, 1, 0.325)	(19, 0, 1, 0.424)	(20, 0, 1, 0.601)
		0.5	ERLANG	(5, 1, 5, 0.550)	(4, 1, 5, 0.565)	(15, 0, 1, 0.787)
			PH-MIX	(6, 1, 5, 0.492)	(5, 1, 5, 0.529)	(16, 0, 1, 0.760)
			HYPEXP	(19, 0, 1, 0.325)	(19, 0, 1, 0.425)	(20, 0, 1, 0.609)
		1	ERLANG	(5, 1, 5, 0.549)	(4, 1, 5, 0.564)	(15, 0, 1, 0.795)
			PH-MIX	(6, 1, 6, 0.491)	(5, 1, 5, 0.528)	(16, 0, 1, 0.767)
			HYPEXP	(19, 0, 1, 0.325)	(19, 0, 1, 0.427)	(20, 0, 1, 0.614)
	0.95	0	ERLANG	(27, 7, 5, 0.136)	(32, 5, 5, 0.122)	(40, 3, 5, 0.123)
			PH-MIX	(24, 7, 5, 0.141)	(28, 6, 5, 0.124)	(33, 4, 5, 0.115)
			HYPEXP	(20, 9, 5, 0.159)	(23, 8, 5, 0.141)	(27, 7, 5, 0.117)
		0.5	ERLANG	(27, 7, 5, 0.136)	(31, 5, 5, 0.124)	(40, 4, 5, 0.119)
			PH-MIX	(25, 7, 5, 0.140)	(28, 6, 5, 0.123)	(34, 4, 5, 0.112)
			HYPEXP	(20, 9, 5, 0.159)	(23, 8, 5, 0.141)	(27, 7, 5, 0.117)
		1	ERLANG	(27, 7, 5, 0.136)	(32, 5, 5, 0.124)	(40, 3, 5, 0.122)
			PH-MIX	(25, 7, 5, 0.140)	(28, 6, 5, 0.124)	(34, 4, 5, 0.116)
			HYPEXP	(20, 9, 5, 0.159)	(23, 8, 5, 0.141)	(28, 7, 5, 0.117)
0.05	0.5	0	ERLANG	(5, 1, 5, 0.547)	(4, 1, 5, 0.573)	(15, 0, 1, 0.733)
			PH-MIX	(6, 1, 5, 0.489)	(5, 1, 5, 0.533)	(16, 0, 1, 0.710)
			HYPEXP	(20, 0, 1, 0.326)	(19, 0, 1, 0.419)	(20, 0, 1, 0.572)
		0.5	ERLANG	(5, 1, 5, 0.546)	(4, 1, 5, 0.568)	(14, 0, 1, 0.774)
			PH-MIX	(6, 1, 5, 0.488)	(5, 1, 5, 0.530)	(15, 0, 1, 0.749)
			HYPEXP	(20, 0, 1, 0.327)	(19, 0, 1, 0.426)	(19, 0, 1, 0.606)
		1	ERLANG	(5, 1, 5, 0.544)	(4, 1, 5, 0.563)	(13, 0, 1, 0.807)
			PH-MIX	(6, 1, 5, 0.487)	(5, 1, 5, 0.526)	(15, 0, 1, 0.782)
			HYPEXP	(21, 0, 1, 0.329)	(19, 0, 1, 0.432)	(19, 0, 1, 0.635)
	0.95	0	ERLANG	(27, 7, 5, 0.136)	(31, 6, 5, 0.124)	(17, 2, 5, 0.227)
			PH-MIX	(25, 7, 5, 0.140)	(28, 6, 5, 0.125)	(22, 3, 5, 0.165)
			HYPEXP	(20, 9, 5, 0.158)	(22, 8, 5, 0.142)	(26, 7, 5, 0.122)
		0.5	ERLANG	(27, 7, 5, 0.136)	(31, 5, 5, 0.126)	(32, 4, 5, 0.128)
			PH-MIX	(25, 7, 5, 0.139)	(28, 6, 5, 0.125)	(30, 5, 5, 0.121)
			HYPEXP	(21, 9, 5, 0.157)	(23, 8, 5, 0.141)	(26, 7, 5, 0.120)
		1	ERLANG	(27, 7, 5, 0.135)	(31, 5, 5, 0.126)	(32, 3, 5, 0.134)
			PH-MIX	(25, 7, 5, 0.139)	(28, 6, 5, 0.126)	(31, 4, 5, 0.121)
			HYPEXP	(21, 9, 5, 0.156)	(23, 8, 5, 0.141)	(27, 7, 5, 0.119)

6. Concluding remarks

In this paper, we studied a multi-server retrial queue in which servers are subject to breakdowns and repairs. The retrial attempts are modeled using a recently introduced threshold-based phase type distributions. Upon a server failure, the customer whose service gets interrupted will be handed over to another available server, if any, in order to complete its ongoing service; otherwise, the customer may choose to join the retrial orbit or depart from the system according to a Bernoulli trial. Using the matrix-analytic method, the steady state analysis of the model including a few illustrative examples are presented.

The model considered in this paper can be studied further in a number of ways. Some specific ones are as follows. First, one can generalize this to include MAP arrivals and/or phase-type services. Secondly, it would be interesting to study the present model using simulation. Thirdly, we can look at the case of having a common repair facility with one or more repairmen to tend to failed servers.

Acknowledgments The authors are grateful to the anonymous referees whose comments helped to improve the presentation of the paper.

References

1. J.R. Artalejo, A. Economou, M.J. Lopez-Herrero, *Algorithmic approximations for the busy period distribution of the $M/M/c$ retrial queue*, European Journal of Operational Research **176** (2007), 1687-1702.
2. J.R. Artalejo, A. Gómez-Corral, *Retrial Queueing Systems-A Computational Approach*, Springer-Verlag, Heidelberg, 2008.
3. S.R. Chakravarthy, *Analysis of $MAP/PH/c$ retrial queue with phase type retrials-Simulation approach*, Communications in Computer and Information Science **356** (2013), 37-49.
4. S.R. Chakravarthy, *A Retrial Queueing Model with Thresholds and Phase Type Retrial times*, Journal of Applied Mathematics & Informatics **38** (2020), 351-373.
5. F.-M. Chang, T.-H. Liu, J.-C. Ke, *On an unreliable-server retrial queue with customer feedback and impatience*, Applied Mathematical Modelling **55** (2018), 171-182.
6. G. Choudhury, J.-C. Ke, *An unreliable retrial queue with delaying repair and general retrial times under Bernoulli vacation schedule*, Applied Mathematics and Computation **230** (2014), 436-450.
7. G. Choudhury, K. Deka, *An $M/G/1$ retrial queueing system with two phases of service subject to the server breakdown and repair*, Performance Evaluation **65** (2008), 714-724.
8. J.E. Diamond, A.S. Alfa, *Approximation method for $M/PH/1$ retrial queues with phase type inter-retrial times*, European Journal of Operational Research **113** (1999), 620-631.
9. S. Dudin, O. Dudina, *Retrial multi-server queuing system with PHF service time distribution as a model of a channel with unreliable transmission of information*, Applied Mathematical Modelling **65** (2019), 676-695.
10. D. Efrosinin, L. Breuer, *Threshold policies for controlled retrial queues with heterogeneous servers*, Annals of Operations Research **141** (2006), 139-162.
11. G.I. Falin, J.G.C. Templeton, *Retrial Queues*, Chapman and Hall, London, 1997.
12. N. Gharbi, C. Dutheillet, *An algorithmic approach for analysis of finite-source retrial systems with unreliable servers*, Computers & Mathematics with Applications **62** (2011), 2535-2546.
13. A. Gomez-Corral, *Stochastic analysis of a single server retrial queue with general retrial times*, Naval Research Logistics **46** (1999), 561-581.
14. Q.-M. He, H. Li, Y.Q. Zhao, *Ergodicity of the $BMAP/PH/s/s+K$ retrial queue with PH-retrial times*, Queueing Systems **35** (2000), 323-347.
15. J.-C. Ke, T.-H. Liu, D.-Y. Yang, *Retrial queues with starting failure and service interruption*, IET Communications **12** (2018), 1431-1437.
16. C. Kim, V.I. Klimenok, D.S. Orlovsky, *The $BMAP/PH/N$ retrial queue with Markovian flow of breakdowns*, European Journal of Operational Research **189** (2008), 1057-1072.
17. A. Krishnamoorthy, P.K. Pramod, S.R. Chakravarthy, *Queues with interruptions: a survey*, Top. **22** (2014), 290-320.
18. M.F. Neuts, *A versatile Markovian point process*, J. Appl. Prob. **16** (1979), 764-779.

19. M.F. Neuts, *Matrix-Geometric Solutions in Stochastic Models-An Algorithmic Approach*, Johns Hopkins University Press, Baltimore, 1981.
20. T. Phung-Duc:Retrial queueing models: A survey on theory and applications. In: T. Dohi,et.al.(eds.): Stochastic Operations Research in Business and Industry, Singapore: World Scientific, (2017) [Online]. Available: http://infoshako.sk.tsukuba.ac.jp/ tuan/papers/ Tuan_chapter_ver3.pdf
21. L. Raiah, N. Oukid, *An $M/M/2$ Retrial Queue with Breakdowns and Repairs*, Romanian Journal of Mathematics and Computer Science **7** (2017), 11-20.
22. P. Rajadurai, M.C. Saravanarajan, V.M. Chandrasekaran, *A study on $M/G/1$ feedback retrial queue with subject to server breakdown and repair under multiple working vacation policy*, Alexandria Engineering Journal **57** (2018), 947-962.
23. Y.W. Shin, *Algorithmic solution for $M/M/c$ retrial queue with PH_2-retrial times*, Journal of Applied Mathematics & Informatics **29** (2011), 803-811.
24. Y.W. Shin, D.H. Moon, *Approximation of $M/M/c$ retrial queue with PH-retrial times*, European journal of operational research **213** (2011), 205-209.
25. W.H. Steeb, Y. Hardy, *Matrix Calculus and Kronecker Product*, World Scientific Publishing, Singapore, 2011.
26. M.G. Subramanian, G. Ayyappan, G. Sekar, *$M/M/c$ Retrial queueing system with breakdown and repair of services*, Asian Journal of Mathematics and Statistics **4** (2011), 214-223.
27. T. Yang, M.J.M Posner, J.G.C. Templeton, H. Li, *An approximation method for the $M/G/1$ retrial queue with general retrial times*, European Journal of Operational Research **76** (1994), 552-562.
28. D.-Y. Yang, F.-M. Chang, J.-C. Ke, *On an unreliable retrial queue with general repeated attempts and J optional vacations*, Applied Mathematical Modelling **40** (2016), 3275-3288.

Srinivas R. Chakravarthy received B.Sc. in Mathematics, and M.Sc. in Statistics from the University of Madras, India; and Ph.D. in Operations Research, from the University of Delaware, Newark, US. Since 1983, he has been in the Departments of Industrial and Manufacturing Engineering & Mathematics, Kettering University, Flint, Michigan, USA. His research interests include queueing, inventory, reliability, matrix-analytical methods, algorithmic probability, and simulation.

Departments of Industrial and Manufacturing Engineering & Mathematics, Kettering University, Flint, Michigan, USA.
`e-mail: schakrav@kettering.edu`

Serife Ozkar received M.Sc. and Ph.D. from Dokuz Eylul University, Turkey. She is currently a professor at Balikesir University since 2018. Her research interests include queueing, inventory, matrix-analytical methods.

Department of International Trade and Logistics, Balikesir University, Balikesir-10145, Turkey.
`e-mail: serife.ozkar@balikesir.edu.tr`

Shruti received her B.Sc. Mathematics degree with 1st class honors, in 2013 and M.Sc. degree with 1st class in Mathematics, in 2015 from University of Delhi, India. Currently, she is working towards a Ph.D. degree from Birla Institute of Technology and Science Pilani, India. Her research interests lie primarily in the area of mobile and wireless communications, including spectrum management, reliability analysis, simulation, optimization, matrix-analytical methods, and modeling and performance evaluation of communication networks, particularly cognitive radio networks.

Department of Mathematics, Birla Institute of Technology and Science Pilani, Pilani Campus, Pilani, Rajasthan-333031, India.

e-mail: shrutigoel93@gmail.com