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THE CONNECTED DOUBLE GEODETIC NUMBER OF A GRAPH

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Abstract. For a connected graph G of order n, a set S of vertices is called a double geodetic set of G if for each pair of vertices x, y in G there exist vertices $u, v \in S$ such that $x, y \in I[u, v]$. The double geodetic number $dg(G)$ is the minimum cardinality of a double geodetic set. Any double godetic set of cardinality $dg(G)$ is called a dg-set of G. A connected double geodetic set of G is a double geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected double geodetic set of G is the connected double geodetic number of G and is denoted by $dg_c(G)$. A connected double geodetic set of cardinality $dg_c(G)$ is called a dg_c-set of G. Connected graphs of order n with connected double geodetic number 2 or n are characterized. For integers n, a and b with $2 \leq a < b \leq n$, there exists a connected graph G of order n such that $dg(G) = a$ and $dg_c(G) = b$. It is shown that for positive integers r, d and $k \geq 5$ with $r < d \leq 2r$ and $k - d - 3 \geq 0$, there exists a connected graph G of radius r, diameter d and connected double geodetic number k.

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1. Introduction

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m respectively. For basic graph theoretic terminology we refer to [4]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest x-y path in G. It is known that the distance is a metric on the vertex set of G. An x-y path of length $d(x, y)$ is called an x-y *geodesic*. A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y. For any vertex u of G, the *eccentricity* of u is $e(u) = \max\{d(u, v) : v \in V\}$. A vertex v is

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an *eccentric vertex of* u if $e(u) = d(u, v)$. The radius *rad* G and diameter *diam* G are defined by *rad* $G = \min\{e(v) : v \in V\}$ and *diam* $G = \max\{e(v) : v \in V\}$ respectively. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced $N(v)$ is complete. A vertex v is a *weak extreme vertex* of G if there exists a vertex u in G such that $u, v \in I[x, y]$ for a pair of vertices x, y in G, then $v = x$ or $v = y$. Equivalently, a vertex v in a connected graph is a weak extreme vertex if there exists a vertex u in G such that v is either an initial vertex or a terminal vertex of any interval containing both u and v. Each extreme vertex of a graph is weak extreme. For the graph G in Figure 1, it is clear that the pair v_2, v_5 lies only on the $v_2 - v_5$ geodesic and so v_2 and v_5 are weak extreme vertices of G. Similarly, the vertices v_4 and v_6 are also weak extreme vertices of G. It is easily seen that v_1 and v_3 are also weak extreme vertices of G.

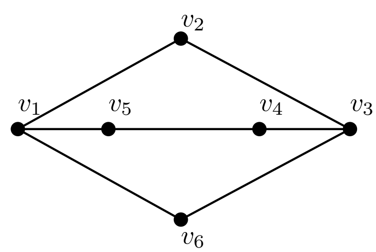

Figure 1 : G

The *closed interval* $I[x, y]$ consists of all vertices lying on some *x-y* geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* of G if $I[S] = V$, and the minimum cardinality of a geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set* of G. A *connected geodetic set* S of G is a geodetic set such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected geodetic set of G is the *connected geodetic number* of G and is denoted by $g_c(G)$. A connected geodetic set of cardinality $g_c(G)$ is called a g_c-set of G. The geodetic number of a graph was introduced in [1, 5] and further studied in [2, 3, 6]. It was shown in [5] that determining the geodetic number of a graph is an NP-hard problem. Let 2^V denote the set of all subsets of V. The mapping $I : V \times V \to 2^V$ defined by $I[u, v] = \{w \in V : w \text{ lies on a } u - v \text{ geodesic in } G\}$ is the *interval function* of G. One of the basic properties of I is that $u,\ v \in I[u, v]$ for any pair $u,\ v \in V$. Hence the interval function captures every pair of vertices and so the problem of double geodetic sets is trivially well-defined while it is clear that this fails in many graphs already for triplets (for example, complete graphs). This motivated us to introduce and study double geodetic sets.

A set S of vertices in G is called a *double geodetic set* of G if for each pair of vertices x, y there exist vertices $u, v \in S$ such that $x, y \in I[u, v]$. The *double geodetic number* $dg(G)$ is the minimum cardinality of a double geodetic set. Any double geodetic of cardinality $dg(G)$ is called dg-set of G. The double geodetic number of graph was introduced and studied in [8]. The following theorems will be used in the sequel.

Theorem 1.1. [8] *Every double geodetic set of a connected graph G contains all the weak extreme vertices of G. In particular, if the set W of all weak extreme vertices is a double geodetic set, then W is the unique dg-set of G.*

Theorem 1.2. [8] *Let G be a connected graph with a cut-vertex v. Then each double geodetic set of G contains at least one vertex from each component of $G - v$.*

2. The connected double geodetic number of a graph

Definition 2.1. *Let G be a connected graph with at least two vertices. A connected double geodetic set of G is a double geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected double geodetic set of G is the connected double geodetic number of G and is denoted by $dg_c(G)$.*

Example 2.1. *For the graph G given in Figure 2.1, $S = \{v_1, v_4, v_5, v_6\}$ is a minimum double geodetic set of G so that $dg(G) = 4$. Since the subgraph induced by S is not connected, S is not a connected double geodetic set of G. It is clear that $T = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a minimum connected double geodetic set of G and so $dg_c(G) = 6$.*

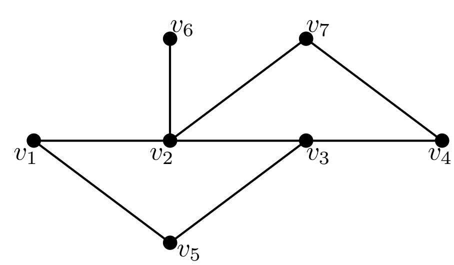

Figure 2.1: G

Theorem 2.1. *Each weak extreme vertex of a connected graph G belongs to every connected double geodetic set of G. In particular, every end-vertex of G belongs to every connected double geodetic set of G.*

Proof. Since every connected double geodetic set is also a double geodetic set, the result follows from Theorem 1.1. □

Corollary 2.1. *For the complete graph K_n $(n \geq 2)$, $dg_c(K_n) = n$.*

Theorem 2.2. *Let G be a connected graph with a cut-vertex v. Then each connected double geodetic set of G contains at least one vertex from each component of $G - v$.*

Proof. This follows from Theorem 1.2. □

Theorem 2.3. *Each cut-vertex of a connected graph G belongs to every connected double geodetic set of G.*

Proof. Let v be any cut-vertex of G and let $G_1, G_2, \ldots, G_r$ $(r \geq 2)$ be the components of $G - \{v\}$. Let S be any connected double geodetic set of G. Then by Theorem 2.2, S contains at least one element from each $G_i (1 \leq i \leq r)$. Since $G[S]$ is connected, it follows that $v \in S$. □

Corollary 2.2. *For a connected graph G with k weak extreme vertices and l cut-vertices, $dg_c(G) \geq max\ \{2, k+l\}$.*

Proof. This follows from Theorems 2.1 and 2.3. □

Corollary 2.3. *For any non-trivial tree T of order n, $dg_c(T) = n$.*

Proof. This follows from Corollary 2.2. □

Theorem 2.4. *For a connected graph G of order n, $2 \leq dg(G) \leq dg_c(G) \leq n$.*

Proof. Any double geodetic set needs at least two vertices and so $dg(G) \geq 2$. Since every connected double geodetic set is also a double geodetic set, it follows that $dg(G) \leq dg_c(G)$. Also, since $V(G)$ induces a connected double geodetic set of G, it is clear that $dg_c(G) \leq n$. □

Remark 2.1. *The bounds in Theorem* 2.4 *are sharp. For any non-trivial path P, $dg(P) = 2$. For the complete graph K_n, $dg(K_n) = dg_c(K_n)$. By Corollary 2.3, $dg_c(T) = n$ for any non-trivial tree T of order n. Also, all the inequalities in Theorem* 2.4 *are strict. For the graph G given Figure* 2.1*, $dg(G) = 4, dg_c(G) = 6$ and $n = 7$ so that $2 < dg(G) < dg_c(G) < n$.*

Corollary 2.4. *Let G be a connected graph. If $dg_c(G) = 2$, then $dg(G) = 2$.*

Proof. This follows from Theorem 2.4. □

Theorem 2.5. *Let G be a connected graph of order $n \geq 2$. Then $dg_c(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then $dg_c(G) = 2$. Conversely, let $dg_c(G) = 2$. Let $S = \{u, v\}$ be a minimum connected double geodetic set of G. Then uv is an edge. If $G \neq K_2$, then there exists a vertex w different from u and v, and w does not lie on any u-v geodesic so that S is not a dg_c-set, which is a contradiction. Thus $G = K_2$. □

Theorem 2.6. *Let G be a connected graph of order n. Then $dg_c(G) = n$ if and only if every vertex of G is either a cut-vertex or a weak extreme vertex.*

Proof. Let G be a connected graph with every vertex of G either a cut-vertex or weak extreme vertex. Then the result follows from Theorems 2.1 and 2.3. Conversely, let G be a connected graph of order n with $dg_c(G) = n$. Suppose that there exists a vertex v which is neither a weak extreme vertex nor a cut-vertex of G. We show that $S = V - \{v\}$ is a connected double geodetic set of G. Since v is not a cut-vertex of G, the subgraph induced by S is connected. Let $u \neq v$ be any vertex of G. Since v is not a weak extreme vertex of G, we have $u, v \in I[x, y]$ for a pair of vertices $x, y \in G$ with $v \neq x$ and $v \neq y$. This shows that S is a double geodetic set of G. Thus S is a connected double geodetic set of G and so $dg_c(G) \leq n - 1$, which is a contradiction. Hence every vertex of G is either a cut-vertex or a weak extreme vertex. □

Theorem 2.7. *If $n, a,$ and b are integers such that $2 \leq a < b \leq n$, then there exists a connected graph G of order n such that $dg(G) = a$ and $dg_c(G) = b$.*

Proof. The theorem is proved by considering three cases.
Case 1. $2 \leq a < b = n$. Let G be any tree of order b with number of end-vertices equal to a. Then by Theorem 1.1, $dg(G) = a$ and by Corollary 2.3, $dg_c(G) = n$.
Case 2. $2 = a < b < n$. Let $P_b : u_1, u_2, \ldots, u_b$ be a path on b vertices. Add $(n - b)$ new vertices $w_1, w_2, \ldots, w_{n-b}$ to P_b and join $w_1, w_2, \ldots, w_{n-b}$ to both u_1 and u_3, thereby producing the graph G of Figure 2.2. Then G has order n and $S = \{u_1, u_b\}$ is the unique minimum double geodetic set of G and so by Theorem 1.1 $dg(G) = 2 = a$. Also, $S_1 = \{u_1, u_3, u_4, \ldots, u_b\}$ is the set of all cut-vertices and weak extreme vertices of G. By Theorems 2.1 and 2.3, every connected double geodetic set contains S_1. It is clear that S_1 is not a connected double geodetic set of G. Since $S_1 \cup \{u_2\}$ is a connected double geodetic set of G, it follows that $dg_c(G) = b$.

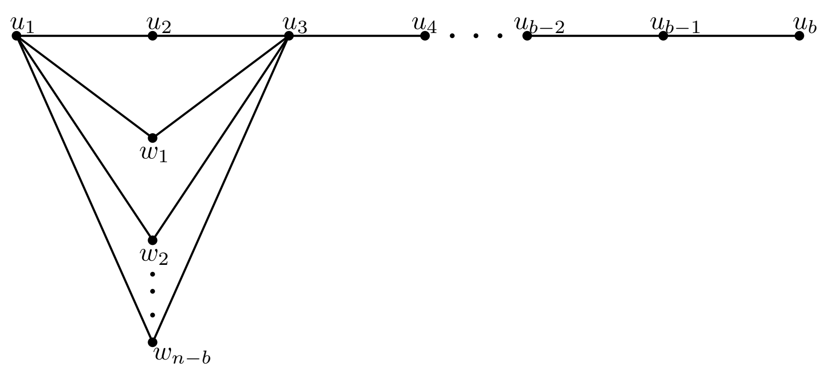

Figure 2.2: G

Case 3. $3 \leq a < b < n$. First assume that $b \neq a + 1$. Let P_{b-a+2} : $u_1, u_2, \ldots, u_{b-a+2}$ be a path on $b - a + 2$ vertices. Add $a - 2 + n - b$ new vertices $v_1, w_1, w_2, \ldots, w_{a-3}, x_1, x_2, \ldots, x_{n-b}$ to P_{b-a+2} and join v_1 to u_2, and join $w_1, w_2, \ldots, w_{a-3}$ to both u_1 and u_3, and join $x_1, x_2, \ldots, x_{n-b}$ to both u_2 and u_4 thereby producing the graph G Figure 2.3. Then G has order n and $S = \{v_1, u_1, w_1, w_2, \ldots, w_{a-3}, u_{b-a+2}\}$ is the unique minimum double geodetic set of G and so by Theorem 1.1 $dg(G) = a$. Also $S_1 = \{u_2, u_4, u_5, \ldots, u_{b-a+2},$

$v_1, u_1, w_1, w_2, \ldots, w_{a-3}\}$ is the set of all cut-vertices and weak extreme vertices of G. By Theorems 2.1 and 2.3, every connected double geodetic set contains S_1. It is clear that S_1 is not a connected double geodetic set of G. Since $S_1 \cup \{u_3\}$ is a connected double geodetic set of G, we have $dg_c(G) = b$.

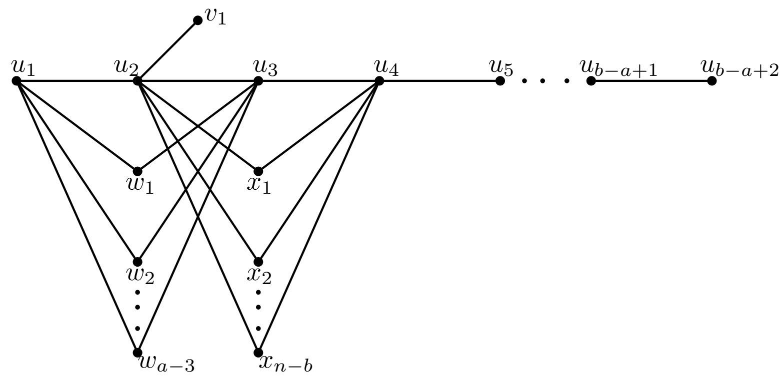

Figure 2.3: G

Next, assume that $b = a + 1$. Let $V(K_{n-a}) = \{u_1, u_2, \ldots, u_{n-a}\}$ and $V(K_a) = \{v_1, v_2, \ldots, v_a\}$. Let $G = \overline{K_a} + K_{n-a}$. Then G has order n and $S = \{v_1, v_2, \ldots, v_a\}$ is the unique minimum double geodetic set of G and so by Theorem 1.1 $dg(G) = a$. By Theorem 2.1, every connected double geodetic set contains S. It is clear that S is not a connected double geodetic set of G. Since $S \cup \{u_1\}$ is a connected double geodetic set of G, it follows that $dg_c(G) = a + 1 = b$. □

For every connected graph G, $rad\ G \leq diam\ G \leq 2\ rad\ G$. Ostrand [7] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the connected double geodetic number can also be prescribed.

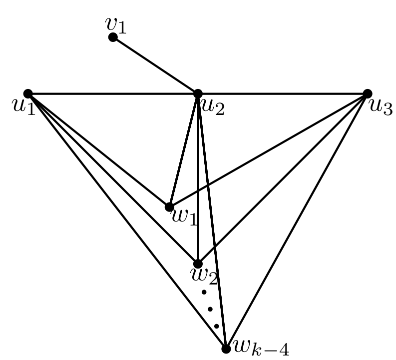

Figure 2.4: G

Theorem 2.8. *For positive integers r, d and $k \geq 4$ with $r \leq d \leq 2r$ and $k-d-1 \geq 0$, there exists a connected graph G with rad $G = r$, diam $G = d$ and $dg_c(G) = k$.*

Proof. If $r = 1$, then $d = 1$ or 2. For $d = 1$, let $G = K_k$. Then $dg_c(G) = k$. For $d = 2$, construct a graph G as follows: Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Add a new vertex v_1 to P_3 and join to the vertex u_2 and obtain the graph H. Also, add $(k-4)$ new vertices $w_1, w_2, \ldots, w_{k-5}$ to H and join each $w_i (1 \leq i \leq k-4)$ to u_1, u_2 and u_3 and obtain the graph G in Figure 2.4. Then *rad* $G = 1$ and *diam* $G = 2$. It is clear that $v_1, u_1, u_3, w_1, w_2, \ldots, w_{k-4}$ are the weak extreme vertices of G and u_2 is the only cut-vertex of G. Hence by Theorem 2.6, $dg_c(G) = k$.

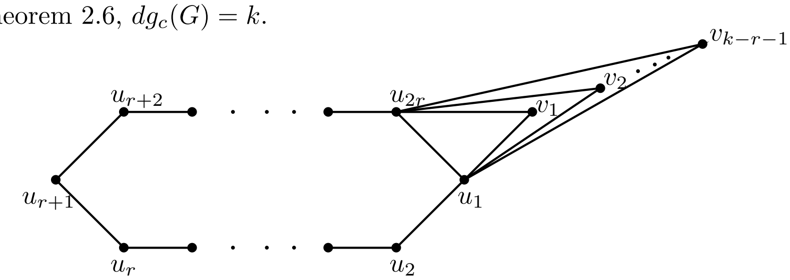

Figure 2.5: G

Now, let $r \geq 2$.

Case 1. $r = d$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$. Let G be the graph given in Figure 2.5, obtained by adding the new vertices $v_1, v_2, \ldots, v_{k-r-1}$ and joining each $v_i (i \leq i \leq k-r-1)$ with u_1 and u_{2r} of C_{2r}. It is easily verfied that the eccentricity of each vertex of G is r so that *rad* $G =$ *diam* $G = r$. Let $S = \{v_1, v_2, \ldots, v_{k-r-1}, u_r, u_{r+1}, u_1\}$ be the set of all weak extreme vertices of G. By Theorem 2.1, every connected double geodetic set of G contains S. It is clear that S is not a connected double geodetic set of G. Since $S_1 = S \cup \{u_2, u_3 \ldots u_{r-1}\}$ is a connected double geodetic set of G, it follows that $dg_c(G) = k$.

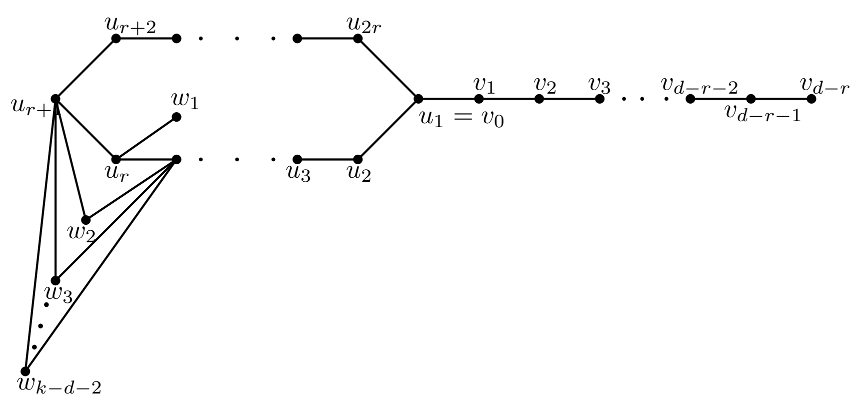

Figure 2.6: G

Case 2. $r < d$. Let $C_{2r} : u_1, u_2, \ldots, u_{2r}, u_1$ be a cycle of order $2r$ and let $P_{d-r+1} : v_0, v_1, \ldots, v_{d-r}$ be a path of order $d - r + 1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying v_0 of P_{d-r+1} and u_1 of C_{2r}. Now, add $k - 6$ new vertices $w_1, w_2, \ldots, w_{k-6}$ to the graph H and join w_1 to u_r, and join each vertex $w_i (2 \leq i \leq k - d - 2)$ to both u_{r+1} and u_{r-1}, thereby obtaining the graph G in Figure 2.6. Then $rad\ G = r$ and $diam\ G = d$. Now, $S_1 = \{w_1, w_2, \ldots, w_{k-d-2}, u_{r+1}, u_{2r}, v_{d-r}\}$ is the set of all weak extreme vertices of G and $S_2 = \{u_r, u_1, v_1, v_2, \ldots, v_{d-r-1}\}$ is the set of all cut-vertices of G. By Theorems 2.1 and 2.3, every connected double geodetic set contains $S_1 \cup S_2$. Although $S_1 \cup S_2$ is a double geodetic set, it is not a connected double geodetic set of G. It is clear that $T = S_1 \cup S_2 \cup \{u_2, u_3, \ldots, u_{r-1}\}$ is a minimum connected double geodetic set of G and so $dg_c(G) = k$. □

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