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ON ALMOST SURE CONVERGENCE OF NEGATIVELY SUPERADDITIVE DEPENDENT FOR SEMI-GAUSSIAN RANDOM VARIABLES†

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Abstract. When $\{X_{ni}|1 \leq i \leq n, n \geq 1\}$ be an array of rowwise negatively superadditive dependent(NSD) for semi-Gaussian random variables and $\{a_{ni}|1 \leq i \leq n, n \geq 1\}$ is an array of constants, we study the almost sure convergence of weighted sums $\sum_{i=1}^{n} a_{ni}X_{ni}$ under some appropriate conditions and we obtain some corollaries.

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1. Introduction

Hsu and Robbins[9] introduced the concept of complete convergence of a sequence $\{X_n\}$ of random variables as follows. A sequence $\{X_n\}$ of random variables is said to converge completely to a constant c if

$$\sum_{n=1}^{\infty} P(|X_n - c| > \epsilon) < \infty, \text{ for every } \epsilon > 0$$

If $X_n \to c$ completely, then the Borel-caltelli Lemma implies that $X_n \to c$ almost sure, but the inverse is not true in general. Moreover, it was proved that the sequence of arithmetic means of independent identically distributed($i.i.d.$) random variables converges completely to the expected value if the variance of the summands is finite by Hsu and Robbins. This result has been generalized and extended in several directions and carefully studied by many authors (see, Chow[4]; Ouy[13]; Taylor and Tien[14]; Gut[8]; Bozorgnia et al[3]; Ghosal and

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Chandra[7]; Hu et al[11]; Ahmed et al[1]; Wang et al[16,17]). Almost sure convergence for a sequence of random variables plays a central role in the area of limit theorems in probability theory and mathematical statistics. Conditions of independence and identical distribution of random variables are basic in historic results due to Bernoulli, Borel or Kolmogorov. Since then, serious attempts have been made to relax these strong conditions. For example, independence has been relaxed to pairwise independence or pairwise negative quadrant dependence or, even replaced by conditions of dependence such as mixing or martingale. In particular, many authors showed that many results could be obtained by replacing *i.i.d.* condition by uniformly bounded condition. We call that an array of random variables $\{X_{ni}|1 \leq i \leq n,\ n \geq 1\}$ is said to be uniformly bounded by a random variable X if for all n and $x \geq 0$,

$$P(|X_{ni}| > x) = O(1)P(|X| > x).$$

Hu[10] was introduced the concept of NSD random variables which is based on the class of superadditive functions.

Definition 1.1. ([10]) A random vector $(X_1, X_2, \cdots, X_n)$ is said to be NSD if

$$E\phi(X_1, X_2, ..., X_n) \leq E\phi(X_1^*, X_2^*, ..., X_n^*)$$

where $X_1^*, X_2^*, \cdots, X_n^*$ are independent such that X_i^* and X_i have the same distribution for each i and ϕ is a superadditive function such that the expectations in the above equation exist.

Definition 1.2. ([10]) A sequence $\{X_n : n \geq 1\}$ of random variables is said to be NSD if for all $n \geq 1, (X_1, X_2, ..., X_n)$ is NSD.

An array $\{X_{ni}|n \geq 1, i \geq 1\}$ of random variables is said to be NSD if for an $n \geq 1, (X_1, X_2, \cdots, X_n)$ is said to be NSD

Since the assumption of NSD for a sequence of random variables is much weaker than an independence, negative association, or negative dependence, a study on a limiting behavior of NSD sequences is of interest.

The concept of NSD random variables was first introduced by Hu and also gave an example illustrating that NSD does not imply NA, and posed an open problem whether NA implies NSD, but Christofides and Vaggelatou[5] was introduced the NA implies NSD and so it is weaker than NA and it's structure is an extension of NA, and sometimes more useful than NA (see Joag-Dev and Proschan[12])

Moreover, the notion of random variables has wide applications in multivariate statistical analysis and reliability theory and is very important for probability inequality(See, Block et al [2], Eghbal et al[6], Shen et al[15], Wang et al[16, 17]). Hence it is of important significance to extend the limit properties of the case of random variables.

The main purpose of this paper is to provide the almost sure convergence results for weighted sums of arrays of rowwise NSD with semi-Gaussian random variables though exponential bounds of semi-Gaussian type under some conditions. The outline of this paper is as follows. We give a definition and some lemmas in section 2 for section 3 and the main results and some corollaries will be provided in section 3.

2. Preliminaries

To prove the main results, we need to introduce a definition and present some lemmas. The statement of the first definition we could found in Chow[4].

Definition 2.1. ([4]) A random variable X with $EX = 0$ is said to be semi-Gaussian, if there exists $\alpha \geq 0$ such that for every real number t,

$$Ee^{tX} \leq e^{\alpha^2 t^2/2}. \tag{1}$$

The minimum of those α satisfying (1) is denoted by $\tau(X)$. See([4])

Lemma 2.2. *([10]) Let $(X_1, X_2, X_3, \cdots, X_n)$ be an NSD random vector and $f_1, f_2, \cdots, f_n$ are non-decreasing functions, then $f_1(X_1), f_2(X_2), \cdots, f_n(X_n)$ are NSD.*

Lemma 2.3. *Let $(X_1, X_2, X_3, \cdots, X_n)$ be an NSD random vector with $\tau(X_i) \leq \alpha_i$ and $\sum_{i=1}^n \alpha_i^2 = \alpha^2$. Then for each $n \geq 1$ and $t > 0$,*

$$Ee^{\sum tX_i} \leq \prod_{i=1}^n Ee^{tX_i} \leq e^{\alpha^2 t^2/2}.$$

Proof. By Definition 1.1,and Lemma 2.2, we can obtain a result of Lemma 2.3. □

Lemma 2.4. *Suppose that $\{X_{ni} | 1 \leq i \leq n,\ n \geq 1\}$ be an array of rowwise NSD for semi-Gausian random variables and let $P(|X_{ni}| > x) = O(1)P(|X| > x)$ with $\tau(X) \leq \alpha$ for $1 \leq i \leq n,\ n \geq 1$. Then for each $\epsilon > 0$, $P(|\sum_{i=1}^n X_{ni}| > \epsilon) \leq O(1)e^{-\epsilon^2/2n\alpha^2}$.*

Proof. By Lemma 2.3,

$$\begin{aligned}
P(|\sum_{i=1}^n X_{ni}| > \epsilon) &\leq e^{-\epsilon t} Ee^{t|\sum_{i=1}^n X_{ni}|} \\
&\leq e^{-\epsilon t}(\prod_{i=1}^n Ee^{tX_{ni}}) + e^{-\epsilon t}(\prod_{i=1}^n Ee^{-tX_{ni}}) \\
&= O(1)e^{-\epsilon t}(\prod_{i=1}^n Ee^{tX}) + e^{-\epsilon t}(\prod_{i=1}^n Ee^{-tX}) \\
&\leq O(1)e^{-\epsilon^2/2n\alpha^2},\ \text{taking } t = \epsilon/n\alpha^2.
\end{aligned}$$

□

Lemma 2.5. *Let $\{X_{ni}|1 \le i \le n, n \ge 1\}$ be an array of rowwise NSD for semi-Gaussian random variables and let $P(|X_{ni}| > x) = O(1)(|X| > x)$ with $\tau(X) \le \alpha$ for all $1 \le i \le n,\ n \ge 1$ and $x \ge 0$. Assume that $\{a_{ni}|1 \le i \le n,\ n \ge 1\}$ is an array of constants, and that $A_n = \sum_{i=1}^{\infty} a_{ni}^2$ for each $n \ge 1$. Then for every $\epsilon > 0$,*

$$P(|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \epsilon) \le O(1)e^{-\frac{\epsilon^2}{2\alpha^2 A_n}}$$

Proof. By Definition 2.1, Lemma 2.2, 2.3 and $t \in R$,

$$\begin{aligned} E(e^{t\sum_{i=1}^{n} a_{ni}X_{ni}}) &\le O(1)E(\prod_{i=1}^{n} e^{ta_{ni}X}) \\ &\le O(1)e^{t^2\alpha^2\sum_{i=1}^{n} a_{ni}^2/2}. \end{aligned}$$

Hence, by Fatou's Lemma,

$$E(e^{t\sum_{i=1}^{\infty} a_{ni}X_{ni}}) \le O(1)e^{t^2\alpha^2 A_n/2}$$

Therefore,

$$\begin{aligned} &P(|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \epsilon) \\ &\le e^{-\epsilon t}E(e^{t\sum_{i=1}^{\infty} a_{ni}X_{ni}}) + e^{-\epsilon t}Ee^{-t\sum_{i=1}^{\infty} a_{ni}X_{ni}} \\ &\le O(1)e^{-\frac{\epsilon^2}{2\alpha^2 A_n}},\ \text{taking } t = \epsilon/\alpha^2 A_n. \end{aligned}$$

□

3. Main Results

With the Definition 2.1, Lemma 2.3 and 2.4, we could now present our first result.

Theorem 3.1. *Let $\{X_{ni}|1 \le i \le n, n \ge 1\}$ be an array of rowwise NSD for semi-Gaussian random variables and let $P(|X_{ni}| > x) = O(1)P(|X| > x)$ with $\tau(X) \le \alpha$ for all $1 \le i \le n,\ n \ge 1$ and $x \ge 0$. If for every n, and $p > 0$, $\frac{1}{n^{1/p}}\sum_{i=1}^{m} a_{ni}X_{ni}$ converges in probability as $m \to \infty$, then $\frac{1}{n^{1/p}}\sum_{i=1}^{m} a_{ni}X_{ni}$ converges almost surely as.*

Proof. If $\sum_{i=1}^{m} a_{ni}X_{ni} \to k_n$ in probability for every n, then there exists a subsequence $\{m_k|k \ge 1\}$ such that $\sum_{i=1}^{m_k} a_{ni}X_{ni} \to k_n$ almost surely. Next, we define that

$$\frac{1}{n^{1/p}}T_{nk} = \frac{1}{n^{1/p}} \max_{m_k < m \le m_{k+1}} |\sum_{i=1}^{m} a_{ni}X_{ni} - \sum_{i=1}^{m_k} a_{ni}X_{ni}|.$$

Then, $p > 0$ and by Lemma 2.5,

$$P(T_{nk} > n^{1/p}\epsilon)$$

$$
\begin{aligned}
&= P(\max_{m_k<m\le m_{k+1}} |\sum_{i=1}^{m} a_{ni}X_{ni} - \sum_{i=1}^{m_k} a_{ni}X_{ni}| > n^{1/p}\epsilon) \\
&\le e^{-n^{1/p}\epsilon t} E e^{\sum_{i=m_k+1}^{m_{k+1}} |a_{ni}X_{ni}|t} \\
&\le e^{-n^{1/p}\epsilon t} \prod_{i=m_k+1}^{m_{k+1}} E e^{(a_{ni}X_{ni})t} + e^{-n^{1/p}\epsilon t} \prod_{i=m_k+1}^{m_{k+1}} E e^{-(a_{ni}X_{ni})t} \\
&\le O(1) e^{-\frac{n^{2/p}\epsilon^2}{2\alpha^2 \sum_{i=m_k+1}^{\infty} a_{ni}^2}}, \text{ taking } t = \frac{n^{1/p}\epsilon}{\alpha^2 \sum_{i=1}^{\infty} a_{ni}^2} \\
&= O(1) e^{-\frac{n^{2/p}\epsilon^2}{2\alpha^2 A_n}} < \infty, \text{ taking } A_n = \sum_{i=1}^{\infty} a_{ni}^2 < \infty.
\end{aligned}
$$

Therefore,

$$P(\frac{1}{n^{1/p}} T_{nk} > \epsilon,\ i.\ o.) = 0$$

follows from the Borel-Cantelli Lemma. Thus

$$
\begin{aligned}
&\frac{1}{n^{1/p}} |\sum_{i=1}^{m} a_{ni}X_{ni} - k_n| = \frac{1}{n^{1/p}} |\sum_{i=1}^{m} a_{ni}X_{ni} - \sum_{i=1}^{m_k} a_{ni}X_{ni} + \sum_{i=1}^{m_k} a_{ni}X_{ni} - k_n| \\
\le\ & \frac{1}{n^{1/p}} \max_{m_k<m\le m_{k+1}} (|\sum_{i=1}^{m} a_{ni}X_{ni} - \sum_{i=1}^{m_k} a_{ni}X_{ni}| + |\sum_{i=1}^{m_k} a_{ni}X_{ni} - k_n|) \\
=\ & \frac{1}{n^{1/p}} T_{nk} + \frac{1}{n^{1/p}} |\sum_{i=1}^{m_k} a_{ni}X_{ni} - k_n| \to 0 \text{ almost surely.}
\end{aligned}
$$

□

Theorem 3.2. *Suppose that* $\{X_{ni} | 1 \le i \le n,\ n \ge 1\}$ *be an array of rowwise NSD for semi-Gaussian random variables and let* $P(|X_{ni}| > x) = O(1)P(|X| > x)$ *with* $\tau(X) \le \alpha$ *for* $1 \le i \le n,\ n \ge 1$ *and* $x \ge 0$. *Assume that* $0 < a_{ni} \le An^{-\beta}$ *for some* $0 < A < \infty,\ i \le n$ *and* $\beta > 0$. *If* $p > \frac{-2}{2\beta-1}$, *then*

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} a_{ni}X_{ni} \to 0 \quad \text{almost surely as } n \to \infty.$$

Proof. For every $\epsilon > 0$, by Lemma 2.5,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni}X_{ni}| > n^{1/p}\epsilon) \\
\le\ & O(1) \sum_{n=1}^{\infty} e^{-\frac{(n^{1/p}\epsilon)^2}{2\alpha^2 \sum_{i=1}^{n} a_{ni}^2}} \\
\le\ & O(1) \sum_{n=1}^{\infty} e^{-\frac{\epsilon^2 n^{2/p+2\beta-1}}{2\alpha^2 A^2}} < \infty.
\end{aligned}
$$

Hence, by Borel-Cantelli Lemma, it follows that

$$\lim_{n\to\infty} P(|\frac{1}{n^{1/p}}\sum_{i=1}^{n} a_{ni}X_{ni}| > \epsilon,\ i.\ o.) = 0.$$

Thus, $\frac{1}{n^{1/p}}\sum_{i=1}^{n} a_{ni}X_{ni} \to 0$ almost surely as $n \to \infty$.

□

The next two corollaries follow immediately from Theorem 3.2.

Corollary 3.3. *Under the Theorem 3.2, if* $n^{-1/p} = 1$ *and* $a_{ni} = n^{-\beta}$ *for* $n \geq 1$, $1 \leq i \leq n$, *for some* $\beta > 1$,

$$\frac{1}{n^{\beta}}\sum_{i=1}^{n} X_{ni} \to 0 \text{ almost surely as } n \to \infty.$$

Proof. For each $\epsilon > 0$,

$$\begin{aligned} & \sum_{n=1}^{\infty} P(|\frac{1}{n^{\beta}}\sum_{i=1}^{n} X_{ni}| > \epsilon) \\ \leq \quad & O(1)\sum_{n=1}^{\infty} e^{\frac{-n^{2\beta}\epsilon^2}{2n\alpha^2}} \\ = \quad & O(1)\sum_{n=1}^{\infty} e^{-\frac{n^{\beta-1}\epsilon^2}{2c}} < \infty. \end{aligned}$$

Hence, by Borel-Cantelli Lemma,

$$\frac{1}{n^{\beta}}\sum_{i=1}^{n} X_{ni} \to 0 \text{ almost surely, as } n \to \infty.$$

□

Corollary 3.4. *Under the Theorem 3.2, if* $\sum_{i=1}^{n} a_{ni}^2 < Bn^{-\beta}$ *for some* $0 < B < \infty$ *and* $\beta > 0$, *if* $0 < p < 2$, *then*

$$\frac{1}{n^{1/p}}\sum_{i=1}^{n} a_{ni}X_{ni} \to 0 \text{ almost surely, as } \ n \to \infty.$$

Proof. We can obtain the result from Theorem 3.2.

□

The next theorem is an application of Lemma 2.5 and we show that the property NSD among random variables can be preserved through suitable combinations and transformations of random variables.

Theorem 3.5. *Suppose that* $\{X_{ni}|1 \leq i \leq n,\ n \geq 1\}$ *be an array of* NSD *for semi-Gaussian random variables and let* $\{Y_{ni}|1 \leq i \leq n,\ n \geq 1\}$ *be an array of independent random variables. Let* $\{X_{ni}|1 \leq i \leq n,\ n \geq 1\}$ *and* $\{Y_{ni}|1 \leq i \leq n, n \geq 1\}$ *be two array of rowwise independent random variables. Assume that* $P(|X_{ni}| > x) = O(1)P(|X| > x)$ *with* $\tau(X) \leq \alpha$ *for all* $1 \leq i \leq n,\ n \geq 1$. *If* $f(x)$ *is a monotone, nonnegative function bounded by* β, *then* $\frac{1}{n^{1/p}} \sum_{i=1}^{n} a_{ni} X_{ni} f(Y_{ni})$ *converges almost surely, where* $\tau(Xf(Y_{ni})) \leq \alpha\beta$.

Proof. If X and Y are NSD for semi-Gaussian random variable and T and W are independent random variables and independent of X and Y, we can check that $h(X,T)$ and $g(Y,W)$ are NSD for semi-Gaussian random variable for nondecreasing functions h and g, and thus $a_{ni}X_{ni}f(Y_{ni})$ is a NSD for semi-Gaussian random variables. Therefore for each $p > 0$ and $\epsilon > 0$, by Lemma 2.5, since

$$\begin{aligned} E(e^{ta_{ni}X_{ni}f(Y_{ni})}) &= E(E(e^{ta_{ni}X_{ni}f(Y_{ni})|Y_{ni}})) \\ &\leq O(1)E(e^{t^2\alpha^2\beta^2\sum_{i=1}^{\infty} a_{ni}^2}) \\ &= O(1)e^{t^2\alpha^2\beta^2\sum_{i=1}^{\infty} a_{ni}^2} \end{aligned}$$

$$\begin{aligned} & \sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni}X_{ni}f(Y_{ni})| > n^{1/p}\epsilon) \\ \leq\ & \sum_{n=1}^{\infty} e^{-n^{1/p}\epsilon t} E(e^{t|\sum_{i=1}^{n} a_{ni}X_{ni}f(Y_{ni})|}) \\ =\ & O(1)\sum_{n=1}^{\infty} e^{-n^{1/p}\epsilon t}(E(e^{t\sum_{i=1}^{n} a_{ni}Xf(Y_{ni})}) + E(e^{-t\sum_{i=1}^{n} a_{ni}Xf(Y_{ni})})) \\ \leq\ & O(1)e^{-n^{1/p}\epsilon t}(\prod_{i=1}^{\infty} Ee^{ta_{ni}Xf(Y_{ni})} + \prod_{i=1}^{\infty} Ee^{-ta_{ni}Xf(Y_{ni})}) \\ \leq\ & O(1)e^{-\frac{n^{2/p}\epsilon^2}{2\alpha^2\beta^2 A_n}} < \infty, \quad \text{taking } A_n = \sum_{i=1}^{\infty} a_{ni}^2 < \infty \end{aligned}$$

Hence, by Borel-Cantelli Lemma,

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} a_{ni}X_{ni}f(Y_{ni}) \to 0 \ \text{ almost surely as } n \to \infty.$$

□

References

1. S.E. Ahmed, R.G. Antonini and A. Volodin, *On the rate of complete convergence for weighted sums of arrays of Banach space valued random elements with application to moving average processes*, Statistics & Probability Letters **58** (2002), 185-194.
2. H.W. Block, W.S. Griffths and T.H. Savits, *L-superadditive structure functions*, Adv. Appl. Probab. **21** (1989), 919-929.
3. A. Bozorgnia, R.E. Patterson and R.L. Taylor, *Limit theorems for ND r.v.'s*, Technical Report; University of Georgia, 1993.
4. Y.S. Chow, *Some convergence theorems for independent r.v.'s*, Ann. Mat. Statist. **37** (1966), 1482-1493.
5. T.C. Chirstofides and E. Vaggelatou, *A connection between supermodular ordering and positive/negative association*, Journal of Multivariate Analysis **88** (2004), 138-151.
6. N. Eghbal, M. Amini and A. Bozorgnia, *Some maximal inequalities for quadra-tic forms of negative superadditive dependence r.v.'s*, Stat. Probab. Lett. **80** (2010), 587-591.
7. S. Ghosal and T.K. Chandra, *Complete convergence of martingale arrays*, Journal of Theoretical Probability **11** (1998), 621-631.
8. A. Gut, *Complete convergence for arrays*, Periodica Mathematica Hungarica **25** (1992), 51-75.
9. P.L. Hsu and H. Robbins, *Complete convergence and the law of large numbers for arrays*, In Proceedings of the National Academy of Sciences of the United States of America **33** (1947), 25-31.
10. T.Z. Hu, *Negatively superadditive dependence of r.v.'s with applications*, Chinese Journal of Applied Probability and Statistics **16** (2000), 133-144.
11. T.C. Hu, A. Rosalsky, D. Szynal and A. Volodin, *On complete convergence for arrays of rowwise independent random elements in Banach spaces*, Stochastic Analysis and Applications **17** (1999), 963-992.
12. D.K. Joag and F. Proschan, *Negative association of random variables with applications*, Ann. Statist. **11** (1983), 286-295.
13. K.C. Ouy, *Some convergence theorems for dependent generalized Gaussian r.v.'s*, J. Natl. Chiao. Tung. University **1** (1976), 227-246.
14. R.L. Taylor and C.H. Tien, *Sub-Gaussian techniques in proving strong law of large numbers*, Teach. Math. **94** (1987), 295-299.
15. A.T. Shen, Y. Zhang, A. Volodin, *Applications of the Rosenthal-type inequality for negatively superadditive dependent r.v.'s*, Metrika **78** (2015), 295-311.
16. X. Wang, T.C. Hu, A. Volodin and S. Hu, *Complete convergence for weighted sums and arrays of rowwise extended negatively dependent random variables*, JKMS. **50** (2013), 379-392.
17. X.J. Wang, X. Deng, L.L. Zheng, S.H. Hu, *Complete convergence for arrays of rowwise negatively superadditive dependent r.v.'s and its applications*, Statistics **48** (2014), 834-850.

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